DIFFRACTION OF A MAGNETOELASTIC WAVE ON STRESS CONCENTRATORS IN CONDUCTING BODIES (ANTIPLANAR DEFORMATION)

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We consider a two-dimensional boundary-value problem of magnetoelasticity for a half-space weakened by tunnel stress concentrators (cracks, holes) in the presence of a static magnetic field. The mechanical stimulus is taken as a magnetoelastic shear wave incident from infinity or a shear load that varies harmonically in time and is prescribed on the edges of the crack or hole. The problem reduces to a singular integral equation that can be solved numerically by the method of mechanical quadratures. We give the results of computation of the coefficient of stress intensity K_{111} for a slit and the stress concentration on the edge of a hole. We conclude that it is necessary to take account of electromagnetic effects in estimating the strength of diamagnetic or paramagnetic bodies. Four figures. Bibliography: 6 titles.

If a diamagnetic (resp. paramagnetic) body in a static magnetic field is subjected to a mechanical stimulus, induced (eddy) currents arise in the body, leading to Lorentz solid forces. Taking account of these forces gives an additional tensor, the Maxwell stress tensor, which introduces significant corrections into the stressed state of the body.

In what follows we consider a boundary-value problem of magnetoelasticity for a half-space weakened by tunnel stress concentrators (cracks or holes)

Starting from the relations of linear magnetoelasticity [1–3], ascribing ideal conductivity to the medium and assuming the electromagnetic field is quasi-static ($\vec{D}=0$, $\partial \vec{D}/\partial t = 0$) we have a complete system of equations

$$\mu\nabla^2 \vec{u} + (\lambda+\mu)\operatorname{grad}\operatorname{div}\vec{u} + \mu_e \operatorname{curl}\vec{h}\times\vec{H}^0 = \rho\frac{\partial^2 \vec{u}}{\partial t^2}; \tag{1}$$

$$\vec{h} = \operatorname{curl}(\vec{u}\times\vec{H}^0), \quad \vec{e} = -\mu_e\left(\frac{\partial \vec{u}}{\partial t}\times\vec{H}^0\right); \quad [\vec{h}]_\tau = 0, \quad [\mu_e\,\vec{h}]_n = 0;$$

$$[\sigma_{ij}+t_{ij}]n_j = X_{in}, \quad \nabla^2 = \partial_1^2+\partial_2^2+\partial_3^2, \quad \partial_i = \frac{\partial}{\partial x_i},$$

$$t_{ij} = \mu_e(H_i^0 h_j + H_j^0 h_i - \delta_{ij}H_k^0 h_k), \quad (i,j,k=\overline{1,3}).$$

Here $\vec{H}^0 = (H_1^0; H_2^0, H_3^0)$ is the intensity of the external magnetic field, $\vec{e}$ and $\vec{h}$ are the fluctuations of the electric and magnetic fields, $\vec{u}$ are the mechanical displacements, σ_{ij} and t_{ij} are respectively the mechanical and Maxwell stresses, X_{in} are the components of the external load, μ_e is the magnetic permeability of the substance, ρ is the density of the substance, μ and λ are the Lamé constants, δ_{ij} is the Kronecker symbol, and x_i are rectangular Cartesian coordinates. The symbol [*] denotes the jump in a quantity at the interface line of the two media.

We assume that the magnetoelastic medium occupies the half-space $x_2 \geq 0$ (Fig. 1) and contains tunnel concentrators of crack type L_j $(j=\overline{1,k})$ or hole type l_i $(i=\overline{1,n})$ along x_3. Suppose the half-space is adjacent to a vacuum in which there is a static magnetic field $\vec{H}_*^0 = (0; H_0^*; 0)$, $H_0^* = \text{const}$. We shall take the mechanical stimulus to be a shear load $X_{3n} = \operatorname{Re}(X_3 e^{-i\omega t})$, $X_3 = X_3(x_1; x_2)$ acting on the surface of the cavity and harmonic with respect to time or a magnetoelastic displacement wave

