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Homogeneous solutions of the electroelasticity equations for piezoceramic layers in R^3

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Abstract We present a procedure for the derivation of homogeneous solutions for piezoceramic layers within the framework of electroelasticity. The proposed approach simplifies considerably the Lurié (J Appl Math Mech 6:151–169, 1942) method. Two cases of mechanical boundary-conditions for piezoceramic layers are examined, namely, when the bases are (a) built in, and (b) free from the influence of forces. In both cases, the bases of the layer are assumed to be covered by grounded electrodes. It is shown that in the case of boundary conditions of the first type and for the symmetric, with respect to the mid-surface of the layer, electro-elastic state, the homogeneous solutions do not contain any biharmonic terms. We also calculate the distribution of the characteristic values of the corresponding spectrum problems for every given type of boundary conditions. The derived homogeneous solutions can be used for solving boundary-value problems for piezoceramic cylinders and layers within the framework of electroelasticity. We illustrate our approach through a practical example considering an oblique-symmetric boundary-value problem for layers which weaken due to a side to side elliptic cavity.

1 Introduction

Over the last 10 years, the research in the area of deformed solid body mechanics, the so called electroelasticity has been intensively developed due to the rapid increase of the production of piezoceramic transformers with a wide range of applications ranging from ultrasonics to radioelectronics and from data acquisition to computer engineering systems. Based on the analysis of certain physical properties of natural crystals and artificial ceramics (which undergo certain treatment), electroelasticity studies the mechanics of bodies subject to the action of coupled electric and mechanical fields.

After the discovery of the piezoeffect by Jack and Pierre Curie and the classical analysis by Voigt [2] the theory of electroelasticity was symmetrically developed (see for example [3–9]). Various methods for the solution of boundary-value problems can be found in [5–7, 10–29]. The fact that the linear state equations for preliminary polarized piezoceramics are analogous to the relations for crystals with hexagonal symmetry allows the formalization of fundamental boundary-value problems within a common framework. The linearization procedure for the electroelasticity equations for piezoceramic media has been introduced by Mason [6]. Experimental data have established the fact that as far as the mechanical and electrical properties are concerned piezoceramics behave like transversely isotropic materials. Over the past years, there have been a large number of publications in the field of static and dynamical theory of electroelasticity. Due to the coupling

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of mechanical and electrical fields with anisotropy, the analysis of dynamical problems is not a simple task. In this paper we demonstrate the use of complex analysis in order to deal with two-dimensional boundary-value problems of electroelasticity. The proposed approach reduces the analysis to the theory of analytical functions. For piezoceramic materials, we express the mechanical and electric field quantities using three analytical functions of complex variables. Such variants of complex representations are suggested by various authors (for a review cf. [20,30,31]). In [32] and making use of the methods appearing in [30], the authors studied the stress concentration on the contour of an elliptic hole on a half-plane under the influence of a point electric charge on its boundary. The boundary-value problem of fracture mechanics for a linear crack located at the boundary between a piezoelectric and a conductive medium is solved in [33]. In the monograph [20], the authors examine fundamental static problems of electroelasticity for a piezoceramic plane and a half-plane. In the same work, the authors have constructed the Green's functions for an infinite plate weakened by a linear crack or a rigid linear inclusion; the problem of averaging the electroelastic properties of piecewise-homogeneous piezoceramic structures using the method of integral equations is also considered.

The solution of three-dimensional (3-D) problems of electroelasticity confronts to certain mathematical difficulties. In the work of Mindlin [34], it was presented a 3-D problem of electroelasticity for a plate with generalized anisotropy. The problem was solved using the variational principle considering both the mechanical displacements and the electric potentials which were expressed as exponential functions of the thickness. An effective method for solving 3-D problems was developed by Lurié [1], who derived for the first time the homogeneous solutions for elastic layers with bases free from stresses. The aim of the present work focuses towards the simplification and generalization of the above mentioned method.

2 Formulation of the problem

Let us consider a Cartesian system of reference and a piezoceramic layer defined by $-\infty < x_1, x_2 < \infty, |x_1| \leq h$. The vector of the electrical field which contributes to the preliminary polarization of the ceramic is directed across the x_3-axis. The complete system of equations, describing the coupled electroelastic layer, reads as follows [3,21]:
Equations of equilibrium:

$$\partial_j \sigma_{ij} = 0, \quad \partial_j = \partial/\partial x_j, \tag{1}$$

Equations of electrostatics:

$$\partial_i D_i = 0, \quad E_i = -\partial_i \varphi, \tag{2}$$

Cauchy relations:

$$\varepsilon_{ij} = (\partial_i u_j + \partial_j u_i)/2 \quad (i, j = 1, 2, 3). \tag{3}$$

The constitutional equations of the preliminary polarized along the Ox_3-axis piezoceramic are given below:

$$\begin{aligned}
&\sigma_{11} = c_{11}^E \varepsilon_{11} + c_{12}^E \varepsilon_{22} + c_{13}^E \varepsilon_{33} - e_{31} E_3, \quad \sigma_{22} = c_{12}^E \varepsilon_{11} + c_{11}^E \varepsilon_{22} + c_{13}^E \varepsilon_{33} - e_{31} E_3,\\
&\sigma_{33} = c_{13}^E (\varepsilon_{11} + \varepsilon_{22}) + c_{33}^E \varepsilon_{33} - e_{33} E_3, \quad \sigma_{12} = (c_{11}^E - c_{12}^E)\varepsilon_{12},\\
&\sigma_{13} = 2c_{44}^E \varepsilon_{13} - e_{15} E_1, \quad \sigma_{23} = 2c_{44}^E \varepsilon_{23} - e_{15} E_2,\\
&D_1 = \varepsilon_{11}^S E_1 + 2e_{15}\varepsilon_{13}, \quad D_2 = \varepsilon_{11}^S E_2 + 2e_{15}\varepsilon_{23},\\
&D_3 = \varepsilon_{33}^S E_3 + e_{31}(\varepsilon_{11} + \varepsilon_{22}) + e_{33}\varepsilon_{33}.
\end{aligned} \tag{4}$$

Here, we deal with the following boundary conditions:

(a) the bases of the layer are built-in and grounded, that is

$$u_1(x_1, x_2, \pm h) = u_2(x_1, x_2, \pm h) = u_3(x_1, x_2, \pm h) = \varphi(x_1, x_2, \pm h) = 0, \tag{5}$$

(b) the bases of the layer are unloaded and grounded, i.e.:

$$\sigma_{13}(x_1, x_2, \pm h) = \sigma_{23}(x_1, x_2, \pm h) = \sigma_{33}(x_1, x_2, \pm h) = \varphi(x_1, x_2, \pm h) = 0. \tag{6}$$

In the relations (1)–(6), $u_i (u_1 = u, u_2 = \upsilon, u_3 = w)$ and σ_{ij}, ε_{ij} are the components of the elastic displacement vector, the stress and strain tensor components, respectively; $c_{ij}^E = c_{ij}$, e_{ij}, $\varepsilon_{ii}^S = \varepsilon_u$ are the elastic constants for a constant electric field, the piezoelectric and the dielectric constants for constant deformation, respectively; E_i, D_i, ϕ are the components of the electric field intensity, electrical displacement and the potential of the electric field, respectively.

