

# Noise-induced oscillations in non-equilibrium steady state systems

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## Abstract

We consider the effect of stochastic sources on the self-organization process being initiated with creation of the limit cycle. The general expressions obtained are applied to the stochastic Lorenz system to show that offset from the equilibrium steady state can destroy the limit cycle at a certain relation between characteristic scales of temporal variation of principal variables. Noise-induced resonance related to the limit cycle is found analytically to appear in the non-equilibrium steady state system if the fastest variations display a principal variable, which is coupled with two different degrees of freedom or more.

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## 1. Introduction

The interplay between noise and nonlinearity of dynamical systems [1] is known to arrive at a crucial change in behavior of systems displaying noise-induced [2, 3] and recurrence [4, 5] phase transitions, stochastic resonance [6, 7], noise-induced pattern formation [8, 9], noise-induced transport [3, 10] etc (see [11] for a review). The constructive role of noise in dynamical systems includes hopping between multiple stable attractors [12, 13] and stabilization of the Lorenz attractor near the threshold of its formation [14, 15]. Such a type of behavior is inherent in finite systems, which involve discrete entities (for instance, in ecological systems, individuals form population stochastically in accordance with random births and deaths). Examples of substantial alteration of finite systems under the effect of intrinsic noises give epidemics [16–18], predator–prey population dynamics [19, 20], opinion dynamics [21], biochemical clocks [22, 23], genetic networks [24], cyclic trapping reactions [25], etc.

Within the phase-plane language, the phase transitions pointed out present the simplest case where a fixed point only appears. We are interested in studying a more complicated situation, when the system under consideration may display oscillatory behavior related to the limit cycle appearing as a result of the Hopf bifurcation [26, 27]. It has long been conjectured [28] that in some situations the influence of noise would be sufficient to produce a cyclic behavior [29]. Recent

consideration [30] allows the relation between the stochastic oscillations in the fixed point phase and the oscillations in the limit cycle phase to be elucidated. Moreover, excitable [31], bistable [32] and close to bifurcation [33] systems display oscillation behavior, whose adjacency to ideally periodic signal depends resonantly on the noise intensity [34], which was a reason to call these oscillations coherence resonance [31] or stochastic coherence [11]. Control of the coherence resonance regime was shown to be achieved with a time-delayed feedback, which enables us to increase or decrease the regularity of motion [35]. Characteristically, a quasioscillatory behavior may be organized without any input signal, provided a stochastic nonlinear system itself has an intrinsic timescale. If this scale is driven by a multiplicative noise, which induces bistable behavior in a deterministically monostable medium, then a doubly stochastic resonance arises [36].

The simplest way to formulate the model related to systems with a finite number  $N < \infty$  of constituents is to consider the sum  $\vec{S} = \sum_{i=1}^N \vec{\xi}_i$  of random state vectors  $\vec{\xi}_i$  with components  $\xi_i^\alpha$ ,  $\alpha = 1, \dots, d$ . Then, the state vector

$$\vec{S} = N\vec{X} + \sqrt{N}\vec{x} \quad (1)$$

is decomposed into a deterministic component that is proportional to the total system size  $N$  and a random one that is proportional to its square root [37]. In the limit of infinite particle numbers  $N \rightarrow \infty$ , such systems are faithfully

described by deterministic equations to find time dependence  $\bar{X}(t)$ , which addresses the behavior of the system on a mean-field level. On the other hand, a systematic study of corrections due to finite system size can capture the behavior of fluctuations  $\vec{x}(t)$  about the mean-field solution. These fluctuations are governed by the Langevin equations; however, in difference of approach [30], we consider multiplicative noises instead of additive ones, on the one hand, and nonlinear forces instead of linear ones, on the other. Within such a framework, the aim of the present paper is to extend analytical descriptions [30] of finite-size stochastic effects to non-equilibrium systems where noises play a crucial role with respect to periodic limit cycle solution creation or its suppression. We will show that the character of the stationary behavior of the non-equilibrium system is determined by the relation between scales of temporal variation of principal variables as well as their coupling. In contrast to the doubly stochastic resonance [36], we consider the case when the multistable state is caused by both multiplicative noise and offset from the equilibrium state.

This paper is organized as follows. In section 2, we obtain the conditions of the limit cycle creation using a pair of stochastic equations with nonlinear forces and multiplicative noises. Sections 3 and 4 are devoted to consideration of these conditions on the basis of the stochastic Lorenz system with different regimes of principal variables slaving. According to section 3 the limit cycle is created only in the case where the fastest variation displays a principal variable, which is coupled nonlinearly with two other degrees of freedom or more. The opposite case is studied in section 4 to show that the limit cycle disappears in the non-equilibrium steady state. Section 5 concludes our consideration.