$$u_3^0 = \operatorname{Re}(U_3^0 e^{-i\omega t}); \tag{2}$$

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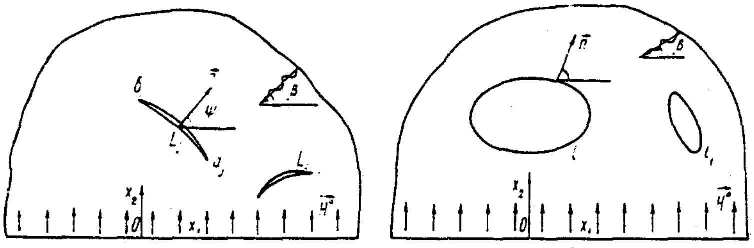

a b

Fig. 1

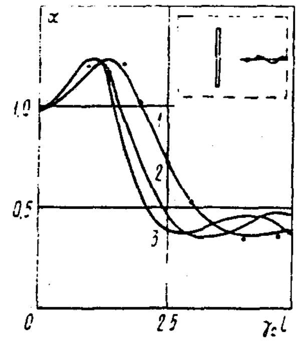

Fig. 2

$$U_3^0 = U \exp(-i\gamma(x_1 \cos\beta + x_2 \sin\beta)), \quad U = \text{const};$$

$$\gamma = \frac{\gamma_2}{\sqrt{1+\chi^2 \sin^2\beta}}, \quad \gamma_2 = \frac{\omega}{c_2}, \quad c_2 = \sqrt{\frac{\mu}{\rho}}, \quad \chi^2 = \frac{\mu_e H_0^2}{\mu},$$

where β is the angle between the normal to the front of the incident wave and the ox_1-axis and ω is the cyclic frequency.

Under these conditions a wave-type mechanical and electromagnetic fields arise in the medium corresponding to a state of anti-planar deformation. The complete system of equations has the following form:

the equations of motion

$$\nabla^2 u_3 + \chi^2 \partial_2^2 u_3 = \frac{1}{c_2^2}\frac{\partial^2 u_3}{\partial t^2}; \tag{3}$$

the components of the electromagnetic field

$$h_1 = h_2 = 0, \quad h_3 = H_0 \partial_2 u_3, \quad e_1 = -\mu_e H_0 \frac{\partial u_3}{\partial t}, \quad e_2 = e_3 = 0; \tag{4}$$

the boundary conditions on the edge of the cavity

$$\sigma_{13}\cos\psi + \sigma_{23}\sin\psi = X_{3n}; \quad h_3^* = h_3; \quad \sigma_{i3} = \mu\partial_i u_3. \tag{5}$$

Here ψ is the angle between the positive normal to the edge and the ox_1-axis (Fig. 1); the asterisk refers to the cavity of the defect.

Consider the half-space with tunnel cracks L_j $(j = \overline{1,k})$ along x_3. In accordance with what was shown above there is a static magnetic field in it $\vec{H}^{\,0} = (0; H_0; 0)$, $H_0 = \mu_0 H_0^*/\mu_e$, where μ_0 is the magnetic permeability of a vacuum.

Under mechanical stimulus a stationary wave (oscillation) process takes place in the body, and the components of the fields σ_{i3}, t_{23} $(i = 1, 2)$ and h_3 have a characteristic fractional-power singularity at the ends of the defects, leading to the need to take account of the influence of electromagnetic effects on the stress of the body.

The mechanical field in the half-space with defect is composed of the incident wave field (2), the reflected wave field

$$u_3^1 = \mathrm{Re}\,(U_3^1 e^{-i\omega t}), \quad U_3^1 = U\exp(-i\gamma(x_1\cos\beta - x_2\sin\beta)) \tag{6}$$

and the scattered field, which, generalizing [4], we represent in the form

$$u_3 = \mathrm{Re}\,(U_3 e^{-i\omega t}), \tag{7}$$

where

$$U_3(x_1; x_2) = \frac{1}{2}\int_L p(\zeta)\left\{\frac{\partial}{\partial\zeta_1}E(\zeta_1; z_1)\,d\zeta_1 - \frac{\partial}{\partial\bar{\zeta}_1}E(\zeta_1; z_1)\,d\bar{\zeta}_1\right\} + \int_L q(\zeta)E(\zeta_1; z_1)\,ds,$$