The system of equations (1)–(4) takes the form [35]:

$$\begin{aligned}
&V\nabla^2 u + c_{44}\partial_3^2 u + \partial_1\theta = 0, \quad \nabla^2 = \partial_1^2 + \partial_2^2,\\
&V\nabla^2 \upsilon + c_{44}\partial_3^2 \upsilon + \partial_2\theta = 0,\\
&c_{44}\nabla^2 w + c_{33}\partial_3^2 w + c\partial_3(\partial_1 u + \partial_2 \upsilon) + e_{15}\nabla^2\varphi + e_{33}\partial_3^2\varphi = 0,\\
&\varepsilon_{11}\nabla^2\varphi + \varepsilon_{33}\partial_3^2\varphi - e\partial_3(\partial_1 u + \partial_2\upsilon) - e_{15}\nabla^2 w - e_{33}\partial_3^2 w = 0,\\
&\theta = U(\partial_1 u + \partial_2 \upsilon) + c\partial_3 w + e\partial_3\varphi,\\
&U = (c_{11} + c_{12})/2, \quad V = (c_{11} - c_{12})/2, \quad c = c_{13} + c_{44}, \quad e = e_{15} + e_{31}.
\end{aligned} \tag{7}$$

On the basis of the Helmholtz theorem a vector $\{u, \upsilon, w\}$ can be expressed as:

$$u = \partial_1\Phi + \partial_2\Psi, \quad \upsilon = \partial_2\Phi - \partial_1\Psi, \quad w = \Omega. \tag{8}$$

Using (8), the relationships (7), (8) take the form:

$$\begin{aligned}
&\partial_1(V\nabla^2\Phi + c_{44}\partial_3^2\Phi + \theta) + \partial_2(V\nabla^2\Psi + c_{44}\partial_3^2\Psi) = 0,\\
&\partial_2(V\nabla^2\Phi + c_{44}\partial_3^2\Phi + \theta) - \partial_1(V\nabla^2\Psi + c_{44}\partial_3^2\Psi) = 0,\\
&c_{44}\nabla^2\Omega + c_{33}\partial_3^2\Omega + c\nabla^2\partial_3\Phi + e_{15}\nabla^2\varphi + e_{33}\partial_3^2\varphi = 0,\\
&\varepsilon_{11}\nabla^2\varphi + \varepsilon_{33}\partial_3^2\varphi - e\nabla^2\partial_3\Phi - e_{15}\nabla^2\Omega - e_{33}\partial_3^2\Omega = 0,\\
&\theta = U\nabla^2\Phi + c\partial_3\Omega + e\partial_3\varphi.
\end{aligned} \tag{9}$$

Solving the first two equations of (9) with respect to the function Ψ we get:

$$V\nabla^2\Psi + c_{44}\partial_3^2\Psi = 0. \tag{10}$$

From the combination and transformation of (9), we obtain a system of differential equations with respect to the functions Φ, Ω, ϕ reading:

$$\begin{aligned}
&c_{11}\nabla^2\Phi + c_{44}\partial_3^2\Phi + c\partial_3\Omega + e\partial_3\varphi = 0,\\
&c_{44}\nabla^2\Omega + c_{33}\partial_3^2\Omega + c\nabla^2\partial_3\Phi + e_{15}\nabla^2\varphi + e_{33}\partial_3^2\varphi = 0,\\
&\varepsilon_{11}\nabla^2\varphi + \varepsilon_{33}\partial_3^2\varphi - e\nabla^2\partial_3\Phi - e_{15}\nabla^2\Omega - e_{33}\partial_3^2\Omega = 0.
\end{aligned} \tag{11}$$

Using the following symbolism for the differential operators $\partial_3 A = A'$, $\partial_3^2 A = A''$, $\nabla^2 A = \beta^2 A$, the system (10) and (11) takes the form:

$$\begin{aligned}
&V\beta^2\Psi + c_{44}\Psi'' = 0,\\
&c_{11}\beta^2\Phi + c_{44}\Phi'' + c\Omega' + e\varphi' = 0,\\
&c_{44}\beta^2\Omega + c_{33}\Omega'' + c\beta^2\Phi' + e_{15}\beta^2\varphi + e_{33}\varphi'' = 0,\\
&\varepsilon_{11}\beta^2\varphi + \varepsilon_{33}\varphi'' - e\beta^2\Phi' - e_{15}\beta^2\Omega - e_{33}\Omega'' = 0.
\end{aligned} \tag{12}$$

Here, we seek solutions of these equations in the following form:

$$\begin{aligned}
&\Phi = \mathrm{ch}(\lambda x_3)A_1, \quad \Omega = \beta^{-1}\mathrm{sh}(\lambda x_3)A_2, \quad \varphi = \beta^{-1}\mathrm{sh}(\lambda x_3)A_3,\\
&\Psi = \cos(\beta x_3/\mu_3)A_4, \quad \mu_3^2 = c_{44}/V, \quad A_i = A_i(x_1, x_2),
\end{aligned} \tag{13}$$

where λ is an operator to be determined. Hence the three relations in (12) read:

$$\begin{aligned}&\mathrm{ch}(\lambda x_3)\lfloor(c_{11}\beta^2+c_{44}\lambda^2)A_1+c\lambda\beta^{-1}A_2+e\lambda\beta^{-1}A_3\rfloor=0,\\&\beta^{-1}\mathrm{sh}(\lambda x_3)\lfloor c\beta^3\lambda A_1+(c_{44}\beta^2+c_{33}\lambda^2)A_2+(e_{15}\beta^2+e_{33}\lambda^2)A_3\rfloor=0,\\&\beta^{-1}\mathrm{sh}(\lambda x_3)\lfloor -e\beta^3\lambda A_1-(e_{15}\beta^2+e_{33}\lambda^2)A_2+(\varepsilon_{11}\beta^2+\varepsilon_{33}\lambda^2)A_3\rfloor=0.\end{aligned}\tag{14}$$

The determinant of the above system is given by

$$D(\lambda)=\beta^{-2}\mathrm{ch}(\lambda x_3)\mathrm{sh}^2(\lambda x_3)(c_1\lambda^6+c_2\beta^2\lambda^4+c_3\beta^4\lambda^2+c_4\beta^6)$$

where:

$$\begin{aligned}&c_1=c_{44}(c_{33}\varepsilon_{33}+e_{33}^2),\quad c_4=c_{11}(c_{44}\varepsilon_{11}+e_{15}^2),\\&c_2=c_{44}c_{33}\varepsilon_{11}+c_{44}^2\varepsilon_{33}+c_{11}c_{33}\varepsilon_{33}-c^2\varepsilon_{33}+e^2c_{33}-2cee_{33}+2e_{15}e_{33}c_{44}+c_{11}e_{33}^2,\\&c_3=c_{11}c_{44}\varepsilon_{33}+c_{44}^2\varepsilon_{11}+c_{33}c_{11}\varepsilon_{11}-c^2\varepsilon_{11}+e^2c_{44}-2cee_{15}+2e_{33}e_{15}c_{11}+c_{44}e_{15}^2.\end{aligned}$$

At this point, we introduce a new variable $\mu=i\beta\lambda^{-1}$. Under this representation, the relationship $D(\lambda)=0$ takes the following form:

$$c_4\mu^6-c_3\mu^4+c_2\mu^2-c_1=0.$$

The roots of this characteristic equation are $\pm\mu_1,\pm\mu_2,\pm\bar{\mu}_2$ [35].

