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JAMMING TRANSITION WITH FLUCTUATIONS OF CHARACTERISTIC ACCELERATION/BRAKING TIME

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Jamming transition in traffic flow (between free and jammed traffic) for homogeneous car following model has been investigated taking into account fluctuations of characteristic acceleration/braking time. These fluctuations are defined by Ornstein–Uhlenbeck process. The behaviour of the most probable deviation of headway from its optimal value has been studied and phase diagram of the system has been calculated for supercritical and subcritical regimes of jam formation. It has been found that for the first regime the fluctuations of characteristic acceleration/braking time result in coexistence of free moving and jammed traffic, that is typical for the first–order phase transition, and in appearance of two steady states for the second mode. These states correspond to non–zero values of headway deviation at which the formation of jam and congested traffic are possible. Using phase–plain portraits method the kinetics of the system transitions has been analyzed for different domains of the phase diagram for both regimes.

Key words: Headway deviation, Lorentz system, Ornstein–Uhlenbeck process, Phase diagram, Langevin equation.

Nowadays traffic problems attract considerable attention. For a study of the traffic jams formation problem thermodynamic [1], stochastic [2], and some hydrodynamic and kinetic theories [3, 4] are used. These theories are based on car–following model [3, 4], Maxwell model [5], and cellular automaton model [3, 6]. Within the framework of thermodynamic approach jamming transition is represented as nonequilibrium first–order phase transition. The mentioned method describes the deterministic picture of traffic flow. Stochastic theory, which is based on the master equation, allows to find a stationary density of spatially–temporal distribution of traffic jams. However, the fluctuations influence of the parameters characterizing the system is not finally studied. It is known that fluctuations do not only play a role of the trigger for the phase transition but also lead to essentially new system behaviour, and are a major cause for system's self–organization [7, 8]. Therefore, at describing the jamming transition the estimation of fluctuations influence of the dynamical car characteristics is rational. This is achieved within the framework of synergetic approach by the most natural way.

One of the simplest schemes that describe the self–organization process is the Lorentz system [8, 9]. At first time it was offered for the atmospheric phenomena description and then used for physics, chemistry, biology, sociology problems and so on. Recently synergetic scheme was proposed to describe the jamming transition in traffic flow [10]. In our work offered in Ref. [10] the synergetic approach will be developed taking into account internal fluctuations of characteristic acceleration/braking time. This parameter is a car characteristic; it indicates a time necessary for car to reach the characteristic velocity and plays a key role at the traffic jam formation. We will show that internal fluctuations, which have additive noise meaning, lead to the complication of traffic flow behaviour. In this work the stationary regime of such system will be considered and influence of the characteristic acceleration/braking time in the most probable headway deviation from its optimal value will be studied. As it will be shown this headway deviation characterizes a phase transition into the state that has traffic jam meaning.

Basic equations. On the basis of the car following model for one–line highway it is possible to show that dissipative dynamics of the homogeneous traffic flow can be represented within the framework of the lorentz scheme [10]. To describe the jamming transition we use the synergetic concept of phase transition, which is realized as a result of mutually coordinated behaviour of three freedom degrees: an order parameter, a conjugate field, and a control parameter. In traffic flow the roles of these quantities are played by the absolute value of headway deviation between vehicular from a safety distance h

$$\eta \equiv \left|\Delta x - h\right|; \tag{1}$$

by the deviation of velocity of the η variation from optimal value $h/t_0 - V$

$$v \equiv \Delta\dot{x} - h/t_0 + V, \tag{2}$$

where t_0 is the nominal time lag, V is the actual value of the velocity, and by acceleration/braking time τ, respectively. Here x is the vehicle coordinate.

Let us consider the simplest homogeneous system with goal of definition of time dependencies $\eta(t)$, $v(t)$, $\tau(t)$. To achieve this we will use a phenomenological approach. In the equations of the motion it is assumed that in the autonomous mode the evolution of the quantities η, v and τ has dissipative character and their relaxation to equilibrium values is described by the Debye equation. Besides a Le Chatelier principle has a great importance: since the decrease of acceleration/baking time τ assists to the formation of stable traffic flow, the headway deviation η and its velocity deviation v should vary so that to prevent the growth of τ, and as a consequence to impede a jam formation. Also the essentially important role is played by positive feedback of values η and τ on v. Namely the availability of this feedback is a reason for self–organization that leads to traffic jam formation.

The Lorentz system takes into account the mentioned above circumstances by the simplest way. Taking under consideration fluctuational addition it is determined as follows

$$\dot{\eta} = -\eta/t_\eta + v, \tag{3}$$

$$\dot{v} = -v/t_v + g_v \tau\eta, \tag{4}$$

$$\dot{\tau} = (\tau_0 - \tau)/t_\tau - g_\tau \eta v + \lambda(t). \tag{5}$$

Here the dot means the time differentiation; t_η, t_v and t_τ are the appropriate relaxation times; g_v and g_τ are the positive constants.

System (3)–(5) represents a base of the self–consistent description for the car–following model. Here the first terms on the right–hand side of the equations describe the relaxation of each quantity to an equilibrium value. In equation (3) the last term is a usual addition. It is easy to see from this equation that in the stationary state headway deviation is proportional to the velocity deviation.