$$E(\zeta_1; z_1) = H_0^{(1)}(\gamma_2 r_1) + H_0^{(1)}(\gamma_2 r_1^*), \quad z_1 = x_1 + \frac{ix_2}{\sqrt{1+\chi^2}}, \quad \zeta_1 = \xi_1 + \frac{i\xi_2}{\sqrt{1+\chi^2}},$$

$$r_1 = |\zeta_1 - z_1|, \quad r_1^* = |\bar{\zeta}_1 - z_1|, \quad \zeta = \xi_1 + i\xi_2 \in \Delta = \bigcup_{j=1}^{k} L_j,$$

$p(\zeta) = \{p_j(\zeta), \zeta \in L_j\}$, $q(\zeta) = \{q_j(\zeta), \zeta \in L_j\}$ are the unknown "densities," $H_n^{(1)}(x)$ is the Hankel function of first kind and order n, and ds is the element of arc length along the boundary of the curve L. The density $p(\zeta)$ is equal to $-0.5[U_3(\zeta)]$, where $[U_3(\zeta)]$ is the jump in the displacement amplitude at L.

The function (7) is a solution of Eq. (3) that automatically satisfies the condition $\sigma_{23} = 0$ on the boundary of the half-space, and also the radiation condition.

Taking account of (2), (6), and (7), we represent the boundary condition (5) as

$$c(\psi)\Big\{\frac{\partial}{\partial z_1}(U_3 + U_3^0 + U_3^1)\Big\}^{\pm} + \overline{c(\psi)}\Big\{\frac{\partial}{\partial \bar{z}_1}(U_3 + U_3^0 + U_3^1)\Big\}^{\pm} = \pm X_3^{\pm}; \tag{8}$$

Here the upper sign corresponds to the left edge of L_j (in moving from its beginning point a_j to its endpoint b_j), Ψ is the angle between the positive normal to the left edge of L_j and the ox_1-axis. Taking account of the continuous extension of the vector of mechanical stresses across the cuts and carrying out the operations prescribed by (8), we find the following connection between the densities:

$$q(\zeta) = \frac{\chi^2 \sin 2\psi}{4i\sqrt{1+\chi^2}} \frac{dp}{ds}. \tag{9}$$

Substituting the values of the functions occurring in the boundary conditions that are reached in the limit on the left edge of the cut, we arrive at the following singular integro-differential equation:

$$\int_L \frac{df}{ds} g(\zeta; \zeta_0)\, ds + \int_L f(\zeta) G(\zeta; \zeta_0)\, ds = N(\zeta_0), \tag{10}$$

where

$$g(\zeta;\zeta_0) = \frac{1}{\pi}\mathrm{Im}\left(\frac{c(\psi_0)}{\zeta_1 - \zeta_{10}}\right) + \frac{\chi^2 \sin 2\psi}{2\pi\sqrt{1+\chi^2}}\mathrm{Re}\left(\frac{c(\psi_0)}{\zeta_1 - \zeta_{10}}\right)$$
$$+ \frac{i\gamma_2\chi^2 \sin 2\psi}{4\sqrt{1+\chi^2}}\{H_1(\gamma_2 r_{10})\mathrm{Re}\,(c(\psi_0)e^{-i\alpha_{10}}) + H_1^{(1)}(\gamma_2 r_{10}^*)\mathrm{Re}(c(\psi_0)e^{-i\alpha_{10}^*})\};$$

$$G(\zeta;\zeta_0) = \frac{\gamma_2^2}{4i}\{H_2(\gamma_2 r_{10})\mathrm{Im}\,(\overline{a(\psi)c(\psi_0)}e^{2i\alpha_{10}}) + \mathrm{Im}\,(a(\psi)c(\psi_0))(H_0^{(1)}(\gamma_2 r_{10}) + H_0^{(1)}(\gamma_2 r_{10}^*))$$
$$+ H_2^{(1)}(\gamma_2 r_{10}^*)\mathrm{Im}\,(\overline{a(\psi)}c(\psi_0)e^{-2i\alpha_{10}^*})\};$$