For the piezoceramic material PZT-4, these take the following values:

$$\mu_1=0.8307013,\quad \mu_2=0.903579+0.1693901i.$$

For this case, the eigenvalues of the system (14) read:

$$\lambda_{1,2}=\pm i\beta/\mu_1,\quad \lambda_{3,4}=\pm i\beta/\mu_2,\quad \lambda_{5,6}=\pm i\beta/\bar{\mu}_2,\quad \lambda_2=\bar{\lambda}_1,\quad \lambda_5=\bar{\lambda}_4,\quad \lambda_6=\bar{\lambda}_3.$$

Consequently, as a total we have six choices for the eigenvectors $A^{(k)}=\{A_1^{(k)},A_2^{(k)},A_3^{(k)}\}$ (for each eigenvalue λ_k),

$$\begin{gathered}\{A_1^{(1)},A_2^{(1)},A_3^{(1)}\},\quad\{\bar{A}_1^{(1)},\bar{A}_2^{(1)},\bar{A}_3^{(1)}\},\\\{A_1^{(2)},A_2^{(2)},A_3^{(2)}\},\quad\{A_1^{(2)},-A_2^{(2)},-A_3^{(2)}\},\quad\{\bar{A}_1^{(2)},-\bar{A}_2^{(2)},-\bar{A}_3^{(2)}\},\quad\{\bar{A}_1^{(2)},\bar{A}_2^{(2)},\bar{A}_3^{(2)}\}\end{gathered}$$

where

$$\begin{aligned}A_2^{(k)}&=y_1^{(k)}(\mu)\beta^2A_1^{(k)},\quad A_3^{(k)}=y_2^{(k)}(\mu)\beta^2A_1^{(k)},\\p(\mu_k)\gamma_1^{(k)}&=-ip_1(\mu_k),\quad p(\mu_k)\gamma_2^{(k)}=ip_2(\mu_k),\\p_1(\mu)&=e_{15}c_{11}\mu^4-(e_{33}c_{11}+e_{15}c_{44}-ec)\mu^2+e_{33}c_{44},\\p_2(\mu)&=c_{11}c_{44}\mu^4-(c_{44}^2+c_{11}c_{33}-c^2)\mu^2+c_{33}c_{44},\\p(\mu)&=(ec_{44}+e_{15}c)\mu^3-(ec_{33}-e_{33}c)\mu.\end{aligned}$$

The calculations for the piezoceramic PZT-4 give:

$$\begin{aligned}&\gamma_1^{(1)}=0.887\,i,\quad \gamma_1^{(2)}=-0.347+0.887\,i,\\&\gamma_2^{(1)}=-5.261\times10^8\,i,\quad \gamma_2^{(2)}=8.007\times10^8+1.278\times10^9\,i.\end{aligned}$$

Using (13), we finally obtain:

$$\begin{aligned}&\Psi=\cos(\beta x_3/\mu_3)A_4,\quad \mu_3^2=c_{44}/V,\\&\Phi=2\cos(\beta x_3/\mu_1)\mathrm{Re}A_1^{(1)}+4\mathrm{Re}\{\cos(\beta x_3/\mu_2)A_1^{(2)}\},\\&\Omega=-2\alpha_{11}\beta\sin(\beta x_3/\mu_1)\mathrm{Re}A_1^{(1)}-4\mathrm{Im}\{\gamma_1^{(2)}\beta\sin(\beta x_3/\mu_2)A_1^{(2)}\},\quad \alpha_{11}=\mathrm{Im}\{\gamma_1^{(1)}\},\\&\varphi=-2\alpha_{21}\beta\sin(\beta x_3/\mu_1)\mathrm{Re}A_1^{(1)}-4\mathrm{Im}\{\gamma_2^{(2)}\beta\sin(\beta x_3/\mu_2)A_1^{(2)}\},\quad \alpha_{21}=\mathrm{Im}\{\gamma_2^{(1)}\}.\end{aligned}\tag{15}$$

Consequently, from (8) and (15) we derive the components of the displacement vector, the electric potential and the dilatation of the layer, respectively, as:

$$\begin{aligned}
u &= 2\cos(\beta x_3/\mu_1)\partial_1 \mathrm{Re}A_1^{(1)} + 4\mathrm{Re}\{\cos(\beta x_3/\mu_2)\partial_1 A_1^{(2)}\} + \cos(\beta x_3/\mu_3)\partial_2 A_4,\\
v &= 2\cos(\beta x_3/\mu_1)\partial_2 \mathrm{Re}A_1^{(1)} + 4\mathrm{Re}\{\cos(\beta x_3/\mu_2)\partial_2 A_1^{(2)}\} - \cos(\beta x_3/\mu_3)\partial_1 A_4,\\
w &= -2\alpha_{11}\beta\sin(\beta x_3/\mu_1)\mathrm{Re}A_1^{(1)} - 4\mathrm{Im}\{\gamma_1^{(2)}\beta\sin(\beta x_3/\mu_2)A_1^{(2)}\},\\
\varphi &= -2\alpha_{21}\beta\sin(\beta x_3/\mu_1)\mathrm{Re}A_1^{(1)} - 4\mathrm{Im}\{\gamma_2^{(2)}\beta\sin(\beta x_3/\mu_2)A_1^{(2)}\},\\
\theta &= -2S_1\beta^2\cos(\beta x_3/\mu_1)\mathrm{Re}A_1^{(1)} - 4\mathrm{Re}\{S_2\beta^2\cos(\beta x_3/\mu_2)A_1^{(2)}\},\\
S_1 &= U - \frac{c\gamma_1^{(1)} + e\gamma_2^{(1)}}{i\mu_1}, \quad S_2 = U - \frac{c\gamma_1^{(2)} + e\gamma_2^{(2)}}{i\mu_2}.
\end{aligned} \tag{16}$$

Taking into account the relationships (16) and the constitutive equations (4) we obtain the following expressions for the mechanical stresses and the vector of the electric induction:

