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New Non-relativistic Bound States Solutions for Modified Kratzer Potential in One-electron Atoms

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In this study, three-dimensional modified time-independent Schrödinger equation of modified Kratzer potential was solved using Bopp's shift method instead to apply star product, in the framework of both noncommutativity three dimensional real space and phase (NC: 3D-RSP). We have obtained the explicit energy eigenvalues for ground and first excited states for interactions in one-electron atoms. Furthermore, the obtained corrections of energies are depended on infinitesimal parameters (Θ,χ) and $(\bar{\theta},\bar{\sigma})$ which are induced by position-position and momentum-momentum noncommutativity, respectively, in addition to the discreet atomic quantum numbers ($j = l \pm 1/2$, $s = 1/2$, l and m). We have also shown that, the usual states in ordinary three dimensional spaces for ordinary Kratzer potential are canceled and have been replaced by new degenerated $2(2l + 1)$ sub-states in the extended new quantum mechanics.

Keywords: The Kratzer potential, Noncommutative space and phase, Star product and Bopp's shift method.

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1. INTRODUCTION

During the last years the energy spectrum of atoms have been studied by several analytic methods, for example: Laplace integral transform, factorization method, proper quantization rule, exact quantization rule, Nikiforov–Uvarov method, supersymmetry quantum mechanics for solving the non-relativistic Schrödinger with central and non-central potentials for describing atoms, nuclei, etc [1-9]. In particular the Kratzer potential (known also by inverse power potential) has well accounted for some observed phenomena in atomic, molecular and chemical physics, in addition to study the Shannon entropy [10], this potential studied in two dimensional spaces by the author Shi-Hai Dong et al. [11] and by Süleyman Özcelik and Mehmet Simsek in three dimensional space [12]. The main goal to this study is to extended our previously study in ref. [13] to the noncommutative three dimensional space-phase to possibility to obtain a new another applications to this potential, we have using the physical terms contained in my previous relevant works, in the context of two-dimensional physical potentials (see, for example, [14, 15]) or in the context of three-dimensional physical potentials (see, for example, [16, 17]. It is important to notice that the author H. Snyder was firstly who introduce this idea (see, for example, [27]). The study of Kratzer potential has now become a very interest field due to their applications in different fields [10], the bound state solutions of the non-relativistic Schrödinger equation, with the modified Kratzer potential has not been obtained yet. This is the priority for this work. The modified Kratzer potential used in this framework takes the form:

$$V_{\text{nc-kp}}(\hat{r}) = V(r) + \left(\frac{2d}{r^6} + \frac{3c}{2r^5} + \frac{b}{r^4} + \frac{a}{2r^3}\right)\vec{\mathbf{L}}\vec{\Theta} + \frac{\vec{\mathbf{L}}\vec{\bar{\boldsymbol{\theta}}}}{2\mu} \quad (1)$$

where $V(r)$ denote to the ordinary Kratzer potential (will be determine in the next section). The crucial purpose of this paper is to determine the energy levels of above potential in (NC: 3D-RSP) symmetries using the generalization Bopp's shift method which depend on the concepts that we present now and in the third section to discover the new symmetries and a possibility to obtain another applications to this potential in different fields. Furthermore, much considerable effort has been expanded on the solutions of Schrödinger, Dirac and Klein-Gordon equations to noncommutative quantum mechanics, to search an a profound interpretation in microscopic scales [16 – 18], which based to new noncommutative canonical commutations relations (NNCCRs) in both Schrödinger and Heisenberg pictures ((SP) and (HP)), respectively, as follows:

$$\begin{aligned}
&[x_i, p_j] = i\delta_{ij} \rightarrow [\hat{x}_i \overset{*}{,} \hat{p}_j] = [\hat{x}_i(t) \overset{*}{,} \hat{p}_j(t)] = i\delta_{ij},\\
&[x_i, x_j] = 0 \rightarrow [\hat{x}_i \overset{*}{,} \hat{x}_j] = [\hat{x}_i(t) \overset{*}{,} \hat{x}_j(t)] = i\theta_{ij} \qquad (2)\\
&[p_i, p_j] = 0 \rightarrow [\hat{p}_i \overset{*}{,} \hat{p}_j] = [\hat{p}_i(t) \overset{*}{,} \hat{p}_j(t)] = i\bar{\theta}_{ij}
\end{aligned}$$

the new operators $(\hat{x}_i(t), \hat{p}_i(t))$ in (HP) are related to the corresponding new operators $(\hat{x}_i, \hat{p}_i)$ in (SP) from the following projections relations:

$$\begin{aligned}
&x_i(t) = \exp(iH(t-t_0))x_i\exp(-iH(t-t_0)) \rightarrow\\
&\hat{x}_i(t) = \exp(iH_{nc-kp}(t-t_0)) * \hat{x}_i * \exp(-iH_{nc-kp}(t-t_0)) \qquad (3)\\
&p_i(t) = \exp(iH(t-t_0))p_i\exp(-iH(t-t_0)) \rightarrow\\
&\hat{p}_i(t) = \exp(iH_{nc-kp}(t-t_0)) * \hat{p}_i * \exp(-iH_{nc-kp}(t-t_0))
\end{aligned}$$