## 2. Statistical picture of limit cycle

According to the theorem of central manifold [26], to achieve a closed description of a limit cycle it is enough to use only two variables  $x_\alpha$ ,  $\alpha = 1, 2$ . In such a case, stochastic evolution of the system under investigation is defined by the Langevin equations [39]

$$\dot{x}_\alpha = f^{(\alpha)} + \mathcal{G}_\alpha \zeta_\alpha(t), \quad \alpha = 1, 2 \quad (2)$$

with forces  $f^{(\alpha)} = f^{(\alpha)}(x_1, x_2)$  and noise amplitudes  $\mathcal{G}_\alpha = \mathcal{G}_\alpha(x_1, x_2)$ , being functions of stochastic variables  $x_\alpha$ ,  $\alpha = 1, 2$ ; white noises  $\zeta_\alpha(t)$  are determined by usual conditions  $\langle \zeta_\alpha(t) \rangle = 0$ ,  $\langle \zeta_\alpha(t) \zeta_\beta(t') \rangle = \delta_{\alpha\beta} \delta(t - t')$ . Within the assumption that microscopic transfer rates are not correlated for different variables  $x_\alpha$  (see below), the probability distribution function  $\mathcal{P} = \mathcal{P}(x_1, x_2; t)$  is determined by the Fokker–Planck equation

$$\frac{\partial \mathcal{P}}{\partial t} + \sum_{\alpha=1}^2 \frac{\partial J^\alpha}{\partial x_\alpha} = 0, \quad (3)$$

where components of the probability current take the form

$$J^{(\alpha)} \equiv \mathcal{F}^{(\alpha)} \mathcal{P} - \frac{1}{2} \sum_{\beta=1}^2 \frac{\partial}{\partial x_\beta} (\mathcal{G}_\alpha \mathcal{G}_\beta \mathcal{P}) \quad (4)$$

with the generalized forces

$$\mathcal{F}^{(\alpha)} = f^{(\alpha)} + \lambda \sum_{\beta=1}^2 \frac{\partial (\mathcal{G}_\alpha \mathcal{G}_\beta)}{\partial x_\beta}, \quad (5)$$

being determined with choice of the calculus parameter  $\lambda \in [0, 1]$  (for the Ito and Stratonovich cases, one has  $\lambda = 0$  and  $1/2$ , respectively). Within the steady state, the components of the probability current take constant values  $J_0^{(\alpha)}$  and the system behavior is defined by the following equations:

$$\frac{\partial}{\partial x_1} (\mathcal{G}_1^2 \mathcal{P}) + \frac{\partial}{\partial x_2} (\mathcal{G}_1 \mathcal{G}_2 \mathcal{P}) - 2\mathcal{F}^{(1)} \mathcal{P} = -2J_0^{(1)}, \quad (6)$$

$$\frac{\partial}{\partial x_1} (\mathcal{G}_1 \mathcal{G}_2 \mathcal{P}) + \frac{\partial}{\partial x_2} (\mathcal{G}_2^2 \mathcal{P}) - 2\mathcal{F}^{(2)} \mathcal{P} = -2J_0^{(2)}. \quad (7)$$

Multiplying the first of these equations by a factor  $\mathcal{G}_2$  and the second one by  $\mathcal{G}_1$  and then subtracting the results, we arrive at an explicit form of the probability distribution function as follows:

$$\begin{aligned} \mathcal{P}(x_1, x_2) &= \frac{J_0^{(1)} \mathcal{G}_2(x_1, x_2) - J_0^{(2)} \mathcal{G}_1(x_1, x_2)}{\mathcal{D}(x_1, x_2)}, \\ \mathcal{D}(x_1, x_2) &\equiv (\mathcal{G}_2 \mathcal{F}^{(1)} - \mathcal{G}_1 \mathcal{F}^{(2)}) \\ &+ \frac{1}{2} \left[ \left( \mathcal{G}_1^2 \frac{\partial \mathcal{G}_2}{\partial x_1} - \mathcal{G}_2^2 \frac{\partial \mathcal{G}_1}{\partial x_2} \right) - \mathcal{G}_1 \mathcal{G}_2 \left( \frac{\partial \mathcal{G}_1}{\partial x_1} - \frac{\partial \mathcal{G}_2}{\partial x_2} \right) \right]. \end{aligned} \quad (8)$$