$$\begin{aligned}
\sigma_{11} &= 2(2V\partial_1^2 + C_1\beta^2)\cos(\beta x_3/\mu_1)\mathrm{Re}A_1^{(1)} + 4\mathrm{Re}\{(2V\partial_1^2 + C_2\beta^2)\cos(\beta x_3/\mu_2)A_1^{(2)}\}\\
&\quad + 2V\cos(\beta x_3/\mu_3)\partial_1\partial_2 A_4,\\
\sigma_{22} &= 2(2V\partial_2^2 + C_1\beta^2)\cos(\beta x_3/\mu_1)\mathrm{Re}A_1^{(1)} + 4\mathrm{Re}\{(2V\partial_2^2 + C_2\beta^2)\cos(\beta x_3/\mu_2)A_1^{(2)}\}\\
&\quad - 2V\cos(\beta x_3/\mu_3)\partial_1\partial_2 A_4,\\
\sigma_{12} &= 4V\cos(\beta x_3/\mu_1)\partial_1\partial_2\mathrm{Re}A_1^{(1)} + 8V\mathrm{Re}\{\cos(\beta x_3/\mu_2)\partial_1\partial_2 A_1^{(2)}\} + V\cos(\beta x_3/\mu_3)(\partial_2^2 - \partial_1^2)A_4,\\
\sigma_{33} &= 2C_3\beta^2\cos(\beta x_3/\mu_1)\mathrm{Re}A_1^{(1)} + 4\mathrm{Re}\{C_4\beta^2\cos(\beta x_3/\mu_2)A_1^{(2)}\},\\
\sigma_{13} &= -2C_5\beta\sin(\beta x_3/\mu_1)\partial_1\mathrm{Re}A_1^{(1)} - 4\mathrm{Re}\{C_6\beta\sin(\beta x_3/\mu_2)\partial_1 A_1^{(2)}\} - c_{44}\mu_3^{-1}\beta\sin(\beta x_3/\mu_3)\partial_2 A_4,\\
\sigma_{23} &= -2C_5\beta\sin(\beta x_3/\mu_1)\partial_2\mathrm{Re}A_1^{(1)} - 4\mathrm{Re}\{C_6\beta\sin(\beta x_3/\mu_2)\partial_2 A_1^{(2)}\} + c_{44}\mu_3^{-1}\beta\sin(\beta x_3/\mu_3)\partial_1 A_4,\\
D_1 &= -2F_1\beta\sin(\beta x_3/\mu_1)\partial_1\mathrm{Re}A_1^{(1)} - 4\mathrm{Re}\{F_2\beta\sin(\beta x_3/\mu_2)\partial_1 A_1^{(2)}\} - e_{15}\mu_3^{-1}\beta\sin(\beta x_3/\mu_3)\partial_2 A_4,\\
D_2 &= -2F_1\beta\sin(\beta x_3/\mu_1)\partial_2\mathrm{Re}A_1^{(1)} - 4\mathrm{Re}\{F_2\beta\sin(\beta x_3/\mu_2)\partial_2 A_1^{(2)}\} + e_{15}\mu_3^{-1}\beta\sin(\beta x_3/\mu_3)\partial_1 A_4,\\
D_3 &= 2F_3\beta^2\cos(\beta x_3/\mu_1)\mathrm{Re}A_1^{(1)} + 4\mathrm{Re}\{F_4\beta^2\cos(\beta x_3/\mu_2)A_1^{(2)}\},\\
C_1 &= c_{12} - \frac{c_{13}\gamma_1^{(1)} + e_{31}\gamma_2^{(1)}}{i\mu_1}, \quad C_2 = c_{12} - \frac{c_{13}\gamma_1^{(2)} + e_{31}\gamma_2^{(2)}}{i\mu_2}, \quad C_3 = c_{13} - \frac{c_{33}\gamma_1^{(1)} + e_{33}\gamma_2^{(1)}}{i\mu_1},\\
C_4 &= c_{13} - \frac{c_{33}\gamma_1^{(2)} + e_{33}\gamma_2^{(2)}}{i\mu_2}, \quad C_5 = \frac{c_{44}}{\mu_1} + \frac{c_{44}\gamma_1^{(1)} + e_{15}\gamma_2^{(1)}}{i}, \quad C_6 = \frac{c_{44}}{\mu_2} + \frac{c_{44}\gamma_1^{(2)} + e_{15}\gamma_2^{(2)}}{i},\\
F_1 &= \frac{e_{15}}{\mu_1} + \frac{e_{15}\gamma_1^{(1)} - \varepsilon_{11}\gamma_2^{(1)}}{i}, \quad F_2 = \frac{e_{15}}{\mu_2} + \frac{e_{15}\gamma_1^{(2)} - \varepsilon_{11}\gamma_2^{(2)}}{i},\\
F_3 &= e_{31} - \frac{e_{33}\gamma_1^{(1)} - \varepsilon_{33}\gamma_2^{(1)}}{i\mu_1}, \quad F_4 = e_{31} - \frac{e_{33}\gamma_1^{(2)} - \varepsilon_{33}\gamma_2^{(2)}}{i\mu_2}.
\end{aligned} \tag{17}$$

3 The case of a piezoceramic layer with a built-in base

In this section, we proceed to the derivation of the homogeneous solutions for the system (7), when its bases are built-in and are covered by thin grounded electrodes. Using (16), the boundary conditions (15) lead to the following system of differential equations:

$$\begin{aligned}
&\partial_1\{2\cos(\beta h/\mu_1)\mathrm{Re}A_1^{(1)} + 4\mathrm{Re}\lfloor\cos(\beta h/\mu_2)A_1^{(2)}\rfloor\} + \partial_2\{\cos(\beta h/\mu_3)A_4\} = 0,\\
&\partial_2\{2\cos(\beta h/\mu_1)\mathrm{Re}A_1^{(1)} + 4\mathrm{Re}\lfloor\cos(\beta h/\mu_2)A_1^{(2)}\rfloor\} - \partial_1\{\cos(\beta h/\mu_3)A_4\} = 0,\\
&2\alpha_{11}\beta\sin(\beta h/\mu_1)\mathrm{Re}A_1^{(1)} + 4\mathrm{Im}\{\gamma_1^{(2)}\beta\sin(\beta h/\mu_2)A_1^{(2)}\} = 0,\\
&2\alpha_{21}\beta\sin(\beta h/\mu_1)\mathrm{Re}A_1^{(1)} + 4\mathrm{Im}\{\gamma_2^{(2)}\beta\sin(\beta h/\mu_2)A_1^{(2)}\} = 0.
\end{aligned} \tag{18}$$

The first two equations of the system (18) are the Cauchy–Riemann conditions [36] which guarantee the analyticity of the function $f(z)$ $(z = x_1 + ix_2)$. Therefore, we have the following relations:

$$\cos(\beta h/u_3)A_4 = \mathrm{Re} f(z),$$
$$2\cos(\beta h/\mu_1)\mathrm{Re}A_1^{(1)} + 4\mathrm{Re}\{\cos(\beta h/\mu_2)A_1^{(2)}\} = \mathrm{Im} f(z).$$

Consequently, we obtain the following system of equations with respect to A_4, $\mathrm{Re}A_1^{(1)}$ and $A_1^{(2)} = \mathrm{Re}A_1^{(2)} + i\mathrm{Im}A_1^{(2)}$:

$$\begin{aligned} &\cos(\beta h/\mu_3)A_4 = \mathrm{Re} f(z), \\ &l_{11}\mathrm{Re}A_1^{(1)} + l_{12}\mathrm{Re}A_1^{(2)} + l_{13}\mathrm{Im}A_1^{(2)} = \mathrm{Im} f(z), \end{aligned} \tag{19}$$

$$\begin{aligned} &l_{21}\mathrm{Re}A_1^{(1)} + l_{22}\mathrm{Re}A_1^{(2)} + l_{23}\mathrm{Im}A_1^{(2)} = 0, \\ &l_{31}\mathrm{Re}A_1^{(1)} + l_{32}\mathrm{Re}A_1^{(2)} + l_{33}\mathrm{Im}A_1^{(2)} = 0, \end{aligned} \tag{20}$$

where:

$$\begin{aligned} &l_{11} = 2\cos(\beta h/\mu_1), \quad l_{12} - il_{13} = 4\cos(\beta h/\mu_2), \\ &l_{21} = h^{-1}\beta\sin(\beta h/\mu_1), \quad l_{22} - il_{23} = 2h^{-1}(\gamma_1^{(2)}/\gamma_1^{(1)})\beta\sin(\beta h/\mu_2), \\ &l_{31} = h^{-1}\beta\sin(\beta h/\mu_1), \quad l_{32} - il_{33} = 2h^{-1}(\gamma_2^{(2)}/\gamma_2^{(1)})\beta\sin(\beta h/\mu_2). \end{aligned}$$