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here H and H_{nc-kp} denote to the ordinary and new quantum Hamiltonian operators in the quantum mechanics and it's extension. The very small two parameters $\theta^{\mu\nu}$ and $\theta^{-\mu\nu}$ (compared to the energy) are elements of two antisymmetric real matrixes and (*) denote to the new star product, which is generalized between two arbitrary functions $f(x,p) \to \hat{f}(\hat{x},\hat{p})$ and $g(x,p) \to \hat{g}(\hat{x},\hat{p})$ to $\hat{f}(\hat{x},\hat{p})\hat{g}(\hat{x},\hat{p}) \equiv (f*g)(x,p)$ instead of the usual product $(fg)(x,p)$ in ordinary three dimensional spaces [13 – 21]:

$$(f*g)(x,p) \equiv \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_\mu^x\partial_\nu^x + \frac{i}{2}\bar{\theta}^{\mu\nu}\partial_\mu^p\partial_\nu^p\right)(fg)(x,p)$$
$$\equiv \left(fg - \frac{i}{2}\theta^{\mu\nu}\partial_\mu^x f\partial_\nu^x g - \frac{i}{2}\bar{\theta}^{\mu\nu}\partial_\mu^p f\,\partial_\nu^p g\right)(x,p)\Big|_{\left(x^\mu = x^\nu,\, p^\mu = p^\nu\right)} +$$
$$+O\left(\theta^2, \bar{\theta}^2\right)$$

where $\hat{f}(\hat{x},\hat{p})$ and $\hat{g}(\hat{x},\hat{p})$ are the new function in (NC: 3D-RSP), the two covariant derivatives $\left(\partial_\mu^x f(x,p), \partial_\mu^p f(x,p)\right)$ denotes to the $\left(\frac{\partial f(x,p)}{\partial x^\mu}, \frac{\partial f(x,p)}{\partial p^\mu}\right)$, respectively while the two following terms $-\frac{i}{2}\theta^{\mu\nu}\partial_\mu^x f(x,p)\partial_\nu^x g(x,p)$ and $-\frac{i}{2}\bar{\theta}^{\mu\nu}\partial_\mu^p f(x,p)\partial_\nu^p g(x,p)$ are induced by (space-space) and (phase-phase) noncommutativity properties, respectively. A Bopp's shift method can be used, instead of solving any quantum systems by using directly star product procedure [18 – 21]:

$$\left[\hat{x}_i, \hat{x}_j\right] = i\theta_{ij} \text{ and } \left[\hat{p}_i, \hat{p}_j\right] = i\bar{\theta}_{ij} \qquad (5)$$

The new generalized positions and momentum coordinates $(\hat{x},\hat{y},\hat{z})$ and $(\hat{p}_x,\hat{p}_y,\hat{p}_z)$ in (NC: 3D-RSP) are depended with corresponding usual generalized positions and momentum coordinates (x,y,z) and (p_x,p_y,p_z) in ordinary quantum mechanics by the following, respectively [16, 17]:

$$\hat{x}_i = x_i - \frac{\theta_{ij}}{2}p_j \Rightarrow \begin{cases} \hat{x} = x - \frac{\theta_{12}}{2}p_y - \frac{\theta_{13}}{2}p_z, \\ \hat{y} = y - \frac{\theta_{21}}{2}p_x - \frac{\theta_{23}}{2}p_z \\ \hat{z} = z - \frac{\theta_{31}}{2}p_x - \frac{\theta_{32}}{2}p_y \end{cases} \qquad (6)$$

and

$$\hat{p}_i = p_i - \frac{\bar{\theta}_{ij}}{2}x_j \Rightarrow \begin{cases} \hat{p}_x = p_x - \frac{\bar{\theta}_{12}}{2}y - \frac{\bar{\theta}_{13}}{2}z, \\ \hat{p}_y = p_y - \frac{\bar{\theta}_{21}}{2}x - \frac{\bar{\theta}_{23}}{2}z \\ \hat{p}_z = p_z - \frac{\bar{\theta}_{31}}{2}x - \frac{\bar{\theta}_{32}}{2}y \end{cases} \qquad (7)$$

which allow us to getting the two operators ($\hat{r}^2$ and $\hat{p}^2$) in (NC-3D: RSP), respectively [16, 17]:

$$\hat{r}^2 = r^2 - \mathbf{L}\Theta \text{ and } \hat{p}^2 = p^2 + \vec{\mathbf{L}}\bar{\boldsymbol{\theta}} \qquad (8)$$

where the two couplings $\mathbf{L}\Theta$ and $\vec{\mathbf{L}}\bar{\boldsymbol{\theta}}$ are given by, respectively $\left(\Theta_{ij} = \theta ij/2\right)$:

$$\mathbf{L}\Theta \equiv L_x\Theta_{12} + L_y\Theta_{23} + L_z\Theta_{13} \text{ and } \vec{\mathbf{L}}\bar{\boldsymbol{\theta}} \equiv L_x\bar{\theta}_{12} + L_y\bar{\theta}_{23} + L_z\bar{\theta}_{13}$$

where L_x, L_y and L_z are the three components of angular momentum operator $\vec{L}$. The organization scheme of the study is given as follows: In next section, we briefly review the Schrödinger equation with Kratzer potential on based to ref. [10]. The Section 3, devoted to studying the three deformed Schrödinger equation by applying both Bopp's shift method to the Kratzer potential. In the fourth section and by applying standard perturbation theory we find the quantum spectrum of the excited states in (NC-3D: RSP) for spin-orbital interaction corresponding the ground states and first excited states. In the next section, we derive the magnetic spectrum for studied potential. In the sixth section, we resume the global spectrum and corresponding noncommutative Hamiltonian for Kratzer potential. Conclusions are drawn in Sect 6.

2. REVIEW OF THE EIGNENFUNCTIONS AND THE ENERGY EIGENVALUES FOR KRATZER POTENTIAL IN ORDINARY THREE DIMENSIONAL SPACES

Let's present a brief review of time independent Schrödinger equation for a fermionic particle like electron of rest mass μ and its energy E moving in Kratzer potential [10]:

$$V(r) = \frac{a}{r} + \frac{b}{r^2} + \frac{c}{r^3} + \frac{d}{r^4} \qquad (10)$$

where a, b, c and d are constant coefficients. If we insert this potential into the Schrödinger equation:

$$\left(-\frac{\Delta}{2\mu} + \frac{a}{r} + \frac{b}{r^2} + \frac{c}{r^3} + \frac{d}{r^4}\right)\Psi(\vec{r}) = E\Psi(\vec{r}) \qquad (11)$$