This function diverges at condition

$$\begin{aligned} 2(\mathcal{G}_1 \mathcal{F}^{(2)} - \mathcal{G}_2 \mathcal{F}^{(1)}) &= \left( \mathcal{G}_1^2 \frac{\partial \mathcal{G}_2}{\partial x_1} - \mathcal{G}_2^2 \frac{\partial \mathcal{G}_1}{\partial x_2} \right) \\ &- \mathcal{G}_1 \mathcal{G}_2 \left( \frac{\partial \mathcal{G}_1}{\partial x_1} - \frac{\partial \mathcal{G}_2}{\partial x_2} \right), \end{aligned} \quad (9)$$

which physically means the appearance of a domain of forbidden values of stochastic variables  $x_\alpha$ , which is bonded with a closed line of the limit cycle. Characteristically, such a line appears only if the denominator  $\mathcal{D}(x_1, x_2)$  of fraction (8) includes even powers of both variables  $x_1$  and  $x_2^3$ .

It is worth noting that the analytical expression (8) of the probability distribution function becomes possible due to the special form of the probability current (4), where effective diffusion coefficient takes the multiplicative form  $\mathcal{D}_{\alpha\beta} = \mathcal{G}_\alpha \mathcal{G}_\beta$ . In the general case, this coefficient is known to be defined with the expression [40]

$$\mathcal{D}_{\alpha\beta} = \sum_{ab} I_{ab} g_\alpha^a g_\beta^b, \quad (10)$$

where kernel  $I_{ab}$  determines transfer rate between microscopic states  $a$  and  $b$ , whereas factors  $g_\alpha^a$  and  $g_\beta^b$  are specific noise amplitudes of values  $x_\alpha$  related to these states. We have considered above the simplest case when the transfer rate  $I_{ab} = I$  is constant for all microscopic states. As a result, the diffusion coefficient (10) takes the needed form  $\mathcal{D}_{\alpha\beta} = \mathcal{G}_\alpha \mathcal{G}_\beta$  with cumulative noise amplitudes  $\mathcal{G}_\alpha \equiv \sqrt{I} \sum_a g_\alpha^a$  and  $\mathcal{G}_\beta \equiv \sqrt{I} \sum_b g_\beta^b$ .

<sup>3</sup> Archetype of closed curves presents the circle  $x_1^2 + x_2^2 = 1$ .

### 3. Noise-induced resonance within the Lorenz system

As the simplest and most popular example of the self-organization induced by the Hopf bifurcation, we consider the modulation regime of spontaneous laser radiation, whose behavior is presented in terms of the radiation strength  $E$ , the matter polarization  $P$  and the difference of level populations  $S$  [37]. Accounting for the stochastic sources related, the self-organization process of this system is described by the Lorenz equations

$$\begin{aligned}\tau_E \dot{E} &= [-E + a_E P - \varphi(E)] + g_E \zeta(t), \\ \tau_P \dot{P} &= (-P + a_P E S) + g_P \zeta(t), \\ \tau_S \dot{S} &= [(S_e - S) - a_S E P] + g_S \zeta(t).\end{aligned}\quad (11)$$

Here, the overdot denotes differentiation over time  $t$ ;  $\tau_{E,P,S}$  and  $a_{E,P,S} > 0$  are timescales and feedback constants of related variables, respectively;  $g_{E,P,S}$  are corresponding noise amplitudes, and  $S_e$  is the driven force. In the absence of noises ( $g_E = g_P = g_S = 0$ ) and at relations  $\tau_P, \tau_S \ll \tau_E$  between timescales, system (11) addresses the limit cycle only in the presence of the nonlinear force [41]

$$\varphi(E) = \frac{\kappa E}{1 + E^2/E_n^2} \quad (12)$$

characterized with parameters  $\kappa > 0$  and  $E_n$ . In this section, we consider the noise effect in the case of opposite relations  $\tau_E \ll \tau_P, \tau_S$  of timescales, when periodic variation of stochastic variables becomes possible even at suppression of the force (12).