The solution of (20) is given by:

$$A_4 = \sum_{n=1}^{\infty} \psi_n(x_1, x_2) + \mathrm{Re}\, f(z)$$

where ψ_n are the metaharmonic functions which satisfy the Helmholtz equation:

$$(\nabla^2 - \delta_n^2)\psi_n = 0, \quad 2h\delta_n = \pi(2n-1)\mu_3, \quad (n = 1, 2, \ldots).$$

By proceeding to the integration of the system (20) we get:

$$\mathrm{Re}A_1^{(1)} = L_{11}\Omega, \quad \mathrm{Re}A_1^{(2)} = L_{12}\Omega, \quad \mathrm{Im}A_1^{(2)} = L_{13}\Omega, \tag{21}$$

where L_{11}, L_{12}, L_{13} are the algebraic complements of the elements $\ell_{11}, \ell_{12}, \ell_{13}$, respectively.

The last equations of the system are satisfied as identities, whereas the first one takes the form:

$$D(\beta^2)\Omega = \mathrm{Im} f(z), \tag{22}$$

where $D(\beta^2)$ is the determinant of the system.

By introducing the function $H = \beta^4\Omega$, and after some transformations, the relationships (21) take the form:

$$\begin{aligned} \mathrm{Re}A_1^{(1)} &= 4h^{-2}a\beta^{-2}\sin(\beta h/\mu_2)\sin(\beta h/\mu_2)\mathrm{H}, \\ A_1^{(2)} &= 2ih^{-2}\bar{b}\beta^{-2}\sin(\beta h/\mu_1)\sin(\beta h/\mu_2)\mathrm{H}, \\ a &= \mathrm{Im}\left\{\frac{\gamma_1^{(2)}\bar{\gamma}_2^{(2)}}{\gamma_1^{(1)}\bar{\gamma}_2^{(1)}}\right\}, \quad b = \left[\frac{\gamma_2^{(2)}\gamma_1^{(2)}}{\gamma_2^{(1)}\gamma_1^{(1)}}\right], \end{aligned} \tag{23}$$

and (22) yields:

$$L(\beta^2)\mathrm{H} = \mathrm{Im}\, f(z), \quad L(\beta^2) = \beta^{-4}D(\beta^2). \tag{24}$$

The operator $L(\beta^2)$ is an integer function of $\beta^2 = \nabla^2$. Consequently, the solution of the homogeneous equation $L(\beta^2)\mathrm{H} = 0$ can be constructed by combining the metaharmonic functions. More specifically, if

Table 1 The values of the real roots of (2.8) for the ceramic PZT-4 for $h = 1$

i	α_i	$\alpha_{i+1} - \alpha_i$
1	0.71464690	2.19985797
2	2.91450487	2.76452159
3	5.67902646	2.66309667
4	8.34212313	2.6203834
5	10.96250653	2.60851654
6	13.57102307	2.60765009
7	16.17867316	2.60875362
8	18.78742678	2.60946094
9	21.39688772	2.60970344
10	24.00659116	2.60974607
11	26.61633723	2.60973903
12	29.22607626	2.60972998
13	31.83580624	2.60972601
14	34.44553225	2.609725
15	37.05525725	2.60972494
16	39.66498219	2.60972503
17	42.27470722	2.6097251
18	44.88443232	2.60972511
19	47.49415743	

Table 2 The values of the complex roots of (2.8) for the ceramic PZT-4 for $h = 1$

i	$\text{Re}\alpha_i$	$\text{Im}\alpha_i$	$\text{Re}\alpha_{i+1} - \text{Re}\alpha_i$	$\text{Im}\alpha_{i+1} - \text{Im}\alpha_i$
1	3.24090725	1.11861773	2.79030712	0.5524113
2	6.03121437	1.67102903	2.82018529	0.52901542
3	8.85139966	2.20004445	2.83404817	0.52882108
4	11.68544783	2.72886553	2.83799078	0.53070892
5	14.52343861	3.25957445	2.83871978	0.53171564
6	17.36215839	3.79129009	2.83875095	0.53205758
7	20.20090934	4.32334767	2.83870879	0.53214261
8	23.03961813	4.85549028	2.83868645	0.53215652
9	25.87830458	5.38764680	2.83867914	0.53215656
10	28.71698372	5.91980336	2.83867741	0.53215551
11	31.55566113	6.45195887	2.83867714	0.532155
12	34.39433827	6.98411387	2.83867716	0.53215485
13	37.23301543	7.51626872	2.83867719	0.53215481
14	40.07169262	8.04842353	2.83867719	0.53215481
15	42.91036981	8.58057834	2.8386772	0.53215481
16	45.74904701	9.11273315		

$(\nabla^2 - a^2)\mu = 0$ the homogeneous equation (24) takes the form of $L(a^2)\text{H} = 0$. Therefore, the problem is reduced to the determination of the roots of the complex equation:

$$\begin{aligned} & L(a^2) = 0, \\ & L(\alpha^2) = 4(h\alpha)^{-2}\{2a\cos(\alpha h/\mu_1)\sin(\alpha h/\mu_2)\sin(\alpha h/\bar{\mu}_2) + i\bar{b}\sin(\alpha h/\mu_1)\cos(\alpha h/\mu_2)\sin(\alpha h/\bar{\mu}_2) \\ & \qquad - ib\sin(\alpha h/\mu_1)\sin(\alpha h/\mu_2)\cos(\alpha h/\bar{\mu}_2)\}. \end{aligned} \tag{25}$$

For the most known materials (PZT-4, PXE-5) there are general particularities regarding the spectrum distribution $\{a_i\}$ of the characteristic equation (25): the spectrum is discrete and symmetrically placed on the complex plane, while its accumulation point is at infinity. There are three asymptotic distributions of $\{a_i\}$, one of which is the real axis, whereas the other two are the lines $\arg(a) = \pm v,\ (v \neq 0)$; the angle v is determined from the electrostatic properties of the material. The magnitudes of the characteristic values $\{a_i\}$ are given in Tables 1 and 2.

For $i > 19$, the real values of a_i can be calculated using the recursive formula:

$$\alpha_{i+1} - \alpha_i = \pi h^{-1}\mu_1 \approx 2.609725115.$$

For $i > 16$, the complex values of a_i can be calculated using the recursive formula:

$$\mathrm{Re}\alpha_{i+1} - \mathrm{Re}\alpha_i = \pi h^{-1}\mathrm{Re}\mu_2 \approx 2.8386772,$$
$$\mathrm{Im}\alpha_{i+1} - \mathrm{Im}\alpha_i = \pi h^{-1}\mathrm{Im}\mu_2 \approx 0.53215481.$$

The solution of (24) reads:

$$\mathrm{H} = 2\mathrm{Re}\sum_n \varphi_n + A\mathrm{Im}\, f(z), \quad A = \frac{\mu_1\mu_2\bar{\mu}_2}{8\{a\mu_1 + \mathrm{Im}(b\bar{\mu}_2)\}}, \tag{26}$$

where ϕ_n are metaharmonic functions that satisfy the Helmholtz equation $(\nabla^2 - a_n^2)\Phi_n = 0$, with a_n being the roots of the characteristic equation (25).