$$N(\zeta) = \frac{2}{\mu}X_3 + 2i\gamma(\cos(\psi - \beta)U_3^0 + \cos(\psi + \beta)U_3^1); \quad f(\zeta) = [U_3(\zeta)], \quad \zeta_0 \in L, \quad a(\psi) = dc(\psi)/d\psi;$$
$$r_{10} = |\zeta_1 - \zeta_{10}|, \quad \alpha_{10} = \arg(\zeta_1 - \zeta_{10}); \quad r_{10}^* = |\bar{\zeta}_1 - \zeta_{10}|; \quad \alpha_{10}^* = \arg(\bar{\zeta}_1 - \zeta_{10}).$$

The integral equation must be solved jointly with the additional condition

$$\int_L df = 0. \tag{11}$$

Relations (10) and (11) determine the solution completely in the class h_0 of functions that are unbounded at the endpoints of L_j [5].

We shall now obtain a formula for determining the coefficient of stress intensity at the vertices of the defect. We introduce a parametrization of the contour L_j:

$$\zeta = \zeta(\delta), \quad \zeta_0 = \zeta(\delta_0), \quad -1 \le \delta, \delta_0 \le 1.$$

Accordingly we put

$$\frac{df}{ds} = \frac{\Omega(\delta)}{s'(\delta)\sqrt{1-\delta^2}}, \quad s'(\delta) = \frac{ds}{d\delta}, \quad \Omega(\delta) \in H[-1,1]. \tag{12}$$

The total coefficient of intensity, counting both the mechanical and the Maxwell parts of the stress tensor, is determined by the singular part of the expression

$$Q_n = (S_{13} + T_{13})\cos\psi + (S_{13} + T_{23})\sin\psi \tag{13}$$

according to the formulas

$$\sigma_n = \mathrm{Re}(Q_n e^{-i\omega t}), \quad t_{ij} = \mathrm{Re}(T_{ij} e^{-i\omega t}), \quad \sigma_{ij} = \mathrm{Re}(S_{ij} e^{-i\omega t}).$$

By (1), (4), and (5) we can write

$$Q_n = \mu(\partial_1 U_3 \cos\psi + (1+\chi^2)\partial_2 U_3 \sin\psi).$$

Taking account of the asymptotics of Q_n, we find the total coefficient of stress intensity:

$$K_{\mathrm{III}}^s = \lim\sqrt{2\pi r}\,\mathrm{Re}(Q_n e^{-i\omega t}) = -\frac{\mu\sqrt{\pi(1+\chi^2)}}{2\sqrt{s'(\mp 1)}}|\Omega(\mp 1)|\cos(\omega t - \arg\Omega(\mp 1)). \tag{14}$$

The coefficient of mechanical stress intensity has the form

$$K_{\mathrm{III}} = \lim\sqrt{2\pi r}\,\mathrm{Re}(S_n e^{-i\omega t}); \tag{15}$$

$$S_n = S_{13}\cos\psi_c + S_{23}\sin\psi_c = -\frac{\mu\Omega(\mp 1)}{2\sqrt{2\pi s'(\mp 1)}}\Big(\sqrt{1+\chi^2} + \frac{\chi^4\sin^2 2\psi_c}{4\sqrt{1+\chi^2}}\Big)(1+\chi^2\sin^2\psi_c)^{-1},$$

where ψ_c is the angle of the normal to the left edge of L_j at the vertex c (where c is either a_j or b_j).

As a first example we consider an unbounded space weakened by a tunnel straight-line crack occupying the interval $[-l;l]$ of the x_2-axis. The surface of the crack is free of forces, and a magnetoelastic wave (2) radiates from infinity along the x_1-axis. The singular equation (10), in which the kernels corresponding to the coupled source were assumed zero, was solved numerically by the method of mechanical quadratures [6].