It is easy, further, to pass to dimensionless variables  $t, \zeta, E, P, S, g_E, g_P, g_S$  by making use of the related scales:

$$\begin{aligned}\tau_P; \quad \zeta_s &= \tau_P^{-1/2}; \quad E_s = (a_P a_S)^{-1/2}, \\ P_s &= (a_E^2 a_P a_S)^{-1/2}, \quad S_s = (a_E a_P)^{-1}; \quad g_E^s = (\tau_P / a_P a_S)^{1/2}, \\ g_P^s &= (\tau_P / a_E^2 a_P a_S)^{1/2}, \quad g_S^s = \tau_P^{1/2} / a_E a_P.\end{aligned}\quad (13)$$

Then, equations (11) take the simple form<sup>4</sup>

$$\begin{aligned}\sigma^{-1} \dot{E} &= -E + P - \varphi(E) + g_E \zeta(t), \\ \dot{P} &= -P + E S + g_P \zeta(t), \\ (\varepsilon / \sigma) \dot{S} &= (S_e - S) - E P + g_S \zeta(t),\end{aligned}\quad (14)$$

where the timescale ratios

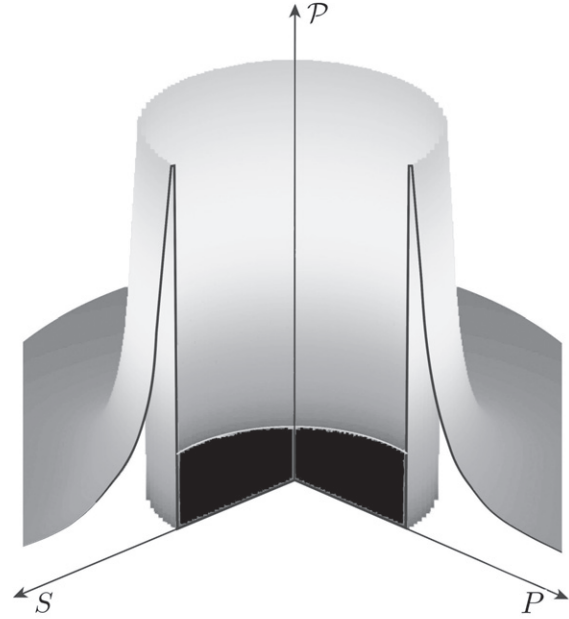
$$\sigma = \tau_P / \tau_E, \quad \varepsilon = \tau_S / \tau_E \quad (15)$$

are introduced. In the absence of the noises, the Lorenz system (14) is known to show the usual bifurcation in the point  $S_e = 1$  and the Hopf bifurcation at the driven force [37, 38]

$$S_e = \frac{\tau_P \tau_E^{-1} + \tau_S^{-1} + 3\tau_P^{-1}}{\tau_E \tau_E^{-1} - \tau_S^{-1} - \tau_P^{-1}}. \quad (16)$$

However, the noiseless limit cycle ( $g_E = g_P = g_S = 0$ ) is unstable and the Hopf bifurcation arrives at the strange attractor only.

<sup>4</sup> These equations are reduced to the initial Lorenz form [38] if we set  $X \equiv \sqrt{\sigma/\varepsilon} E, Y \equiv \sqrt{\sigma/\varepsilon} P, Z \equiv S_e - S, r \equiv S_e, b \equiv \sigma/\varepsilon$  and  $g_E = g_P = g_S = 0$ .



**Figure 1.** Steady state distribution function (8) at  $J_0^{(P)} = 1$ ,  $J_0^{(S)} = 10$ ,  $\tau_P = \tau_S$ ,  $S_e = 0.5$ ,  $g_E = 0.5$ ,  $g_P = 1.376$ ,  $g_S = 2.5$ .

With switching on of the noises, the condition  $\tau_E \ll \tau_P$  allows us to set the lhs of the first equation (14) to zero. Then, the radiation strength is expressed with the equality

$$E = P + g_E \zeta(t), \quad (17)$$

whose insertion into system (14) reduces it into two-dimensional form

$$\begin{aligned}\dot{P} &= -P(1 - S) + \mathcal{G}_P \zeta(t), \\ \dot{S} &= (\sigma/\varepsilon) [(S_e - S) - P^2] + \mathcal{G}_S \zeta(t)\end{aligned}\quad (18)$$

with the effective amplitudes of multiplicative noises

$$\begin{aligned}\mathcal{G}_P &= \sqrt{g_P^2 + g_E^2 S^2}, \\ \mathcal{G}_S &= (\tau_P / \tau_S) \sqrt{g_S^2 + g_E^2 P^2}\end{aligned}\quad (19)$$