The second term on the right–hand side of Eq.(4) describes the positive feedback of headway deviation η and acceleration/braking time τ on velocity deviation v. This feedback leads to the increase of v and is the reason for traffic jam formation.

Equation (5) differs from (3), (4) because relaxation of τ occurs not to zero but to finite value τ_0 which represents the time necessary for automobile to reach a characteristic velocity (the car property). Minus sign before the last term on the right–hand side of equation (5) may be considered as a demonstration of the Le Chatelier principle.

Quantity $\lambda(t)$ represents an influence of fluctuations of the characteristic acceleration/braking time and is defined as Ornstein–Uhlenbeck process:

$$\langle \lambda(t) \rangle = 0,\ \langle \lambda(t)\lambda(t') \rangle \equiv C(t,t') = \left(\delta^2/\tau_\lambda\right)\exp\left(-|t-t'|/\tau_\lambda\right), \tag{6}$$

where δ^2 is the noise intensity, τ_λ is the correlation time of the process $\lambda(t)$.

Within the framework of the mentioned parametrization the formation of traffic jams is represented as a result of spontaneous headway and velocity deviations if the characteristic acceleration/braking time exceeds a critical value. It is reflected by the appearance of the minimum of the effective potential which corresponds to the stationary value of the headway deviation η_0 [10]. Therefore we will be interested in η evolution further.

In the general case system (3)–(5) have no analytical solution therefore we should use the following approximation:

$$t_\eta >> t_\tau,\ t_\eta \approx t_v\,. \tag{7}$$

This condition implies that in the course of evolution the acceleration/braking time τ is coordinated by variation of the headway and velocity deviations. Owing to this condition in equation (5) a small parameter can be eliminated that allows us to assume $t_\tau \dot{\tau} \approx 0$. As a result we derive expression for the control parameter in the following form

$$\tau = \tau_0 - g_\tau t_\tau \eta v + t_\tau \lambda(t). \tag{8}$$

To form simpler system let us reduce initial one to the one–parameter model. For that it is necessary to express v and τ via η. Equation for $\dot{v}$ is determined by differentiating with respect to time equation for the velocity deviation v obtained from Eq. (3). Substituting expressions for v, $\dot{v}$, and equation (8) into (4), and introducing measure scales t_η, $\eta_m=(g_v g_\tau t_\tau t_\eta)^{-1/2}$, $v_m=t_\eta^{-3/2}(g_v g_\tau t_\tau)^{-1/2}$, $\tau_c=(g_v t_\eta^2)^{-1}$, and $g_v t_\tau t_\eta^2$ for time, headway deviation η, velocity deviation v, acceleration/braking time τ, and for the noise of the characteristic acceleration/braking time τ_0, respectively, we get:

$$\ddot{\eta} + \dot{\eta}\left(1 + \sigma + \eta^2\right) = \eta(\varepsilon - \sigma) - \eta^3 + \eta\lambda(t). \tag{9}$$

Here the denotations are introduced:

$$\sigma \equiv t_\eta / t_v\,,\ \varepsilon \equiv \tau_0/\tau_c\,. \tag{10}$$

Obtained expression allows us to write the evolution equation in the canonical form of the motion equation for the nonlinear stochastic oscillator of the van der Pole generator type

$$\ddot{\eta} + \gamma(\eta)\dot{\eta} = f(\eta) + g(\eta)\lambda(t), \tag{11}$$

where

$$\gamma(\eta) = 1 + \sigma + \eta^2, \quad f(\eta) = \eta(\varepsilon - \sigma) - \eta^3, \quad g(\eta) = \eta. \tag{12}$$

Note that equation (11) takes into account reactive behaviour of the system. As is known the task of the statistical physics is solved using distribution function $P(\eta, \dot{\eta}, t)$ that represents probability density of the availability of the corresponding values of headway deviation η and its rate of change $\dot{\eta}$ at a given instant of time t. Since jam in the car flow is defined by headway η and time t we have to consider projection of the distribution function in the half–space (η, t). For that the kinetic equation for $P(\eta, \dot{\eta}, t)$ is reduced to the Fokker–Planck equation with respect to $P(\eta,t)$ function.

Langevin and fokker–planck equations. For derivation of the evolution equation for the probability density $P(\eta, \dot{\eta}, t)$ we use continuity equation for function $\rho(\eta, \dot{\eta}, t)$ which is connected with p by following equality $P(\eta, \dot{\eta}, t) = \langle \rho(\eta, \dot{\eta}, t) \rangle_\lambda$. Here $\langle \ \rangle_\lambda$ stands for averaging over noise λ. Continuity equation is constructed by standard manner:

$$\left(\frac{\partial}{\partial t} + \hat{L}(\dot{\eta}, \eta) \right) \rho(\dot{\eta}, \eta, t) = -g(\eta)\lambda(t) \frac{\partial}{\partial \dot{\eta}} \rho(\dot{\eta}, \eta, t), \tag{13}$$

where the operator

$$\hat{L}(\dot{\eta}, \eta) = -\gamma(\eta) \frac{\partial}{\partial \dot{\eta}} \dot{\eta} + \dot{\eta} \frac{\partial}{\partial \eta} + f(\eta) \frac{\partial}{\partial \dot{\eta}} \tag{14}$$

is introduced. Averaging equation (13) and using decomposition technique in cumulants [11, 12] we obtain kinetic equation for P:

$$\left\{ \frac{\partial}{\partial t} + \hat{L}(\dot{\eta}, \eta) \right\} P(\dot{\eta}, \eta, t) = \hat{\Lambda}(\dot{\eta}, \eta, t) P(\dot{\eta}, \eta, t). \tag{15}$$