In spherical coordinates, the complete wave function $\Psi_{n,l,\mu}(\vec{r}) = \frac{\Phi_{n,l}(r)}{r}Y_{l,\mu}(\theta,\phi)$, thus the radial function $\Phi_{n,l}(r)$ satisfied the following equation, in three dimensional spaces space [10]:

$$\left(\frac{d^2}{dr^2} + \varepsilon - \frac{A}{r} - \frac{B}{r^2} - \frac{C}{r^3} - \frac{D}{r^4} - \frac{l(l+1)}{r^2}\right)\Phi_{n,l}(r) = 0 \qquad (12)$$

Here $(\varepsilon, A, B, C, D) \equiv 2\mu(E, a, b, c, d)$, the complete orthonormalization eignenfunctions and the energy eigenvalues respectively in three dimensional spaces for Kratzer potential for ground state and first excited state, respectively [10]:

$$\Psi_0(r,\theta,\phi) = N_{0,l'} r^{\beta-1} \exp\left(\frac{\gamma}{r} + \alpha r\right) Y_{l',\mu}(\theta,\phi)$$
$$E_0 = \frac{A^2 D}{2\left(2\sqrt{D} - C\right)^2} \tag{13}$$

and

$$\Psi_1(r,\theta,\phi) = N_{1l'}\left(r - \alpha_1^{(1)}\right) r^{\lambda} \exp\left(-\sqrt{-\varepsilon} r - \sqrt{D}/r\right) Y_{l',\mu}(\theta,\phi),$$

where $\alpha = -\sqrt{-\varepsilon}$, $\gamma = -\sqrt{D}$, $\beta = -\frac{A}{2\sqrt{-\varepsilon}}$,

$l' = -1/2 + \sqrt{(l+1/2)^2 + B}$ and $\lambda = (2\sqrt{D} + C)/2\sqrt{D} - 1$,

while $N_{0,l'}$ and $N_{1l'}$ are two normalizations constants.

3. THREE DIMENSIONAL NONCOMMUTATIVE REAL SPACE-PHASE FOR KRATZER POTENTIAL

In this section, we shall study the Kratzer potential in (NC: 3D-RSP), to perform this task the physical form of Schrödinger equation should be written as [16, 17]:

Ordinary two dimensional Hamiltonian operators $\hat{H}_{kp}(p_i, x_i)$ will be replaced by new two Hamiltonian operators $\hat{H}_{nc-kp}(\hat{p}_i, \hat{x}_i)$,

– ordinary complex wave function $\Psi(\vec{r})$ will be replacing by new complex wave function $\hat{\Psi}(\vec{\hat{r}})$,

– ordinary energy E will be replaced by new values E_{nc-kp}, and the last step corresponds to replace the ordinary old product by new star product (*), which allow us to constructing the modified Schrödinger equations in both (NC-3D: RSP) as:

$$\hat{H}_{nc-kp}(\hat{p}_i, \hat{x}_i) * \hat{\Psi}(\vec{\hat{r}}) = E_{nc-kp}\hat{\Psi}(\vec{\hat{r}}) \tag{15}$$

Now, we apply the Bopp's shift method on the above equation to obtain the reduced Schrödinger equation:

$$H(\hat{p}_i, \hat{x}_i)\psi(\vec{r}) = E_{nc-kp}\psi(\vec{r}) \tag{16}$$

Where the new operator of Hamiltonian $H(\hat{p}_i, \hat{x}_i)$ can be expressed as:

$$H_{nc-kp}(\hat{p}_i, \hat{x}_i) \equiv H\left(\hat{x}_i = x_i - \frac{\theta_{ij}}{2} p_j \text{ and } \hat{p}_i = p_i - \frac{\bar{\theta}_{ij}}{2} x_j\right)$$

After straightforward calculations, we can obtain the five important terms, which will be use to determine the modified Kratzer potential in (NC: 3D- RSP):

$$d\hat{r}^{-4} = dr^{-4} + \frac{2d\vec{\mathbf{L}}\vec{\Theta}}{r^6}, c\hat{r}^{-3} = cr^{-3} + \frac{3c}{2r^5}\vec{\mathbf{L}}\vec{\Theta}$$
$$b\hat{r}^{-2} = br^{-2} + \frac{b}{r^4}\vec{\mathbf{L}}\vec{\Theta}, a\hat{r}^{-1} = dr^{-1} + \frac{d\vec{\mathbf{L}}\vec{\Theta}}{2r^3} \tag{18}$$
$$\frac{\hat{p}^2}{2\mu} = \frac{p^2}{2\mu} + \frac{\vec{\mathbf{L}}\vec{\bar{\boldsymbol{\theta}}}}{2\mu}$$

From above relations, one can write the deformed operator $V_{\text{kp}}(\hat{r})$ for Kratzer potential and the noncommutative kinetic term $\frac{\hat{p}^2}{2\mu}$, respectively:

$$V_{\text{kp}}(\hat{r}) = \frac{a}{\hat{r}} + \frac{b}{\hat{r}^2} + \frac{c}{\hat{r}^3} + \frac{d}{\hat{r}^4},$$
$$\frac{\hat{p}^2}{2\mu} = \frac{\vec{p}^2}{2\mu} + \frac{\vec{\mathbf{L}}\vec{\bar{\boldsymbol{\theta}}}}{2\mu}, \tag{19}$$

which allow us to obtaining the global potential operator $H_{\text{nc-kp}}(\hat{r})$ for Kratzer potential in (NC: 3D-RSP) as:

$$H_{\text{nc-kp}}(\hat{r}) = \frac{a}{r} + \frac{b}{r^2} + \frac{c}{r^3} + \frac{d}{r^4} + \left(\frac{2d}{r^6} + \frac{3c}{2r^5} + \frac{b}{r^4} + \frac{a}{2r^3}\right)\vec{\mathbf{L}}\vec{\Theta} + \frac{\vec{\mathbf{L}}\vec{\bar{\boldsymbol{\theta}}}}{2\mu}.$$