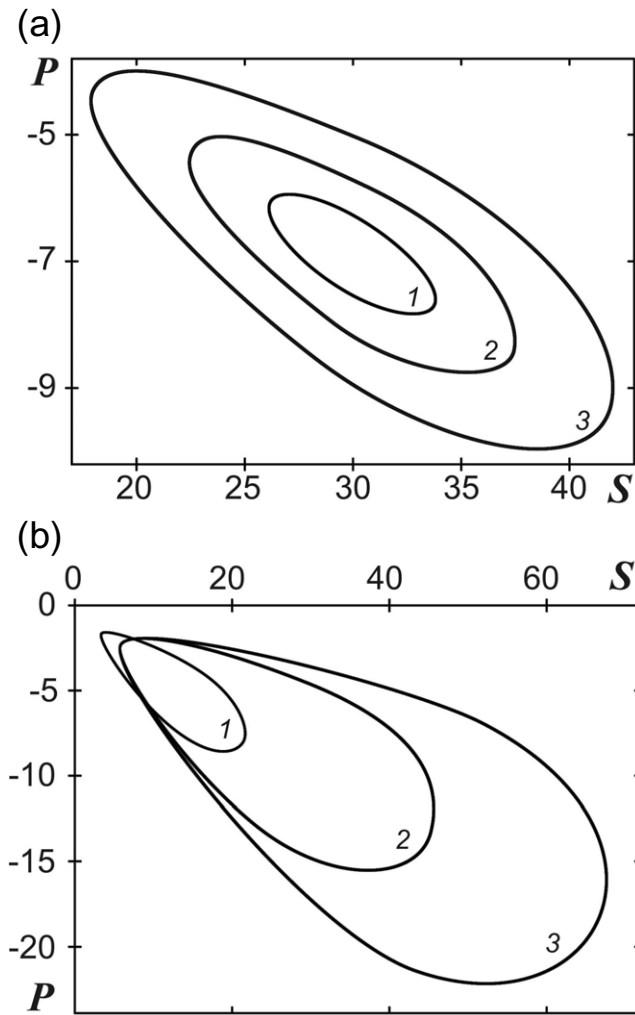
and the generalized forces

$$\mathcal{F}^{(P)} = -P(1 - S) + \lambda \frac{g_E^2}{\tau_S / \tau_P} S \sqrt{\frac{(g_S / g_E)^2 + P^2}{(g_P / g_E)^2 + S^2}}, \quad (20)$$

$$\mathcal{F}^{(S)} = (\tau_P / \tau_S) [(S_e - S) - P^2] + \lambda \frac{g_E^2}{\tau_S / \tau_P} P \sqrt{\frac{(g_P / g_E)^2 + S^2}{(g_S / g_E)^2 + P^2}}.$$

In this way, the probability density (8) takes infinite values at condition

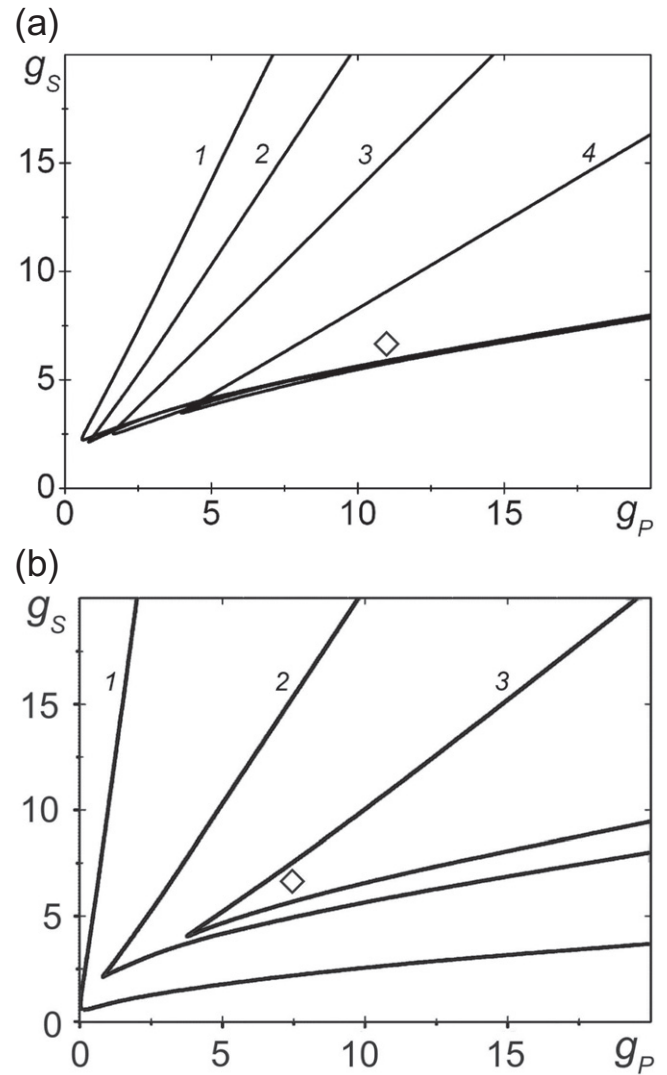
$$\begin{aligned}& \left( \frac{g_S^2}{g_E^2} + P^2 \right) \sqrt{\frac{g_P^2}{g_E^2} + S^2} P(1 - S) \\ & + \left( \frac{g_P^2}{g_E^2} + S^2 \right) \sqrt{\frac{g_S^2}{g_E^2} + P^2} [(S_e - S) - P^2] \\ & + \frac{g_E^2 \sigma}{2 \varepsilon} \left( \frac{g_S^2}{g_E^2} + P^2 \right)^{3/2} S - \frac{g_E^2}{2} \left( \frac{g_P^2}{g_E^2} + S^2 \right)^{3/2} P = 0, \quad (21)\end{aligned}$$