Here

$$\hat{\Lambda}(\dot{\eta}, \eta, t) = g(\eta) \frac{\partial}{\partial \dot{\eta}} \sum_{k=0}^{\infty} C^{(k)} \hat{L}^{(k)}(\dot{\eta}, \eta), \tag{16}$$

and moments of the correlation function are defined as follows:

$$C^{(k)}(t) = \frac{1}{k!} \int_0^\infty \tau_\lambda^k C(t, t - \tau_\lambda) d\tau_\lambda . \tag{17}$$

Operators $\hat{L}^{(k)}$ are defined by recurrent formula

$$\hat{L}^{(k)} = \left[\hat{L}^{(k-1)}, \hat{L} \right], \; \hat{L}^{(0)} = g(\eta) \partial / \partial \dot{\eta}. \tag{18}$$

Here $\left[\hat{A}, \hat{B} \right] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is the denotation for the commutator.

To pass into the half–space (η, t) the distribution function moments are used

$$P_n(\eta, t) = \int_0^\infty (\dot{\eta})^n P(\dot{\eta}, \eta, t) d\dot{\eta}. \tag{19}$$

Namely these moments give us an opportunity to find an effective Fokker–Planck equation. Taking into account components of integer orders, instead of (15) according to [13], we get

$$\frac{\partial}{\partial t}P(\eta,t)=-\frac{\partial}{\partial\eta}D^{(1)}(\eta)P(\eta,t)+\frac{\partial^2}{\partial\eta^2}D^{(2)}(\eta)P(\eta,t),\tag{20}$$

where

$$D^{(1)}(\eta)=\left(f(\eta)+C^{(1)}\,\partial g^2(\eta)/\partial\eta+C^{(0)}g^2(\eta)\nabla\left(\gamma^{-1}(\eta)\right)\right)/\gamma(\eta),\tag{21}$$

$$D^{(2)}(\eta)=g^2(\eta)/\gamma^2(\eta)\tag{22}$$

are the drift and diffusion coefficients, respectively.

Equation (20) corresponds to the Langevin equation governing the η evolution

$$\dot\eta=D^{(1)}(\eta)+\sqrt{2C^{(0)}D^{(2)}(\eta)}\xi(t),\tag{23}$$

where $\xi(t)$ is a white noise with standard properties

$$\langle\xi(t)\rangle=0,\ \langle\xi(t),\xi(t')\rangle=\delta(t-t').\tag{24}$$

For studying the transitions between behaviour regimes of the system let us use path integral formalism. To achieve this aim we write Langevin equation in the form of stochastic differential equation

$$d\eta=D^{(1)}(\eta)dt+\sqrt{2C^{(0)}D^{(2)}}dw\,,\tag{25}$$

where $dw=\xi(t)dt$ represents a Winner process. This notation allows us to get a new process $y(t)$ with transition Jacobean $dy/d\eta=\left(\sqrt{2C^{(0)}D^{(2)}}\right)^{-1}$. Since we use a white noise then for $y(t)$ the stochastic differentiation operator can be written

$$dy=\frac{dy}{d\eta}d\eta+\frac{1}{2}\frac{d^2y}{d\eta^2}(d\eta)^2.\tag{26}$$

Constructing by such way the evolution equation for $y(t)$ process we obtain expression for the white noise

$$\xi(t)=\frac{\dot\eta}{\sqrt{2C^{(0)}D^{(2)}}}-\frac{D^{(1)}}{\sqrt{2C^{(0)}D^{(2)}}}+\frac{1}{2}\left(\sqrt{2C^{(0)}D^{(2)}}\right)'\tag{27}$$

with probability density $P(\xi(t))\propto\exp\left(-1/2\int\xi^2(t)\right)dt$. Here and further accent means differentiation with respect to η. Taking into account the relationship between distributions $P(\eta)=P(\xi)J$, where J is the Jacobean of transition from ξ to η field, according to [14,15], we get following expression

$$P(\dot\eta,\eta,t)\propto\exp\left(-\frac{1}{2}\int\mathbf{L}dt\right),\tag{28}$$

where Onsager–Machlup function **L** acts as Lagrangian in Euclidean field theory

$$\mathbf{L}=\frac{\dot\eta^2}{2C^{(0)}D^{(2)}}+U.\tag{29}$$

Here the effective potential energy U is given by expression

$$U=\left[\frac{D^{(1)}}{\sqrt{2C^{(0)}D^{(2)}}}-\frac{1}{2}\left(\sqrt{2C^{(0)}D^{(2)}}\right)'\right]^2.\tag{30}$$

Thus, the system's kinetics will be characterized by Euler–Lagrange equation with respect to the function (29).