It's clearly, that the four first terms are given the ordinary Kratzer potential in three dimensional space, while the rest terms are proportional's with two infinitesimals parameters (Θ and $\bar{\theta}$) and then gives the terms of perturbations $H_{\text{per-kp}}(r)$ in (NC: 3D-RSP) as:

$$H_{\text{per-kp}}(r) = \left(\frac{2d}{r^6} + \frac{3c}{2r^5} + \frac{b}{r^4} + \frac{a}{2r^3}\right)\vec{\mathbf{L}}\vec{\Theta} + \frac{\vec{\mathbf{L}}\vec{\bar{\boldsymbol{\theta}}}}{2\mu}. \tag{21}$$

4. THE EXACT SPIN-ORBITAL SPECTRUM MODIFICATIONS FOR KRATZER POTENTIAL IN BOTH (NC:3D- RSP):

Again, the perturbative term $H_{\text{per-kp}}(r)$ can be rewritten to the equivalent physical form:

$$H_{\text{per-ip}}(r) = 2\left(\Theta\left(\frac{2d}{r^6} + \frac{3c}{2r^5} + \frac{b}{r^4} + \frac{a}{2r^3}\right) + \frac{\bar{\theta}}{2\mu}\right)\vec{S}\vec{L} \tag{22}$$

Furthermore, the above perturbative terms $H_{\text{per-kp}}(r)$ can be rewritten to the following new form:

$$H_{\text{per-kp}}(r) = \left(\Theta\left(\frac{2d}{r^6} + \frac{3c}{2r^5} + \frac{b}{r^4} + \frac{a}{2r^3}\right) + \frac{\bar{\theta}}{2\mu}\right)\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$$

We just replace $\vec{S}\vec{L}$ by the expression $\frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$, in quantum mechanics, this operator traduces the coupling between spin and orbital momentum. After profound straightforward calculation, one can show that, the radial function $\Phi_{n,l}(r)$ satisfied the following two equations, in (NC: 3D-RSP), respectively:

$$\begin{pmatrix} \frac{d^2}{dr^2}+\varepsilon-\frac{A}{r}-\frac{B}{r^2}-\frac{C}{r^3}-\frac{D}{r^4}- \\ -\left(\Theta\left(\frac{2d}{r^6}+\frac{3c}{2r^5}+\frac{b}{r^4}+\frac{a}{2r^3}\right)+\frac{\bar{\theta}}{2m_0}\right)\left(\vec{J}^2-\vec{L}^2-\vec{S}^2\right)-\frac{l(l+1)}{r^2} \end{pmatrix}\Phi_{n,l}(r)=0$$

The set ($H_{\text{so-ikp}}(r)$, J^2, L^2, S^2 and J_z) forms a complete of conserved physics quantities and the eigenvalues of the spin orbital coupling operator are $k_\pm \equiv \frac{1}{2}\left\{\left(l\pm\frac{1}{2}\right)(l\pm\frac{1}{2}+1)+l(l+1)-\frac{3}{4}\right\}$ corresponding: $j=l+1/2$ (spin up) and $j=l-1/2$ (spin down), respectively, then, one can form a diagonal (3×3) matrix, with non null elements are $(H_{sc-kp})_{11}$, $H_{sc-kp})_{22}$ and $(H_{sc-kp})_{33}=0$ for Kratzer potential in (NC: 3D-RSP) as:

$$\begin{aligned} &\left(H_{so-kp}\right)_{11}=k_+\left(\Theta\left(\frac{2d}{r^6}+\frac{3c}{2r^5}+\frac{b}{r^4}+\frac{a}{2r^3}\right)+\frac{\bar{\theta}}{2m_0}\right) \\ &\text{if } j=l+\tfrac{1}{2} \Rightarrow \text{spin -up} \\ &\left(H_{so-ip}\right)_{22}=k_-\left(\Theta\left(\frac{2d}{r^6}+\frac{3c}{2r^5}+\frac{b}{r^4}+\frac{a}{2r^3}\right)+\frac{\bar{\theta}}{2m_0}\right) \\ &\text{if } j=l-\tfrac{1}{2} \Rightarrow \text{spin -down} \end{aligned} \quad (25)$$

4.1 The Exact Spin-orbital Spectrum Modifications for Kratzer Potential in Both (NC: 3D-RSP) for Ground States

In this sub section, we are going to study the modifications to the energy levels for ground states $E_{u\text{-}0kp}$ and $E_{d\text{-}0kp}$ for spin up and spin down, respectively, at first order of two parameters Θ and $\bar{\theta}$ obtained by applying the standard perturbation theory:

$$E_{\text{u-0kp}}=\left|N_{0,l'}\right|^2 k_+\int_0^{+\infty} r^{2\beta-2}\exp\left(\frac{2\gamma}{r}+2\alpha r\right)\left(\Theta\left(\frac{2d}{r^6}+\frac{3c}{2r^5}+\frac{b}{r^4}+\frac{a}{2r^3}\right)+\frac{\bar{\theta}}{2\mu}\right)r^2dr \quad (26)$$

$$E_{\text{d-0kp}}=\left|N_{0,l'}\right|^2 k_-\int_0^{+\infty} r^{2\beta-2}\exp\left(\frac{2\gamma}{r}+2\alpha r\right)\left(\Theta\left(\frac{2d}{r^6}+\frac{3c}{2r^5}+\frac{b}{r^4}+\frac{a}{2r^3}\right)+\frac{\bar{\theta}}{2\mu}\right)r^2dr \quad (27)$$