**Figure 2.** The form of the limit cycle determined with equation (21) at  $\varepsilon = 1, \sigma = 1$  and: (a)  $g_E = 0.5, g_P = 11, g_S = 6$  (curves 1–3 relate to  $S_e = 0.5, 1.0$  and  $2.0$ , respectively) and (b)  $S_e = 0.5, g_P = 7.5, g_S = 6.5$  (curves 1–3 relate to  $g_E = 1.0, 0.6$  and  $0.5$ , respectively).

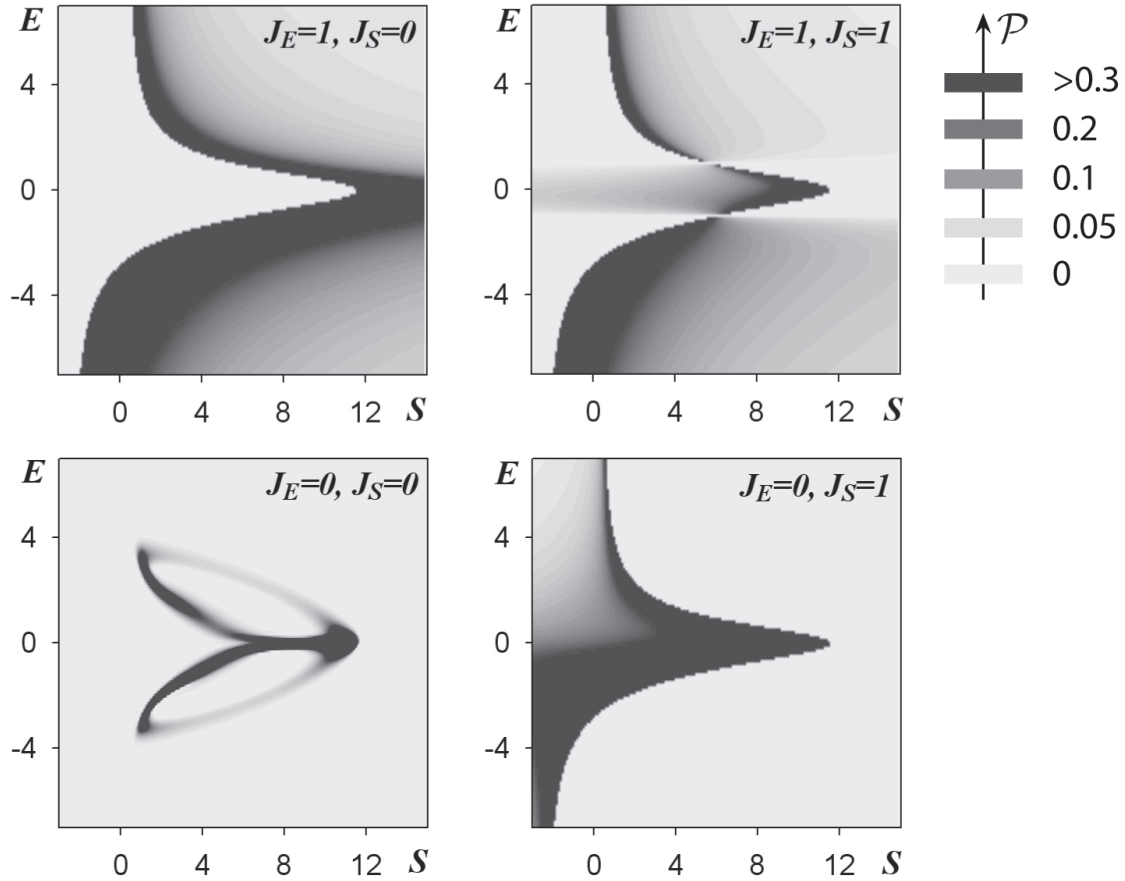
where we choose the simplest case of the Ito calculus ( $\lambda = 0$ ).