From (23), the relation (26) yields:

$$\begin{aligned}
\mathrm{Re}A_1^{(1)} &= 4ah^{-2}\sum_n (\alpha_n^{-2}\sin(\alpha_n h/\mu_2)\sin(\alpha_n h/\bar{\mu}_2)\varphi_n + \bar{\alpha}_n^{-2}\sin(\bar{\alpha}_n h/\mu_2)\sin(\bar{\alpha}_n h/\bar{\mu}_2)\bar{\varphi}_n)\\
&\quad + 4Aa(\mu_2\bar{\mu}_2)^{-1}\mathrm{Im}\, f(z),\\
A_1^{(2)} &= 2i\bar{b}h^{-2}\sum_n (\alpha_n^{-2}\sin(\alpha_n h/\mu_1)\sin(\alpha_n h/\bar{\mu}_2)\varphi_n + \bar{\alpha}_n^{-2}\sin(\bar{\alpha}_n h/\mu_1)\sin(\bar{\alpha}_n h/\bar{\mu}_2)\bar{\varphi}_n)\\
&\quad + 2iA\bar{b}(\mu_1\bar{\mu}_2)^{-1}\mathrm{Im}\, f(z).
\end{aligned}$$

After all these procedures, we can now determine the homogeneous solutions of the problem; the components of the complex electroelastic field in the layer read:

$$\begin{aligned}
u &= h^{-2}\mathrm{Re}\sum_n \alpha_n^{-2}\{a_{\mathrm{ln}}\cos(\alpha_n x_3/\mu_1) + a_{2n}\cos(\alpha_n x_3/\mu_2) - a_{3n}\cos(\alpha_n x_3/\bar{\mu}_2)\}\partial_1\varphi_n\\
&\quad + \sum_{n=1}^{\infty}\cos(\delta_n x_3/\mu_3)\partial_2\psi_n,\\
\upsilon &= h^{-2}\mathrm{Re}\sum_n \alpha_n^{-2}\{a_{\mathrm{ln}}\cos(\alpha_n x_3/\mu_1) + a_{2n}\cos(\alpha_n x_3/\mu_2) - a_{3n}\cos(\alpha_n x_3/\bar{\mu}_2)\}\partial_2\varphi_n\\
&\quad - \sum_{n=1}^{\infty}\cos(\delta_n x_3/\mu_3)\partial_1\psi_n,\\
w &= -h^{-2}\mathrm{Im}\sum_n \alpha_n^{-1}\{\gamma_1^{(1)}a_{1n}\sin(\alpha_n x_3/\mu_1) + \gamma_1^{(2)}a_{2n}\sin(\alpha_n x_3/\mu_2) + \bar{\gamma}_1^{(2)}a_{3n}\sin(\alpha_n x_3/\bar{\mu}_2)\}\varphi_n,\\
\sigma_{13} &= -h^{-2}\mathrm{Re}\sum_n \alpha_n^{-1}\{C_5 a_{\mathrm{ln}}\sin(\alpha_n x_3/\mu_1) + C_6 a_{2n}\sin(\alpha_n x_3/\mu_2) - \bar{C}_6 a_{3n}\sin(\alpha_n x_3/\bar{\mu}_2)\}\partial_2\varphi_n\\
&\quad - c_{44}\mu_3^{-1}\sum_{n=1}^{\infty}\delta_n\sin(\delta_n x_3/\mu_3)\partial_1\psi_n,\\
\sigma_{23} &= -h^{-2}\mathrm{Re}\sum_n \alpha_n^{-1}\{C_5 a_{\mathrm{ln}}\sin(\alpha_n x_3/\mu_1) + C_6 a_{2n}\sin(\alpha_n x_3/\mu_2) - \bar{C}_6 a_{3n}\sin(\alpha_n x_3/\bar{\mu}_2)\}\partial_2\varphi_n\\
&\quad + C_{44}\mu_3^{-1}\sum_{n=1}^{\infty}\delta_n\sin(\delta_n x_3/\mu_3)\partial_1\psi_n,\\
\sigma_{33} &= h^{-2}\mathrm{Re}\sum_n \{C_3 a_{\mathrm{ln}}\cos(\alpha_n x_3/\mu_1) + C_4 a_{2n}\cos(\alpha_n x_3/\mu_2) - \bar{C}_4 a_{3n}\cos(\alpha_n x_3/\bar{\mu}_2)\}\varphi_n,\\
\varphi &= -h^{-2}\mathrm{Im}\sum_n \alpha_n^{-1}\{\gamma_2^{(1)}a_{\mathrm{ln}}\sin(\alpha_n x_3/\mu_1) + \gamma_2^{(2)}a_{2n}\sin(\alpha_n x_3/\mu_2) + \bar{\gamma}_2^{(2)}a_{3n}\sin(\alpha_n x_3/\bar{\mu}_2)\}\varphi_n,\\
a_{\mathrm{ln}} &= 16a\sin(\alpha_n h/\mu_2)\sin(\alpha_n h/\bar{\mu}_2), \quad a_{2n} = 8i\bar{b}\sin(\alpha_n h/\mu_1)\sin(\alpha_n h/\bar{\mu}_2),\\
a_{3n} &= 8ib\sin(\alpha_n h/\mu_1)\sin(\alpha_n h/\mu_2),\\
&\quad (\nabla^2 - \delta_n^2)\psi_n = 0, \quad 2h\delta_n = \pi(2n-1)\mu_3, \quad (\nabla^2 - \delta_n^2)\varphi_n = 0.
\end{aligned}$$

The metaharmonic functions ϕ_n which appear in the above solution determine the potential solution, whereas the metaharmonic function ψ_n determine the vortex solution and a_n are the roots of (25). As can be seen from the structure of the homogeneous solution, for this particular case of the boundary conditions the biharmonic solution is absent.