A direct simplification gives:

$$\begin{aligned} E_{\text{u-0kp}}&=\left|N_{0,l'}\right|^2 k_+\left\{\Theta\sum_{i=1}^{4}T_{0i}+\frac{\bar{\theta}}{2\mu}T_{05}\right\} \\ E_{\text{d-0kp}}&=\left|N_{0,l'}\right|^2 k_-\left\{\Theta\sum_{i=1}^{4}T_{0i}+\frac{\bar{\theta}}{2\mu}T_{05}\right\} \end{aligned} \quad (28)$$

where, the five terms $T_i\left(i=\overline{1,5}\right)$ are given by:

$$\begin{aligned} T_{01}&=2d\int_0^{+\infty} r^{(2\beta-5)-1}\exp\left(\frac{2\gamma}{r}+2\alpha r\right)dr, \\ T_{02}&=\frac{3c}{2}\int_0^{+\infty} r^{(2\beta-4)-1}\exp\left(\frac{2\gamma}{r}+2\alpha r\right)dr, \\ T_{03}&=b\int_0^{+\infty} r^{(2\beta-3)-1}\exp\left(\frac{2\gamma}{r}+2\alpha r\right), \\ T_{04}&=\frac{a}{2}\int_0^{+\infty} r^{(2\beta-2)-1}\exp\left(\frac{2\gamma}{r}+2\alpha r\right)dr, \\ T_{05}&=\int_0^{+\infty} r^{(2\beta+1)-1}\exp\left(\frac{2\gamma}{r}+2\alpha r\right)dr \end{aligned} \quad (29)$$

After straightforward calculations, we can obtain the explicitly results:

$$\begin{aligned} &T_{01}=2\left(\frac{\gamma}{\alpha}\right)^{\frac{2\beta-5}{2}}K_{2\beta-5}\left(4\sqrt{\gamma\alpha}\right), T_{02}=2\left(\frac{\gamma}{\alpha}\right)^{\frac{v}{2}}K_{2\beta-4}\left(4\sqrt{\gamma\alpha}\right) \\ &T_{03}=2\left(\frac{\gamma}{\alpha}\right)^{\frac{2\beta-3}{2}}K_{2\beta-3}\left(4\sqrt{\gamma\alpha}\right), T_{04}=2\left(\frac{\gamma}{\alpha}\right)^{\beta-1}K_{2\beta-2}\left(4\sqrt{\gamma\alpha}\right) \\ &T_{05}=2\left(\frac{\gamma}{\alpha}\right)^{\frac{2\beta+1}{2}}K_{2\beta+1}\left(4\sqrt{\gamma\alpha}\right) \end{aligned}$$

which allow us to obtaining the exact modifications of ground states $E_{u\text{-}0kp}$ and $E_{d\text{-}0kp}$ produced by spin-orbital effect:

$$E_{\text{u-0kp}}=\left|N_{0,l'}\right|^2 k_+\left\{\Theta T_{nc-0skp}+\frac{\bar{\theta}}{m_0}T_{nc-0pkp}\right\} \quad (31)$$

$$E_{\text{d-0kp}}=\left|N_{0,l'}\right|^2 k_-\left\{\Theta T_{nc-0skp}+\frac{\bar{\theta}}{m_0}T_{nc-0pkp}\right\} \quad (32)$$

We have introduced new parameters $T_{nc-0skp}$ and $T_{nc-0pkp}$ for the sake of simplicity:

$$\begin{aligned} T_{nc-0skp}\equiv\, &2\left(\frac{\gamma}{\alpha}\right)^{\frac{2\beta-5}{2}}K_{2\beta-5}\left(4\sqrt{\gamma\alpha}\right)+2\left(\frac{\gamma}{\alpha}\right)^{\frac{v}{2}}K_{2\beta-4}\left(4\sqrt{\gamma\alpha}\right)+ \\ &+2\left(\frac{\gamma}{\alpha}\right)^{\frac{2\beta-3}{2}}K_{2\beta-3}\left(4\sqrt{\gamma\alpha}\right)+2\left(\frac{\gamma}{\alpha}\right)^{\beta-1}K_{2\beta-2}\left(4\sqrt{\gamma\alpha}\right) \end{aligned} \quad (33)$$

$$T_{nc-0pkp}\equiv 2\left(\frac{\gamma}{\alpha}\right)^{\frac{2\beta+1}{2}}K_{2\beta+1}\left(4\sqrt{\gamma\alpha}\right) \quad (34)$$

The first term T_{nc-skp} produced with the noncommutative geometry of space, while the term $T_{nc-0pkp}$ produced from the noncommutativity of phases.

It is important to notice that, the above calculations are obtained by applying the following special integral [22]:

$$\int_0^{+\infty} r^{v-1}\exp\left(-\left(\frac{\lambda_2}{r}+\lambda_1 r\right)\right)dr=2\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{v}{2}}K_v\left(2\sqrt{\lambda_1\lambda_2}\right) \quad (35)$$

Where $(\lambda_1 \text{ and } \lambda_2)$ are positive numbers and $\left|2\sqrt{\lambda_1\lambda_2}\right|\langle\frac{\pi}{2}$ and K_v the modified function of second kind and order v.

4.2 The Exact Spin-orbital Spectrum Modifications for Kratzer Potential in Both (NC: 3D-RSP) for First Excited States

Now, we turn to the modifications to the energy levels for first excited states $E_{\text{u1-kp}}$ and $E_{\text{d1-kp}}$ for spin up and spin down, respectively, at first order of two parameters Θ and $\bar{\theta}$, which obtained by applying the standard perturbation theory:

$$E_{\text{u-1kp}} = \left|N_{1l'}\right|^2 k_+ \int_0^{+\infty} \left(r^{2\lambda+4} - 2\alpha_1^1 r^{2\lambda+3} + \left(\alpha_1^1\right)^2 r^{2\lambda+2}\right) \exp\left(\frac{2\gamma}{r} + 2\alpha r\right) \begin{pmatrix} \Theta\left(\frac{2d}{r^6} + \frac{3c}{2r^5} + \frac{b}{r^4} + \frac{a}{2r^3}\right) \\ + \frac{\bar{\theta}}{2\mu} \end{pmatrix} dr \quad (36)$$

and

$$E_{\text{d-1kp}} = \left|N_{1l'}\right|^2 k_- \int_0^{+\infty} \left({}^{2\lambda+4} - 2\alpha_1^1 r^{2\lambda+3} + \left(\alpha_1^1\right)^2 r^{2\lambda+2}\right) \exp\left(\frac{2\gamma}{r} + 2\alpha r\right) \begin{pmatrix} \Theta\left(\frac{2d}{r^6} + \frac{3c}{2r^5} + \frac{b}{r^4} + \frac{a}{2r^3}\right) \\ + \frac{\bar{\theta}}{2\mu} \end{pmatrix} dr \quad (37)$$