Figure 2 shows the results of computations of the quantity $\alpha^+ = \alpha^- = \alpha$. The coefficient of stress intensity K_{III}^s can be expressed in terms of $\alpha^{\mp}$ by the following formula (where $2l$ is the length of the crack)

$$K_{\mathrm{III}}^s = P_h\sqrt{\pi l}\,\alpha^{\mp}\cos(\omega t - \arg\alpha^{\mp}), \quad P_h = -i\mu U\gamma\sin\beta. \tag{16}$$

Curves *1, 2, 3* are constructed for the values $\chi = 1$, 0.5, and 0 respectively ($l = 1$) The dots give the results obtained by a different method [1].

As a second example consider the half-space $x_2 \ge 0$ weakened by a horizontal crack ($2l = 2$) for the case when there is no incident wave and on the edges of the crack there is a time-harmonic shear load $X_{3n}^{\pm} = \mathrm{Re}(X_3 e^{-i\omega t})$, $X_3 = \mathrm{const}$. The variation of the quantity $\alpha^+ = \alpha^- = \alpha$ in the same correspondence as above is presented in Fig. 3. We assume that the distance of the crack from the boundary $x_2 = 0$ is equal to its length. The coefficient of stress intensity K_{III} is determined by formula (16), in which it is necessary to set X_3 in place of P_h.

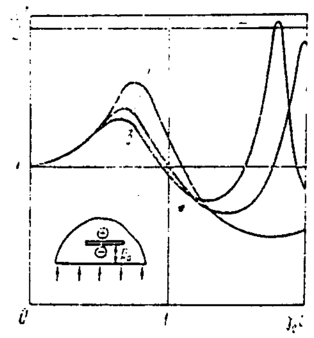

Fig. 3

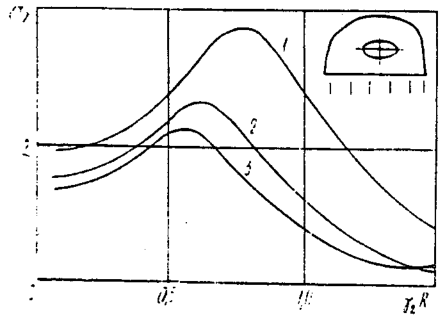

Fig. 4

Now let the half-space $x_2 \geq 0$ adjacent to a vacuum be weakened by cylindrical hole-cavities l_i along x_3 (Fig. 1). Under the conditions of the formulation given above the mechanical field in the half-space is composed of the field of the incident wave (2), the field of the reflected wave (6), and the scattered field, which we represent as

$$u_3 = \mathrm{Re}\,(U_3 e^{-i\omega t}), \quad l = \bigcup_{i=1}^{n} l_i, \tag{17}$$

where

$$U_3(x_1; x_2) = \int_L p(\zeta)(H_0^{(1)}(\gamma_2 r_1) + H_0^{(1)}(\gamma_2 r_1^*))\, ds,$$

and $p(\zeta) = \{p_1(\zeta),\ \zeta \in l_1\}$ is the unknown "density." The integration is carried out counterclockwise. The representation (17) automatically satisfies the condition $\sigma_{23} = 0$ on the boundary of the half-space and the radiation condition, and the function u_3 is a solution of Eq. (3).

We represent the boundary condition (5) on l in amplitudes

$$(S_{13} + S_{13}^0 + S_{13}^1)\cos\psi + (S_{23} + S_{23}^0 + S_{23}^1)\sin\psi = X_3. \tag{18}$$

Here S_{i3}, S_{i3}^0, and S_{i3}^1 are respectively the amplitudes of the quantities σ_{i3}, σ_{i3}^0, and σ_{i3}^1.