The reduced Lorenz system (18) has two-dimensional form (2), where the role of variables  $x_1$  and  $x_2$  is played by the matter polarization  $P$  and the difference of level populations  $S$ . According to the distribution function (8) shown in figure 1, the stochastic variables  $P$  and  $S$  are realized with nonzero probabilities out of the limit cycle only, whereas in its interior the domain of forbidden values  $P, S$  appears. That is the principal difference from the deterministic limit cycle, which bounds a domain of unstable values of related variables. The form of this domain is shown in figure 2 at different values of the noise amplitudes  $g_E, g_P, g_S$  and the driven force  $S_e$ . It is seen that this domain grows with increase of the driven force  $S_e$ , whereas an increase of the force fluctuations  $g_E$  shrinks it. On the other hand, phase diagrams depicted in figure 3 show that increasing the noise amplitudes of both polarization and difference of level populations enlarges the domain of the limit cycle creation (more exactly, the noise amplitude  $g_E$  shrinks this domain from both above and below, whereas an increase of the driven force  $S_e$  makes the same from above only).



**Figure 3.** Phase diagrams of the limit cycle creation at  $\varepsilon = 1, \sigma = 1$  and: (a)  $g_E = 0.5$ , curves 1–4 correspond to  $S_e = 0.0, 0.5, 1.0$  and  $2.0$ , respectively, and (b)  $S_e = 0.5$ , curves 1–3 correspond to  $g_E = 0.1, 0.5$  and  $1.0$ , respectively (diamonds relate to the values  $g_P$  and  $g_S$ , for which limit cycles in figure 2 are depicted).

The principal peculiarity of the limit cycles obtained is that their form, determined with equation (21), does not depend on a non-equilibrium degree fixed by the stationary probability currents  $J_0^{(1,2)}$ , whereas the probability (8) itself does not equal zero at conditions  $J_0^{(1,2)} \neq 0$  only. In this connection, one should point out the non-triviality of the problem of numerical solution of the reduced Lorenz system (18), which determines these limit cycles initially. Indeed, resolving this problem proposes the following steps: (i) direct solution of the stochastic equations (18) to find a set of the time dependencies  $P(t)$  and  $S(t)$ ; (ii) numerical determination of the time-dependent probability  $\mathcal{P}(P, S; t)$  to realize the entire set of possible solutions of the equations (18); (iii) selection of non-equilibrium solutions, which obey the steady state condition  $J^{(\alpha)} = J_0^{(\alpha)}$ ,  $\alpha = 1, 2$ , determined with the probability current (4); (iv) calculation of the probability distribution  $\mathcal{P}_s(P, S)$  of the steady state solutions; (v) determination of the stochastic limit cycle according to the condition  $\mathcal{P}_s(P, S) = \infty$ . Realization of this program is in progress.



**Figure 4.** Steady state probability distribution function dependent on the radiation strength  $E$  and the difference of level populations  $S$  at conditions  $\tau_P \ll \tau_E = \tau_S$ ,  $\kappa = 10$ ,  $S_e = 11.6$ ,  $g_E = 0.2$ ,  $g_P = 0.2$ ,  $g_S = 0.2$  and different probability currents  $J_0^{(E)}$ ,  $J_0^{(S)}$  (shown on the panels related).

#### 4. Lorenz system without limit cycle

According to [41], at conditions  $\tau_P \ll \tau_E, \tau_S$ , the deterministic system ( $g_{E,P,S} = 0$ ) has a limit cycle only at large intensity  $\kappa$  of nonlinear force (12). In this case, it is easy to measure the time  $t$  in the scale  $\tau_E$  and replace  $\tau_P$  by  $\tau_E$  in the set of scales (13). Then, one obtains instead of equation (17) the relation

$$P = ES + g_P \zeta(t), \quad (22)$$

due to which the Lorenz system (14) is reduced to two-dimensional form

$$\begin{aligned} \dot{E} &= -[E(1-S) + \varphi(E)] + \mathcal{G}_E \zeta(t), \\ \dot{S} &= \varepsilon^{-1} [S_e - S(1+E^2)] + \mathcal{G}_S \zeta(t) \end{aligned} \quad (23)$$

with the effective noise amplitudes

$$\begin{aligned} \mathcal{G}_E &= \sqrt{g_P^2 + g_E^2}, \\ \mathcal{G}_S &= \varepsilon^{-1} \sqrt{g_S^2 + g_P^2 E^2}. \end{aligned} \quad (24)$$