A direct simplification gives:

$$E_{\text{u-1kp}} = \left|N_{1l'}\right|^2 k_+ \left\{\Theta \sum_{i=1}^{12} L_{1i} + \frac{\bar{\theta}}{2m_0} \sum_{i=13}^{16} T_{1i}\right\} \quad (38)$$

$$E_{\text{d-1kp}} = \left|N_{1l'}\right|^2 k_- \left\{\Theta \sum_{i=1}^{12} L_{1i} + \frac{\bar{\theta}}{2m_0} \sum_{i=13}^{16} T_{1i}\right\} \quad (39)$$

where, the 15- terms $L_i\left(i = \overline{1,15}\right)$ are given by:

$$\begin{aligned}
L_{11} &= 2d \int_0^{+\infty} r^{(2\lambda-1)-1} xp\left(\frac{2\gamma}{r} + 2\alpha r\right) dr, \\
L_{12} &= \frac{3c}{2} \int_0^{+\infty} r^{2\lambda-1} xp\left(\frac{2\gamma}{r} + 2\alpha r\right) dr \\
L_{13} &= b \int_0^{+\infty} r^{(2\lambda+1)-1} xp\left(\frac{2\gamma}{r} + 2\alpha r\right) dr, \\
L_{14} &= \frac{a}{2} \int_0^{+\infty} r^{(2\lambda+2)-1} xp\left(\frac{2\gamma}{r} + 2\alpha r\right) dr \\
L_{15} &= -4d\alpha_1^1 \int_0^{+\infty} r^{(2\lambda-2)-1} xp\left(\frac{2\gamma}{r} + 2\alpha r\right) dr, \\
L_{16} &= -6\alpha_1^1 c \int_0^{+\infty} r^{(2\lambda-1)-1} xp\left(\frac{2\gamma}{r} + 2\alpha r\right) dr
\end{aligned} \quad (40)$$

$$\begin{aligned}
L_{17} &= -2\alpha_1^1 b \int_0^{+\infty} r^{2\lambda-1} xp\left(\frac{2\gamma}{r} + 2\alpha r\right) dr, \\
L_{18} &= -\frac{\alpha_1^1 a}{2} \int_0^{+\infty} r^{(2\lambda+1)-1} xp\left(\frac{2\gamma}{r} + 2\alpha r\right) dr \\
L_{19} &= 2d\left(\alpha_1^1\right)^2 \int_0^{+\infty} r^{(2\lambda-3)-1} xp\left(\frac{2\gamma}{r} + 2\alpha r\right) dr, \\
L_{110} &= \frac{3c\left(\alpha_1^1\right)^2}{2} \int_0^{+\infty} r^{(2\lambda-2)-1} xp\left(\frac{2\gamma}{r} + 2\alpha r\right) dr \\
L_{111} &= b\left(\alpha_1^1\right)^2 \int_0^{+\infty} r^{(2\lambda-1)-1} xp\left(\frac{2\gamma}{r} + 2\alpha r\right) dr, \\
L_{012} &= \frac{a\left(\alpha_1^1\right)^2}{2} \int_0^{+\infty} r^{2\lambda-1} xp\left(\frac{2\gamma}{r} + 2\alpha r\right) dr
\end{aligned} \quad (41)$$

$$\begin{aligned}
L_{113} &= \int_0^{+\infty} r^{(2\lambda+5)-1} \exp\left(\frac{2a}{r} + 2br\right) dr, \\
L_{114} &= -2\alpha_1^1 \int_0^{+\infty} r^{(2\lambda+4)-1} \exp\left(\frac{2a}{r} + 2br\right) dr \\
L_{115} &= a\left(\alpha_1^1\right)^{22} \int_0^{+\infty} r^{(2\lambda+3)-1} \exp\left(\frac{2a}{r} + 2br\right) dr
\end{aligned} \quad (42)$$

Now we apply the special integral which represents by eq. (35) to obtain the following results:

$$\begin{aligned}
L_{11} &= 4d\left(\frac{\gamma}{\alpha}\right)^{\frac{2\lambda-1}{2}} K_{2\lambda-1}\left(4\sqrt{\gamma\alpha}\right), \\
L_{12} &= \frac{3c}{2}\left(\frac{\gamma}{\alpha}\right)^{\lambda} K_{2\lambda}\left(4\sqrt{\gamma\alpha}\right), \\
L_{13} &= b\left(\frac{\gamma}{\alpha}\right)^{\frac{2\lambda+1}{2}} K_{2\lambda+1}\left(4\sqrt{\gamma\alpha}\right), \\
L_{14} &= \frac{a}{2}\left(\frac{\gamma}{\alpha}\right)^{\lambda+1} K_{2\lambda+2}\left(4\sqrt{\gamma\alpha}\right), \\
L_{15} &= -4d\alpha_1^1\left(\frac{\gamma}{\alpha}\right)^{\lambda-1} K_{2\lambda-2}\left(4\sqrt{\gamma\alpha}\right), \\
L_{16} &= -6\alpha_1^1 c\left(\frac{\gamma}{\alpha}\right)^{\frac{2\lambda-1}{2}} K_{2\lambda-1}\left(4\sqrt{\gamma\alpha}\right)
\end{aligned} \quad (43)$$