4 The case of piezoceramic layers with unloaded bases

In this section, we proceed to the derivation of the homogeneous solution for the system (7) for piezoceramic layers, the bases of which are unloaded and are covered by thin grounded electrodes. Using (16) and (17) the boundary conditions (6) reduce to the following system of differential equations:

$$\begin{aligned}
&\partial_1\{2C_5\beta\sin(\beta h/\mu_1)\mathrm{Re}A_1^{(1)} + 4\mathrm{Re}\lfloor C_6\beta\sin(\beta h/\mu_2)A_1^{(2)}\rfloor\} + \partial_2\{c_{44}\mu_3^{-1}\beta\sin(\beta h/\mu_3)A_4\} = 0,\\
&\partial_2\{2C_5\beta\sin(\beta h/\mu_1)\mathrm{Re}A_1^{(1)} + 4\mathrm{Re}\lfloor C_6\beta\sin(\beta h/\mu_2)A_1^{(2)}\rfloor\} - \partial_1\{c_{44}\mu_3^{-1}\beta\sin(\beta h/\mu_3)A_4\} = 0,\\
&2C_3\beta^2\cos(\beta h/\mu_1)\mathrm{Re}A_1^{(1)} + 4\mathrm{Re}\{C_4\beta^2\cos(\beta h/\mu_2)A_1^{(2)}\} = 0,\\
&2\alpha_{21}\beta\sin(\beta h/\mu_1)\mathrm{Re}A_1^{(1)} + 4\mathrm{Im}\{\gamma_2^{(2)}\beta\sin(\beta h/\mu_2)A_1^{(2)}\} = 0.
\end{aligned} \tag{27}$$

Thus, as in the previous case the first two equations of the system (27) represent the Cauchy–Riemann condition [36] which guarantees the analyticity of the function $hC_5 f''(z)$ with complex variable $z = x_1 + ix_2$. Consequently, we obtain the following relations:

$$c_{44}\mu_3^{-1}\beta\sin(\beta h/\mu_3)A_4 = hC_5\mathrm{Re}f''(z),$$
$$2C_5\beta\sin(\beta h/\mu_1)\mathrm{Re}A_1^{(1)} + 4\mathrm{Re}\{C_6\beta\sin(\beta h/\mu_2)A_1^{(2)}\} = hC_5\mathrm{Im}f''(z).$$

At this point we introduce the following two biharmonic functions:

$$\Phi_0 = \frac{1}{4}\mathrm{Im}\{\bar{z}f'(z) + f_1(z)\}, \quad \Psi_0 = \frac{1}{4}\mathrm{Re}\{\bar{z}f'(z) + f_2(z)\}$$

where $f_1(z)$, $f_2(z)$ are arbitrary analytic functions. It is obvious that the following relationships are valid:

$$\nabla^2\Phi_0 = \mathrm{Im}f''(z), \quad \nabla^2\Psi_0 = \mathrm{Re}f''(z),$$

on the basis of which we get the following differential equation with respect to A_4:

$$c_{44}\mu_3^{-1}\beta\sin(\beta h/\mu_3)A_4 = hC_5\nabla^2\Psi_0, \tag{28}$$

as well as a system of equations with respect to the functions $\mathrm{Re}A_1^{(1)}$ and $A_1^{(2)} = \mathrm{Re}A_1^{(2)} + i\mathrm{Im}A_1^{(2)}$. These read as follows:

$$\begin{aligned}
&l_{11}\mathrm{Re}A_1^{(1)} + l_{12}\mathrm{Re}A_1^{(2)} + l_{13}\mathrm{Im}A_1^{(2)} = \nabla^2\Phi_0,\\
&l_{21}\mathrm{Re}A_1^{(1)} + l_{22}\mathrm{Re}A_1^{(2)} + l_{23}\mathrm{Im}A_1^{(2)} = 0,\\
&l_{31}\mathrm{Re}A_1^{(1)} + l_{32}\mathrm{Re}A_1^{(2)} + l_{33}\mathrm{Im}A_1^{(2)} = 0,\\
&l_{11} = 2h^{-1}\beta\sin(\beta h/\mu_1), \quad l_{12} - il_{13} = 4h^{-1}(C_6/C_5)\beta\sin(\beta h/\mu_2),\\
&l_{21} = \beta^2\cos(\beta h/\mu_1), \quad l_{22} - il_{23} = 2(C_4/C_3)\beta^2\cos(\beta h/\mu_2),\\
&l_{31} = h^{-1}\beta\sin(\beta h/\mu_1), \quad l_{32} - il_{33} = 2h^{-1}(\gamma_2^{(2)}/\gamma_2^{(1)})\beta\sin(\beta h/\mu_2).
\end{aligned} \tag{29}$$

The solution of the Eq. (28) has the following form:

$$A_4 = \sum_{n=1}^{\infty}\delta_n^{-2}\psi_n(x_1, x_2) + C_5V^{-1}\Psi_0$$

where ψ_n are metaharmonic functions satisfying the Helmholtz equation $(\nabla^2 - \delta_n^2)\psi_n = 0$, $h\delta_n = \pi n\mu_3$, $(n = 1, 2, \ldots)$.

Table 3 The values of the real roots of (3.8) for the ceramic PZT-4, when h = 1

	α_i	$\alpha_{i+1} - \alpha_i$
1	2.77412489	2.69187262
2	5.46599751	2.62695527
3	8.09295278	2.6060386
4	10.69899138	2.60519743
5	13.30418881	2.60771139
6	15.91190020	2.60919939
7	18.52109959	2.60969099
8	21.13079058	2.60977159
9	23.74056217	2.60975423
10	26.35031640	2.60973499
11	28.96005139	2.60972681
12	31.56977820	2.60972481
13	34.17950301	2.60972474
14	36.78922775	2.60972495
15	39.39895270	2.60972508
16	42.00867778	2.60972511
17	44.61840289	

Next, we proceed to the integration of the system (29). By introducing the function Ω by

$$\mathrm{Re}A_1^{(1)} = L_{11}\Omega, \quad \mathrm{Re}A_1^{(2)} = L_{12}\Omega, \quad \mathrm{Im}A_1^{(2)} = L_{13}\Omega, \tag{30}$$

where L_{11}, L_{12}, L_{13} are algebraic complements of the elements l_{11}, l_{12}, l_{13}, we reduce the first equation of the system to the form

$$D(\beta^2)\Omega = \nabla^2\Phi_0, \tag{31}$$

where $D(\beta^2)$ is the determinant of the system.

If now we set $\mathrm{H} = \beta^4\Omega$, after some transformations, the expressions (30) read:

$$\begin{aligned} &\mathrm{Re}A_1^{(1)} = 4h^{-1}\mathrm{Im}\{a\bar{b}\beta^{-1}\cos(\beta h/\mu_2)\sin(\beta h/\bar{\mu}_2)\}\mathrm{H}, \quad aC_3 = C_4, \quad \gamma_2^{(1)}b = \gamma_2^{(2)}, \\ &A_1^{(2)} = 2ih^{-1}(\bar{b}\beta^{-1}\cos(\beta h/\mu_1)\sin(\beta h/\bar{\mu}_2) - \bar{a}\beta^{-1}\sin(\beta h/\mu_1)\cos(\beta h/\bar{\mu}_2))\mathrm{H}. \end{aligned} \tag{32}$$

Equation (31) takes the form

$$L(\beta^2)\mathrm{H} = \nabla^2\Phi_0 \quad L(\beta^2) = \beta^{-4}D(\beta^2). \tag{33}$$

As in the previous case the operand $L(\beta^2)$ is an integer function of $\beta^2 = \nabla^2$ and thus the solution of the homogeneous equation $L(\beta^2)\mathrm{H} = 0$ can be expressed as a combination of the metaharmonic functions. Namely, the problem is reduced to the determination of the roots of the complex transcendent equation

$$\begin{aligned} &L(\alpha^2) = 0, \\ &L(\alpha^2) = 4h^{-2}\{2f\cos(\alpha h/\mu_1)\sin(\alpha h/\mu_2)\sin(\alpha h/\bar{\mu}_2) + id\sin(\alpha h/\mu_1)\cos(\alpha h/\mu_2)\sin(\alpha h/\bar{\mu}_2) \\ &\qquad -i\bar{d}\sin(\alpha h/\mu_1)\sin(\alpha h/\mu_2)\cos(\alpha h/\bar{\mu}_2)\}. \end{aligned} \tag{34}$$

The distribution of the spectrum $\{\alpha_i\}$ of the characteristic equation (34) has the same properties (characteristics) as the distribution of the spectrum of the characteristic equation (25) for the first type of the boundary conditions. The characteristic values $\{\alpha_i\}$ are given in Tables 3 and 4.