Stationary states, phase diagrams and phase portraits. At first let us consider a stationary states. Assuming $\dot{\eta} = 0$ in the euler–lagrange equation, we obtain

$$\gamma^2 \left(f + C^{(1)} \nabla g^2 \right) - (1/2) C^{(0)} \gamma \nabla g^2 = 0, \tag{31}$$

$$\gamma^3 \left(\nabla f g - \nabla g f + 2C^{(1)} g^2 \nabla^2 g \right) + C^{(0)} g^2 \left[(3\nabla g - 2)\gamma \nabla \gamma - \gamma^2 \nabla^2 g \right] = 0. \tag{32}$$

Headway deviation stationary values η_0, which correspond to the extremum of the effective potential energy function (30), are determined as solutions of equation (31). The solution of equation (32) gives the point of plateau appearance in U vs η dependence. Further using definitions for $\gamma(\eta)$, $f(\eta)$, and $g(\eta)$, we analyze the transitions for stochastic system in details. The dependence of the order parameter stationary values on noise intensity and characteristic control parameter is given by equation

$$\eta^4 + \eta^2 \left(1 + 2\sigma - \varepsilon - 2C^{(1)} \right) - \left(\varepsilon - \sigma - 2C^{(1)} \right)(1 + \sigma) + C^{(0)} = 0. \tag{33}$$

In this case the correlation function moments (6), which are determined by equation (17), take the following form

$$C^{(0)} = \delta^2, \; C^{(1)} = \delta^2 \tau_\lambda . \tag{34}$$

Then from Eqs. (33) and (34) for the phase diagram curve, that is the boundary of the existence region of disordered phase (η_0=0) corresponding to the free traffic, we get

$$\varepsilon = \delta^2 / (1 + \sigma) + \sigma - 2\delta^2 \tau_\lambda . \tag{35}$$

First of all we should note that separation of the low–frequency domain in noise spectrum results in decrease of the effective time that is necessary for car to reach a characteristic velocity.

The dependence of the headway deviation stationary values on the characteristic acceleration/braking time $\eta_0(\varepsilon)$ is the solution of equation (33) and is shown in Fig.1 a.

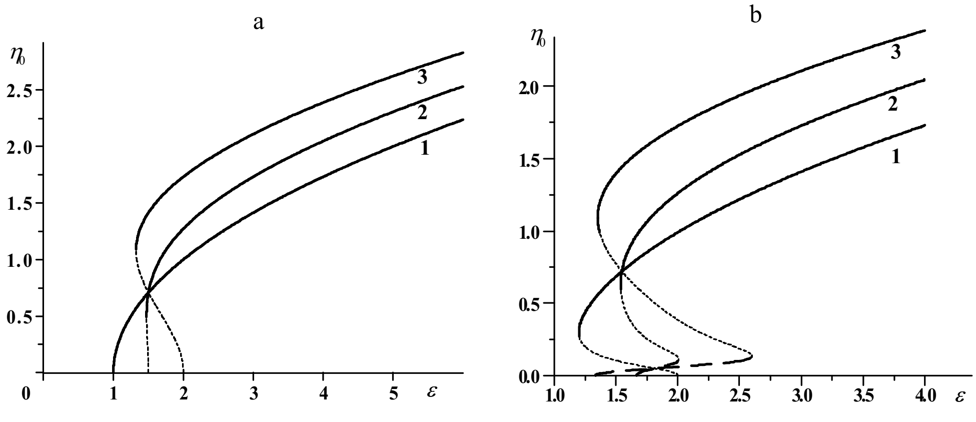

Fig. 1. Dependence of stationary value of headway deviation η_0 on characteristic acceleration/braking time $\varepsilon=\tau_0/\tau_c$ for σ=1 and τ_λ=0.2:
a – for supercritical regime; *b* – for subcritical regime at α = 0.1 and k =1.

According to Fig. 1, *a* there are zero minimum in the dependence of effective potential energy on headway deviation, which corresponds to disordered state (free moving traffic), and minimum at nonzero stationary value η_0, which meets the ordered state (traffic jam formation). Dotted line in Fig.1, *a* shows that these minimums are separated by maximum, corresponding to unstable state of the system. Thus, the increase of the noise intensity results in coexistence of disordered and ordered states inherent in first–order phase transition.

Phase diagram of the system for different values of the noise correlation time τ_λ is represented in Fig. 2. There are three domains in it. Domain 1 corresponds to disordered state of the system, i.e. free traffic. Domain 2 is characterized by coexistence of disordered and ordered states, i.e. free and jammed traffic can coexist at such parameters. Last domain 3 corresponds to ordered state of the system. Here only traffic jam exists (minimum of the effective potential energy) while free traffic is unstable (maximum of the effective potential energy).