$$\begin{aligned}
L_{17} &= -2\alpha_1^1 b\left(\frac{\gamma}{\alpha}\right)^{\frac{2\lambda-1}{2}} K_{2\lambda-1}\left(4\sqrt{\gamma\alpha}\right), \\
L_{18} &= -\frac{a\alpha_1^1}{2}\left(\frac{\gamma}{\alpha}\right)^{\frac{2\lambda+1}{2}} K_{2\lambda+1}\left(4\sqrt{\gamma\alpha}\right) \\
L_{19} &= 2d\left(\alpha_1^1\right)^2\left(\frac{\gamma}{\alpha}\right)^{\frac{2\lambda-3}{2}} K_{2\lambda-3}\left(4\sqrt{\gamma\alpha}\right), \\
L_{110} &= \frac{3c\left(\alpha_1^1\right)^2}{2}\left(\frac{\gamma}{\alpha}\right)^{\lambda-1} K_{2\lambda-2}\left(4\sqrt{\gamma\alpha}\right), \\
L_{111} &= b\left(\alpha_1^1\right)^2\left(\frac{\gamma}{\alpha}\right)^{\frac{2\lambda-1}{2}} K_{2\lambda-1}\left(4\sqrt{\gamma\alpha}\right), \\
L_{112} &= \frac{a\left(\alpha_1^1\right)^2}{2}\left(\frac{\gamma}{\alpha}\right)^{\frac{2\lambda-1}{2}} K_{2\lambda-1}\left(4\sqrt{\gamma\alpha}\right)
\end{aligned} \quad (44)$$

$$L_{113} = \left(\frac{\gamma}{\alpha}\right)^{\frac{2+5l}{2}} K_{2\lambda+5}\left(4\sqrt{\gamma\alpha}\right),$$

$$L_{114} = -2\alpha_1^1\left(\frac{\gamma}{\alpha}\right)^{\lambda+2} K_{2\lambda+4}\left(4\sqrt{\gamma\alpha}\right), \quad (45)$$

$$L_{115} = \left(\alpha_1^1\right)^2\left(\frac{\gamma}{\alpha}\right)^{\frac{2\lambda+3}{2}} K_{2\lambda+3}\left(4\sqrt{\gamma\alpha}\right)$$

which allow us to obtaining the exact modifications $E_{\text{u1-ip}}$ and $E_{\text{d1-ip}}$ of degenerated first excited states produced for spin-orbital effect:

$$E_{\text{u-1kp}} = \left|N_{1l'}\right|^2 k_+ \left\{\Theta L_{nc-1skp} + \frac{\bar{\theta}}{2\mu} L_{nc-1pkp}\right\} \quad (46)$$

$$E_{\text{d-1kp}} = \left|N_{1l'}\right|^2 k_- \left\{\Theta L_{nc-1skp} + \frac{\bar{\theta}}{2\mu} L_{nc-1pkp}\right\} \quad (47)$$

Where the two obtained factors $T_{nc-1skp}$ and $L_{nc-1pkp}$ are given by:

$$L_{nc-1skp} \equiv \sum_{i=1}^{12} L_{1i} \text{ and } L_{nc-1pkp} \equiv \sum_{i=13}^{15} L_{1i} \quad (48)$$

5. THE EXACT MAGNETIC SPECTRUM MODIFICATIONS FOR KRATZER POTENTIAL IN (NC:3D- RSP):

5.1 The Exact Magnetic Spectrum Modifications for Kratzer Potential in (NC: 3D RSP) for Fundamental States

We now consider physically meaningful phenomena, it's possible to found another automatically symmetry for the production of the perturbative terms of Kratzer potential related to the influence of an external uniform magnetic field, it's sufficient to apply the following replacements:

$$\begin{cases}\vec{\Theta} \to \chi\vec{B} \\ \vec{\bar{\theta}} \to \bar{\sigma}\vec{B}\end{cases} \Rightarrow \left(\left(\frac{2d}{r^6}+\frac{3c}{2r^5}+\frac{b}{r^4}+\frac{a}{2r^3}\right)\vec{\mathbf{L}}\vec{\Theta}+\frac{\vec{\mathbf{L}}\vec{\bar{\theta}}}{2\mu}\right) \to \left(\chi\left(\frac{2d}{r^6}+\frac{3c}{2r^5}+\frac{b}{r^4}+\frac{a}{2r^3}\right)+\frac{\bar{\sigma}}{2\mu}\right)\vec{B}\vec{L} \quad (49)$$

Here χ and $\bar{\sigma}$ are infinitesimal real proportional's constants, and we choose the magnetic field $\overleftrightarrow{B} = B\vec{k}$, which allow us to introduce the modified new magnetic Hamiltonian H_{m-kp} in (NC: 3D-RSP) as:

$$H_{m-kp} = \left(\chi\left(\frac{2d}{r^6}+\frac{3c}{2r^5}+\frac{b}{r^4}+\frac{a}{2r^3}\right)+\frac{\bar{\sigma}}{2\mu}\right)\left(\vec{B}\vec{J}-\vec{S}\vec{B}\right) \quad (50)$$

Here $\left(-\vec{S}\vec{B}\right)$ denote to the ordinary Hamiltonian of Zeeman Effect. To obtain the exact noncommutative magnetic modifications of energy ($E_{\text{mag-0kp}}$, $E_{\text{mag-1kp}}$) for Kratzer potential, we replace: k_+, Θ and $\bar{\theta}$ in the Eqs.(31) and (46) by the following parameters: m, χ and $\bar{\sigma}$, respectively, to obtains:

$$E_{\text{mag-0kp}} = \left|N_{0l'}\right|^2 Bm\left\{\chi T_{nc-0skp} + \frac{\bar{\sigma}}{2\mu} T_{nc-0pkp}\right\} \quad (51)$$

$$E_{\text{mag-1kp}} = \left|N_{1l'}\right|^2 Bm\left\{\chi L_{nc-1skp} + \frac{\bar{\sigma}}{2\mu} L_{nc-1pkp}\right\} \quad (52)$$

Where $E_{\text{mag-0kp}}$ and $E_{\text{mag-1kp}}$ are the exact magnetic modifications of spectrum corresponding the fundamental states and first excited states and we have $-l \le m \le +l$, which allow us to fixing $(2l+1)$ values for discreet number m.