Computing the stresses taking account of (2), (6), and (17), we substitute their limiting values as $z \to \zeta_0 \in l$ into the boundary condition (18) and arrive at the following integral equation with respect to $p(\zeta)$:

$$p(\zeta_0) + \int_l p(\zeta) G(\zeta; \zeta_0)\, ds = N(\zeta_0), \tag{19}$$

where

$$G(\zeta; \zeta_0) = \frac{2}{i\pi\eta(\psi_0)}\mathrm{Re}\left(\frac{c(\psi_0)}{\zeta_1 - \zeta_{10}}\right) + \frac{\gamma_2}{\eta(\psi_0)}(H_1(\gamma_2 r_{10})\mathrm{Re}\,(c(\psi_0)e^{-i\alpha_{10}}) + H_1^{(1)}(\gamma_2 r_{10}^*)\mathrm{Re}\,(c(\psi_0)e^{-i\alpha_{10}^*}));$$

$$N(\zeta) = \frac{1}{\mu\eta(\psi)}X_3 + \frac{i\gamma}{\eta(\psi)}(\cos(\beta - \psi)U_3^0 + \cos(\beta + \psi)U_3^1); \quad \eta(\psi) = -2i\,\mathrm{Im}\left(\frac{c(\psi)}{a(\psi)}\right).$$

In the case when there is no preliminary magnetic field ($\chi = 0$) Eq. (19) is a Fredholm integral equation of second kind. And if $\chi > 0$, we obtain a singular integral equation of second kind.

The shear stresses on the plane perpendicular to the edge of the hole are determined by the formulas

$$\tau_g = -(\sigma_{13} + \sigma_{13}^0 + \sigma_{13}^1 + t_{13})\sin\psi + (\sigma_{23} + \sigma_{23}^0 + \sigma_{23}^1 + t_{23})\cos\psi, \quad \tau_s = \mathrm{Re}\,(Te^{-i\omega t}). \tag{20}$$

In our case the Maxwell stresses are

$$t_{13} = 0, \quad t_{23} = \mu_e H_0^2 \partial_2 \mu_3. \tag{21}$$

Applying relations (21), (17), (6), and (2) in formula (20), we obtain the amplitude of the stresses at the point $\zeta_0 \in l$:

$$T = -2i\mu\sqrt{1+\chi^2}p(\zeta_0)\mathrm{Re}\left(\frac{c(\psi_0)}{a(\psi_0)}\right) + i\mu\gamma(\sin(\psi_0 - \beta)U_3^0 + \sin(\psi_0 + \beta)U_3^1) + \int\limits_l p(\zeta)K(\zeta;\zeta_0)\,ds; \quad (22)$$

$$K(\zeta;\zeta_0) = \frac{2i\mu\sqrt{1+\chi^2}}{\pi}\mathrm{Im}\left(\frac{c(\psi_0)}{\zeta_1 - \zeta_{10}}\right) - \mu\gamma_2(H_1(\gamma_2 r_{10})b + H_1^{(1)}(\gamma_2 r_{10}^*)b^*).$$

Here

$$b = \cos\alpha_{10}\sin\psi_0 - \sqrt{1+\chi^2}\cos\psi_0\sin\alpha_{10};$$

$$b^* = \cos\alpha_{10}^*\sin\psi_0 - \sqrt{1+\chi^2}\cos\psi_0\sin\alpha_{10}^*.$$

As an example we consider a half-space weakened by a cavity of elliptic cross-section whose parametric equation is $\xi_1 = a_1\cos\varphi$, $\xi_2 = h + b_1\sin\varphi$. There is no incident wave from infinity, but there is a shear load $X_{3n} = \mathrm{Re}(X_3 e^{-i\omega t})$, $X_3 = X_3^0\sin\varphi$, $X_3^0 = \mathrm{const}$, on the surface of the cavity. Figure 4 shows the results of computing the quantity $\langle T\rangle = T/X_3^0$ at the point $\varphi = 0$ for $h = 1.75$, $b_1/a_1 = 0.75$. The curves are arranged in the same correspondence as above, and $R = (a_1 + b_1)/2$.

Thus taking account of the preliminary magnetic field is necessary for a reliable estimate of the stress of a body with stress concentrators.

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