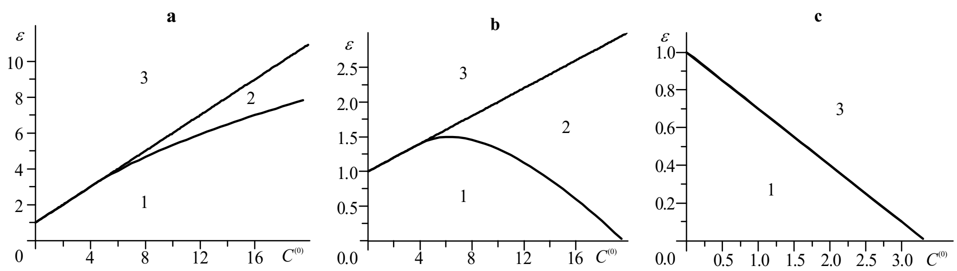

Fig. 2. Phase diagrams of the system in supercritical regime for different values of noise correlation time at σ=1: *a* – τ_λ=0.0; *b* – τ_λ=0.2; *c* – τ_λ=0.4. Domain 1 corresponds to disordered state (free moving traffic). In domain 2 ordered (traffic jam) and disordered states coexist. Domain 3 meets ordered state.

As it is obvious from Fig. 2 with increase of the noise correlation time the domain corresponding to free traffic decreases while domain of traffic jam increases.

For the consideration of the system kinetics let us use Euler–Lagrange equation

$$\frac{\partial \mathbf{L}}{\partial \eta} - \frac{d}{dt}\frac{\partial \mathbf{L}}{\partial \dot{\eta}} = \frac{\partial R}{\partial \dot{\eta}}, \qquad (36)$$

which is supplemented by dissipative function contribution $R = \dot{\eta}^2/2$. Its form is typical for generating functional method. Taking into account equations (29), (30), we find differential equation of second–order

$$\ddot{\eta} - \frac{1}{2}\frac{\left(D^{(2)}\right)'}{D^{(2)}}\dot{\eta}^2 + C^{(0)}D^{(2)}\dot{\eta} - \left[\frac{D^{(1)}}{\sqrt{D^{(2)}}} - C^{(0)}\left(\sqrt{D^{(2)}}\right)'\right]\times$$
$$\left[\left(D^{(1)}\right)'\sqrt{D^{(2)}} - D^{(1)}\left(\sqrt{D^{(2)}}\right)' - C^{(0)}D^{(2)}\left(\sqrt{D^{(2)}}\right)''\right] = 0. \qquad (37)$$

Equation (37) can be represented as a system of two first–order differential equations. This reorganization simplifies the investigation of our model. Such equations set can be solved using phase–plane method which allows us to consider the kinetic behaviour of the system on the basis of phase portraits in the $(\dot{\eta};\eta)$ plane.

Phase–plane portraits of such system are shown in Fig. 3 for each domain of phase

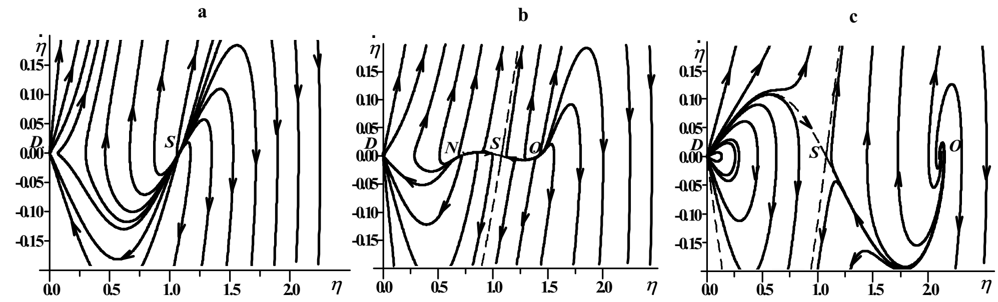

Fig. 3. Phase portraits of the system in supercritical regime for different domains of phase diagram at $C^{(0)}$=10, σ=1 and τ_λ=0.2: *a* – domain 1 at ε=0.5; *b* – domain 2 at ε=1.5; *c* – domain 3 at ε=3.

diagram in Fig. 2, *b* for τ_λ=0.2. Domain 1 has two singular points: *D*, and unstable node *S* (Fig. 3, *a*). Here singular point *D* corresponds to free traffic flow (disordered state). This point has complex character of stability because phase trajectories at $\dot{\eta} < 0$ converge to *D* and diverge from it at $\dot{\eta} > 0$ [16]. Coordinate of singular point *S* is determined by solution of equation (32). Solutions of equations (31), (32) for noise intensity $C^{(0)}$=10 are pictured in Fig. 4, *a*. As is obvious from this figure the solution of equation (32) is independent on characteristic acceleration/braking time ε, i.e. it is shown by horizontal line. Thus, coordinate of singular point *S* for all domains of phase diagram may be defined by intersection of given horizontal line with vertical line for appropriate ε value.