The generalized forces are as follows:

$$\begin{aligned} \mathcal{F}^{(E)} &= -[E(1-S) + \varphi(E)], \\ \mathcal{F}^{(S)} &= \varepsilon^{-1} [(S_e - S) - SE^2] + \lambda \frac{g_P^2}{\varepsilon} E \sqrt{\frac{1 + (g_E/g_P)^2}{(g_S/g_P)^2 + E^2}}. \end{aligned} \quad (25)$$

The probability distribution function (8) diverges at condition

$$\begin{aligned} &\frac{(g_S/g_P)^2 + E^2}{1 + (g_E/g_P)^2} [\varphi(E) + E(1-S)] + \sqrt{\frac{(g_S/g_P)^2 + E^2}{1 + (g_E/g_P)^2}} \\ &\times [S_e - S(1+E^2)] \left( \lambda - \frac{1}{2} \right) g_P^2 E = 0, \end{aligned} \quad (26)$$

being the equation that does not include even powers of the variable  $S$ .

As a result, one can conclude that offset from the equilibrium steady state destroys a deterministic limit cycle at the relations  $\tau_P \ll \tau_E, \tau_S$  between characteristic scales. This conclusion is confirmed by figure 4, which shows divergence of the probability distribution function on the limit cycle of variation of the radiation strength  $E$  and the difference of level populations  $S$  at zero probability currents  $J_0^{(E)}$  and  $J_0^{(S)}$  only. With an increase of these currents the system escapes from the equilibrium steady state and the maximum of the distribution function shifts to non-closed curves to be determined with equation (26).

#### 5. Conclusion

We have considered the effect of stochastic sources on the self-organization process being initiated with creation of the limit cycle. In sections 3 and 4, we have applied general

relations obtained in section 2 to the stochastic Lorenz system. We have shown that offset from the equilibrium steady state can destroy or create the limit cycle dependent on the relation between characteristic scales of temporal variation of principal variables.

Investigation of the Lorenz system with different regimes of principal variables slaving shows that additive noises can take multiplicative character if one of these noises has a much shorter timescale than others. In such a case, the limit cycle may be created if the fastest variable is coupled with more than two slow ones. However, the case considered in section 4 shows that such a dependence is not necessary to arrive at limit cycle. The formal reason is that within adiabatic condition  $\tau_P \ll \tau_E$  both noise amplitude  $\mathcal{G}_S(E)$  and generalized force  $\mathcal{F}^{(S)}(E)$ , determined with equations (24) and (25), enclose the squared strength  $E^2$ , but do not include the square  $S^2$ .

The limit cycle is created if the fastest variations display a principal variable, which is coupled with two different degrees of freedom or more. Indeed, at the relations  $\tau_E \ll \tau_P, \tau_S$  of relaxation times considered in section 3, the strength  $E$  evolves according to the stochastic law of motion (17). Accounting for this relation in the nonlinear terms of the last two equations of (14), we arrive at dependencies of the noise amplitudes of the polarization  $P$  and the difference of level populations  $S$  on both variables  $S$  and  $P$  themselves. Due to the Gaussian nature of the noises their variances are additive values [39, 40], so that effective noise amplitudes  $\mathcal{G}_P$  and  $\mathcal{G}_S$  of the principal variables are defined by equations (19), which include both the squares  $S^2$  and  $P^2$ . As a result, solutions of equation (21) become double-valued to be related to the limit cycle.

This cycle appears physically as stochastic coherence, which has been observed both numerically [14, 31] and analytically [42]. Analogously to the doubly stochastic resonance [36], such a resonance may be organized if the stochastic nonlinear system has two noises, but both of them must be multiplicative in nature. Moreover, the system under study acquires an intrinsic timescale related to the multistable state only far away from the equilibrium statistical state. In contrast to the deterministic limit cycle, which bounds a domain of unstable values of related variables, in our case stochastic variables evolve out of the limit cycle only, whereas in its interior the domain of forbidden values appears.

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