5.2 The Exact Modified of the Lowest Excitations Spectrum for Kratzer Potential in (NC:3D-RSP)

Let us resume the eigenenergies of the modified Schrödinger equation obtained in this paper, the total modified energies ($E_{\text{nc u0-ip}}$ - $E_{\text{nc d0-ip}}$) and ($E_{\text{nc u1-ip}}$ - $E_{\text{nc d1-ip}}$) of a particle fermionic with spin up and spin down are determined corresponding ground and first excited states, respectively, for Kratzer potential in (NC: 3D-RSP), on based to original results presented on the Eqs. (32), (33), (51) and (52):

$$E_{\text{nc u0-kp}} = \frac{A^2 D}{2\left(2\sqrt{D}-C\right)^2} + \left|N_{0,l'}\right|^2 k_+\left\{\Theta T_{nc-0skp} + \frac{\bar{\theta}}{2\mu} T_{nc-0pkp}\right\} + + \left|N_{0l'}\right|^2 Bm\left\{\chi T_{nc-0skp} + \frac{\bar{\sigma}}{2\mu} T_{nc-0pkp}\right\}$$

$$E_{\text{nc d0-kp}} = \frac{A^2 D}{2\left(2\sqrt{D}-C\right)^2} + \left|N_{0,l'}\right|^2 k_-\left\{\Theta T_{nc-0skp} + \frac{\bar{\theta}}{2\mu} T_{nc-0pkp}\right\} + + \left|N_{0l'}\right|^2 Bm\left\{\chi T_{nc-0skp} + \frac{\bar{\sigma}}{2\mu} T_{nc-0pkp}\right\}$$

and

$$E_{\text{nc u1-kp}} = \frac{4\beta^2}{A} + \left|N_{1l'}\right|^2 k_+\left\{\Theta L_{nc-1skp} + \frac{\bar{\theta}}{2\mu} L_{nc-1pkp}\right\} + \left|N_{1l'}\right|^2 Bm\left\{\chi L_{nc-1skp} + \frac{\bar{\sigma}}{2\mu} L_{nc-1pkp}\right\} \quad (55)$$

$$E_{\text{nc d1-kp}} = \frac{4\beta^2}{A} + \left|N_{1l'}\right|^2 k_-\left\{\Theta L_{nc-1skp} + \frac{\bar{\theta}}{2\mu} L_{nc-1pkp}\right\} + \left|N_{1l'}\right|^2 Bm\left\{\chi L_{nc-1skp} + \frac{\bar{\sigma}}{2\mu} L_{nc-1pkp}\right\} \quad (56)$$

It is evident to consider the quantum number m can be takes $(2l + 1)$ values and we have also two values for $j = l \pm 1/2$, thus every state in usually three dimensional space of energy for Kratzer potential will be

$2(2l+1)$ sub-states in (NC: 3D-RSP). It's clearly, that the obtained eigenvalues of energies are real's and then the noncommutative diagonal Hamiltonian H_{nc-kp} is Hermitian, furthermore it's possible to writing the three elements: $(H_{nc-kp})_{11}$, $(H_{nc-kp})_{22}$ and $(H_{nc-kp})_{33}$ as follows:

$$\left(H_{nc-kp}\right)_{11}=-\frac{\Delta}{2\mu}+\frac{a}{r}+\frac{b}{r^2}+\frac{c}{r^3}+\frac{d}{r^4}+k_+\left(\Theta\left(\frac{2d}{r^6}+\frac{3c}{2r^5}+\frac{b}{r^4}+\frac{a}{2r^3}\right)+\frac{\bar{\theta}}{2\mu}\right)$$
$$+\left(\chi\left(\frac{2d}{r^6}+\frac{3c}{2r^5}+\frac{b}{r^4}+\frac{a}{2r^3}\right)+\frac{\bar{\sigma}}{2\mu}\right)\left(\vec{B}\vec{J}-\vec{S}\vec{B}\right) \tag{57}$$

$$\left(H_{nc-kp}\right)_{22}=-\frac{\Delta}{2\mu}+\frac{a}{r}+\frac{b}{r^2}+\frac{c}{r^3}+\frac{d}{r^4}+k_-\left(\Theta\left(\frac{2d}{r^6}+\frac{3c}{2r^5}+\frac{b}{r^4}+\frac{a}{2r^3}\right)+\frac{\bar{\theta}}{2\mu}\right)$$
$$+\left(\chi\left(\frac{2d}{r^6}+\frac{3c}{2r^5}+\frac{b}{r^4}+\frac{a}{2r^3}\right)+\frac{\bar{\sigma}}{2\mu}\right)\left(\vec{B}\vec{J}-\vec{S}\vec{B}\right) \tag{58}$$

$$\left(H_{nc-kp}\right)_{33}=-\frac{\Delta}{2\mu}+\frac{a}{r}+\frac{b}{r^2}+\frac{c}{r^3}+\frac{d}{r^4} \tag{59}$$

It is possible physically to gives interpretations to the above obtained results as Hamiltonian operator describing atom which has two permanent dipoles: the first is electric dipole moment and the second is magnetic moment in external stationary electromagnetic field as it's shown in our work [13].

6. CONCLUSION

In this work, we reviewed the exact solutions of the Schrödinger equation with the Kratzer potential and the formalism of Bopp's shift method. Then, we have solved the Schrodinger equation for modified Kratzer potential in (NC: 3D-RSP), we have obtained the exact energy spectrum for ground and first excited states. We shown that the old states are changed radically and replaced by degenerated new states, describing two new original spectrums, the first new one, produced by spin-orbital interaction H_{so-kp} while the second new spectrum produced by an external magnetic field. Finally, we have shown that, every state in usually three dimensional space of energy for Kratzer potential will be $2(2l+1)$ sub-states in (NC: 3D-RSP).

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