Phase portrait for domain 2 of the phase diagram is shown in Fig. 3, *b*. It has four singular points: *D*, unstable nodes *N* and *O*, and saddle *S*. Similar to the first case singular point *D* corresponds to free traffic. Unstable node *N* corresponds to unstable state of the system, namely, to the maximum of the effective potential energy. Saddle *S* is determined by the solution of equation (32). Unstable node *O* characterizes the traffic jam formation (ordered state).

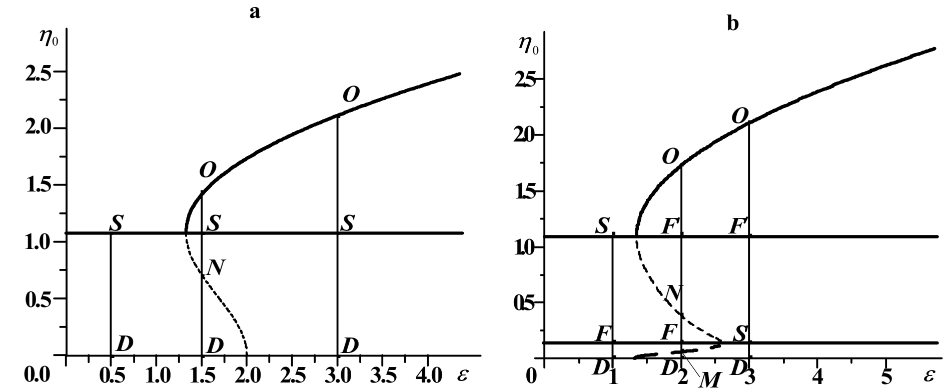

Fig. 4. Dependence of solutions η_0 of equations (31) and (32) for stationary states on characteristic acceleration/braking time ε=τ_0/τ_c at $C^{(0)}$=10, τ_λ=0.2, and σ =1 *a* – for supercritical regime; *b* – for subcritical regime at α = 0.1 and *k* =1.

In Fig. 3, *c* phase portrait meeting the domain 3 is demonstrated. There are three singular points: *D*, saddle *S*, and unstable node *O*. Here point *D* corresponds to free moving traffic also, but now phase trajectories, which pass through it, have closed form at its vicinity [16]. As previously solution of equation (32) is presented by saddle *S*. Unstable node *O* corresponds to traffic jam.

The above consideration represents supercritical regime of the traffic jam formation corresponding to second–order phase transition. However we see that for given system due to fluctuations of the characteristic acceleration/braking time free moving traffic and traffic jam can coexist that inherent in the first–order phase transition (sub–critical regime). In addition the phase portraits (Fig. 3, *a–c*) show that singular point *O* meeting the traffic jam state is unstable. Thus, we should consider subcritical regime of the traffic jam formation which is a true reason of self–organization and analogous to the first–order phase transition.

For this let us assume that the relaxation time of headway deviation is the function $t_\eta(\eta)$ increasing with η from initial value $t_\eta(1+k)^{-1}$, $k>0$ to the final one t_η, and is determined by the simplest approximation

$$t_\eta / t_\eta(\eta) = 1 + k\big/\left(1 + (\eta/\eta_t)^2\right), \tag{38}$$

where $0<\eta_t<1$; k and η_t are dispersion constant and scale. Then doing the same operations as in the supercritical case we obtain equation (11), but now $\gamma(\eta)$ and $f(\eta)$ are defined by the following expressions:

$$\begin{gathered} \gamma(\eta) = 1 + \sigma + k\frac{1-\eta^2/\alpha^2}{\left(1+\eta^2/\alpha^2\right)^2} + \eta^2, \\ f(\eta) = \left[\varepsilon - \sigma\left(1 + k\big/\left(1+\eta^2/\alpha^2\right)\right)\right]\eta - \left(1 + k\big/\left(1+\eta^2/\alpha^2\right)\right)\eta^3, \end{gathered} \tag{39}$$

where $\alpha \equiv \eta_t/\eta_m$. Further Fokker–Planck equation, Langevin equation, and Onsager–Machlup function are determined by the similar manner. However, insertion of Eqs. (39) into Eq. (31) gives the following expression for the stationary values of headway deviation:

$$\eta^4 d + \eta^2\left[d(m+\sigma) - \varepsilon - 2C^{(1)}\right] - m\left(\varepsilon - \sigma d + 2C^{(1)}\right) + C^{0(\,)} = 0, \tag{40}$$

where

$$d = 1 + k\big/\left(1+\eta^2/\alpha^2\right), \quad m = 1 + \sigma + k\left(1-\eta^2/\alpha^2\right)\big/\left(1+\eta^2/a^2\right)^2. \tag{41}$$

Equations (34), (40) and (41) define the boundary of the existence domain of disordered phase ($\eta_0=0$ in phase diagram

$$\varepsilon = \delta^2/(1+\sigma+k) + \sigma(1+k) - 2\delta^2\tau_\lambda. \tag{42}$$

According to this the separation of the low–frequency region in the noise spectrum leads to decrease of the time needed for car to reach a characteristic velocity.

Dependence $\eta_0(\varepsilon)$, that is a solution of equation (40), is pictured in Fig.1, *b*. This figure show that increase of the noise intensity $C^{(0)}$ results in appearance of two stationary states corresponding to minimums of the dependence of effective potential energy on headway deviation η. Hence, we can conclude that two stationary values of headway deviation from optimal value exist. At these values traffic jam and congested traffic can be formed [10]. The smaller value corresponds to the metastable state (dashed curve) while lager value (solid line) meets the stable ordered state of the system. These states are divided by unstable state (dotted line) corresponding to maximum of the effective potential energy.

Phase diagrams of the system are shown in Fig. 5 for different values of the noise correlation time τ_λ. Here domain 1 corresponds to disordered state of the system, in other words, only one minimum of the dependence of effective potential energy on headway deviation exists meeting zero value. This minimum characterizes the free traffic. In domain 2 disordered and ordered states coexist, so that in addition to zero minimum the minimum appears meeting the nonzero value of headway deviation. This minimum corresponds to traffic jam formation. Domain 3 meets the most complex form of effective potential energy. Here the metastable and ordered states of the system coexist. Metastable state appearance is caused by displacement of zero minimum (disordered state) along the axis of headway deviation of the dependence of effective potential energy. It implies that in this domain traffic jam can appear at two different values of headway deviation from optimal value. And at the same time zero value of headway deviation corresponds to maximum of effective potential energy, i.e. free traffic becomes unstable. Domain 4 characterizes the ordered state of the system, i.e. traffic jam at a given headway deviation value and unstable free traffic. Last domain 5 describes

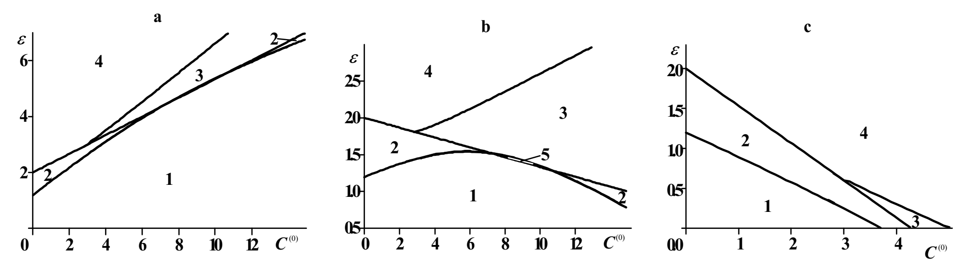

Fig. 5. Phase diagrams of the system in subcritical regime for different values of noise correlation time at α=0.1, σ=1 and k=1: a – τ_λ=0.0; b – τ_λ=0.2; c – τ_λ=0.4. Domain 1 corresponds to disordered state (free moving traffic). In domain 2 ordered (traffic jam) and disordered states coexist. In domain 3 metastable (congested traffic) and ordered states coexist. Domain 4 meets ordered state, and domain 5 meets metastable one.

metastable system's state and is located near the boundary between domain 1 and 3. At crossing the boundary from domain 1 to domain 5 transition of the displacement type from disordered to metastable state occurs. It corresponds to appearance of only one minimum of the effective potential energy and means the opportunity of congested traffic formation at small value of headway deviation. Free traffic is unstable here too.

Let us now analyze kinetics of the system using Euler–Lagrange equation (36). Phase–plane portraits of the system for four domains, that are shown in phase diagram for τ_λ=0.2 (Fig. 5, b), are demonstrated in Fig. 6. Domain 1 is characterized by the presence of three singular points: D, stable focus F and saddle S (Fig. 6, a). Here singular point D corresponds to free traffic and has the same complex character with singular point D for supercritical regime. Coordinates of the stable focus F and saddle S are determined by the solutions of equation (32) with taking into account Eqs. (39). Solutions of Eqs. (31), (32) for subcritical regime for noise intensity $C^{(0)}$=10 are shown in Fig. 4, b. As is obvious from the figure the solutions of equation (32) are independent on characteristic acceleration/braking time ε. Consequently, we may obtain the coordinates of singular points F and S by intersection of corresponding horizontal lines with vertical lines for appropriate ε value. Thus the coordinates of singular points S, S', F, F' can be defined for all phase diagram domains.

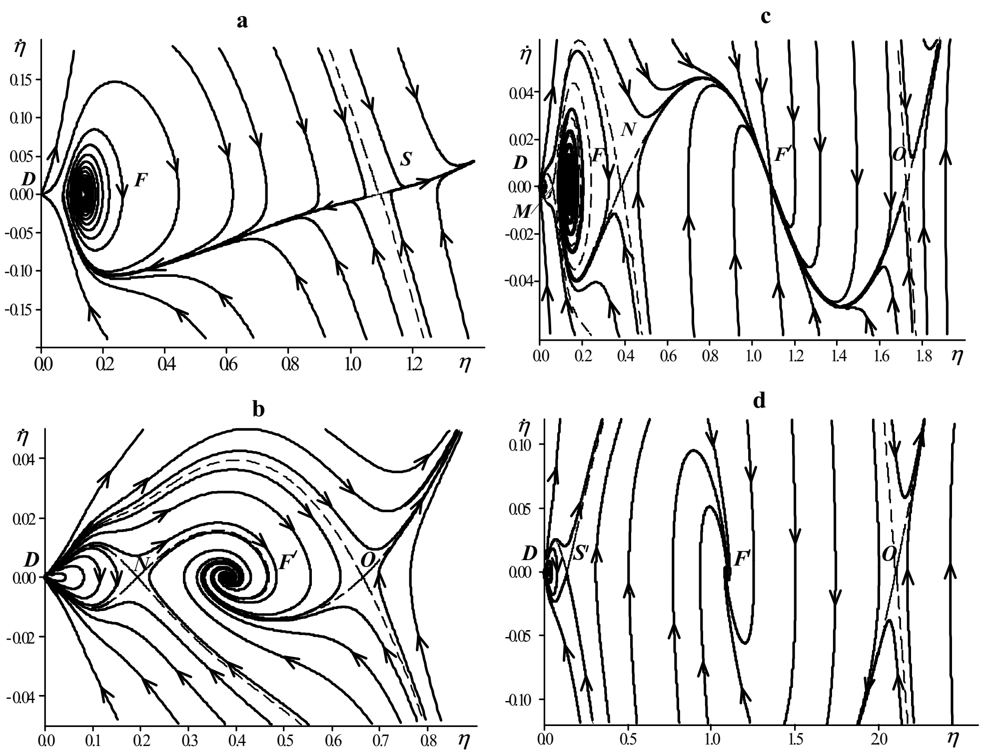

Fig. 6. Phase portraits of the system in subcritical regime for different domains of phase diagram at α=0.1, σ=1 k=1 and τ_λ=0.2: *a* – domain 1 at $C^{(0)}$=10, ε=1; *b* – domain 2 at $C^{(0)}$=2, ε=1.5; *c* – domain 3 at $C^{(0)}$=10, ε=2; *d* – domain 4 at $C^{(0)}$=10, ε=3.

Phase portrait of the system for domain 2 of phase diagram is shown in Fig. 6, *b*. There are four singular points: *D*, saddles *N* and *O*, and stable focus F'. As in the Fig. 6, *a* singular point *D* corresponds to free traffic, but now phase trajectories, which pass through it, have closed form at its vicinity [16]. Point F' is determined by the solution of equation (32). Saddle *O* characterizes the traffic jam formation. Saddle *N* corresponds to unstable state of the system, namely, to maximum of the dependence of effective potential energy on headway deviation η.

In Fig. 6, *c* phase portrait for the most complex domain 3 of phase diagram is demonstrated. The maximal number of singular points is placed here: *D*, saddles *M*, *N* and *O*, stable focus *F*, and stable node F'. As in the Fig. 6a coordinates of points *F* and F' are determined by solutions of equation (32). Singular points *N* and *O* have the same meaning as in Fig. 6, *b*. Saddle *M* corresponds to metastable state of the system, namely, to the formation of congested traffic.

Phase portrait in Fig. 6, *d* corresponds to domain 4 of phase diagram. As is shown above the solutions of equation (32) are represented by saddle S' and stable node F'. Saddle *O* corresponds to jammed traffic.

Discussion. The above consideration shows that for supercritical regime, which corresponds to the second–order phase transition, due to fluctuations of the characteristic acceleration/bra–king time free and jammed traffic can coexist. As is known such a picture is inherent in first–order phase transition. For subcritical regime, which meets the first–order phase transition, due to regulation of fluctuations intensity of characteristic acceleration/braking time a given system can be passed from free traffic regime to traffic jam formation. The last state can appear at different values of headway deviation between transport facilities. State of the system corresponding to the smaller value of headway deviation may be represented as a congested traffic regime. However, the correct answer can be defined using additional investigation of dynamical car characteristics, but such study is out of our consideration framework.

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ПЕРЕХІД МІЖ РЕЖИМАМИ ТРАНСПОРТНОГО ПОТОКУ З УРАХУВАННЯМ ФЛУКТУАЦІЙ ХАРАКТЕРНОГО ЧАСУ ПРИСКОРЕННЯ/ГАЛЬМУВАННЯ

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Досліджено перехід між режимами транспортного потоку (ідеальним рухом автомобілів та затором на дорозі) для однорідної моделі послідовного руху автомобілів з урахуванням флуктуацій характерного часу прискорення/гальмування, які описує процес Орнштейна–Улєнбєка. Досліджено поведінку найбільш імовірного відхилення інтервалу між машинами від його оптимального значення та побудовано фазову діаграму системи для супер– та субкритичного режимів формування транспортного затору. Для першого режиму флуктуації характерного часу прискорення/гальмування приводять до співіснування ідеального руху автомобілів та транспортного затору, що характерно для фазового переходу першого роду. Для другого режиму знайдено два стійкі стани, що відповідають ненульовим значенням відхилення інтервалу, за яких можливе утворення затору та стислого руху машин. Використовуючи метод фазової площини, проаналізовано кінетику переходів системи, що відповідають різним ділянкам фазової діаграми для обох режимів.

Ключові слова: відхилення інтервалу між автомобілями, система Лоренця, процес Орнштейна–Уленбека, фазові діаграми, рівняння Ланжевена.

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