The recursive formulae for $\{\alpha_i\}$ have the same form as for the case examined in the previous section.

The resulting distribution of the spectrum $\{\alpha_i\}$ is in agreement with the well known results derived for analogous boundary conditions [1].

Now, returning to Eq. (33) we express its solution in the form:

$$\mathrm{H} = 2\mathrm{Re}\sum_n \alpha_n^{-2}\varphi_n + A\Phi_0, \quad A = \frac{\mu_1\mu_2\bar{\mu}_2}{8\{f\mu_1 - \mathrm{Im}(d\mu_2)\}},$$

where φ_n are metaharmonic functions satisfying the Helmholtz equation $(\nabla^2 - \alpha_n^2)\varphi_n = 0$, with α_n being the roots of the characteristic equation (34).

Table 4 The values of the complex roots of (3.8) for the ceramic PZT-4, when h = 1

	$\mathrm{Re}\alpha_i$	$\mathrm{Im}\alpha_i$	$\mathrm{Re}\alpha_{i+1} - \mathrm{Re}\alpha_i$	$\mathrm{Im}\alpha_{i+1} - \mathrm{Im}\alpha_i$
1	1.85450478	1.03764675	2.85632368	0.57868631
2	4.71082846	1.61633306	2.82988502	0.52074007
3	7.54071348	2.13707313	2.83654894	0.52610355
4	10.37726242	2.66317668	2.83881525	0.53029076
5	13.21607767	3.19346744	2.83895429	0.53173438
6	16.05503196	3.72520182	2.83880135	0.53209699
7	18.89383331	4.25729881	2.83871478	0.53215942
8	21.73254809	4.78945823	2.83868541	0.5321614
9	24.57123350	5.32161963	2.83867818	0.53215757
10	27.40991168	5.85377720	2.83867703	0.5321556
11	30.24858871	6.38593280	2.83867705	0.53215498
12	33.08726576	6.91808778	2.83867714	0.53215482
13	35.92594290	7.45024260	2.83867718	0.53215481
14	38.76462008	7.98239741	2.8386772	0.5321548
15	41.60329728	8.51455221	2.8386772	0.53215481
16	44.44197448	9.04670702		

The relationships (32) take the following form:

$$\begin{aligned}
\mathrm{Re}A_1^{(1)} &= 2ih^{-1}\sum_n \alpha_n^{-3}(\bar{a}b\sin(\alpha_n h/\mu_2)\cos(\alpha_n h/\bar{\mu}_2) - a\bar{b}\cos(\alpha_n h/\mu_2)\sin(\alpha_n h/\bar{\mu}_2))\varphi_n \\
&+ 2ih^{-1}\sum_n \bar{\alpha}_n^{-3}(\bar{a}b\sin(\bar{\alpha}_n h/\mu_2)\cos(\bar{\alpha}_n h/\bar{\mu}_2) - a\bar{b}\cos(\bar{\alpha}_n h/\mu_2)\sin(\bar{\alpha}_n h/\bar{\mu}_2))\bar{\varphi}_n \\
&+ 4A\mathrm{Im}\left\{\frac{a\bar{b}}{\bar{\mu}_2}\right\}\Phi_0 - 2Ah^2\mathrm{Im}\left\{\frac{a\bar{b}}{\bar{\mu}_2}\left(\frac{1}{\mu_2^2}+\frac{1}{3\bar{\mu}_2^2}\right)\right\}\nabla^2\Phi_0, \\
A_1^{(2)} &= 2ih^{-1}\sum_n \alpha_n^{-3}(\bar{b}\cos(\alpha_n h/\mu_1)\sin(\alpha_n h/\bar{\mu}_2) - \bar{a}\sin(\alpha_n h/\mu_1)\cos(\alpha_n h/\bar{\mu}_2))\varphi_n \\
&+ 2ih^{-1}\sum_n \bar{\alpha}_n^{-3}(\bar{b}\cos(\bar{\alpha}_n h/\mu_1)\sin(\bar{\alpha}_n h/\bar{\mu}_2) - \bar{a}\sin(\bar{\alpha}_n h/\mu_1)\cos(\bar{\alpha}_n h/\bar{\mu}_2))\bar{\varphi}_n \\
&+ 2iA\left(\frac{\bar{b}}{\bar{\mu}_2}-\frac{\bar{a}}{\mu_1}\right)\Phi_0 - iAh^2\left(\frac{\bar{b}}{\bar{\mu}_2}\left(\frac{1}{\mu_1^2}+\frac{1}{3\bar{\mu}_2^2}\right)-\frac{\bar{a}}{\mu_1}\left(\frac{1}{3\mu_1^2}+\frac{1}{\bar{\mu}_2^2}\right)\right)\nabla^2\Phi_0.
\end{aligned}$$

After this and some transformations, we can find the homogeneous solutions that we seek. The vortex part of the solution reads:

$$\begin{aligned}
u &= \sum_{n=1}^{\infty}\delta_n^{-2}\cos(\delta_n x_3/\mu_3)\partial_2\psi_n + C_5V^{-1}\partial_2\Psi_0 - C_5(2c_{44})^{-1}x_3^2\nabla^2\partial_2\Psi_0, \\
v &= -\sum_{n=1}^{\infty}\delta_n^{-2}\cos(\delta_n x_3/\mu_3)\partial_1\psi_n - C_5V^{-1}\partial_1\Psi_0 + C_5(2c_{44})^{-1}x_3^2\nabla^2\partial_1\Psi_0, \\
w &= 0, \quad \varphi = 0.
\end{aligned}$$

The potential part of the solution is given by

$$\begin{aligned}
u &= h^{-1}\mathrm{Im}\sum_n \alpha_n^{-3}\{a_{1n}\cos(\alpha_n x_3/\mu_1) - a_{2n}\cos(\alpha_n x_3/\mu_2) + a_{3n}\cos(\alpha_n x_3/\bar{\mu}_2)\}\partial_1\varphi_n \\
&\quad -B_1\partial_1\Phi_0 + B_2x_3^2\nabla^2\partial_1\Phi_0 + B_3h^2\nabla^2\partial_1\Phi_0, \\
v &= h^{-1}\mathrm{Im}\sum_n \alpha_n^{-3}\{a_{1n}\cos(\alpha_n x_3/\mu_1) - a_{2n}\cos(\alpha_n x_3/\mu_2) + a_{3n}\cos(\alpha_n x_3/\bar{\mu}_2)\}\partial_2\varphi_n \\
&\quad -B_1\partial_2\Phi_0 + B_2x_3^2\nabla^2\partial_2\Phi_0 + B_3h^2\nabla^2\partial_2\Phi_0,
\end{aligned}$$

$$w = h^{-1}\mathrm{Re}\sum_n \alpha_n^{-2}\{\gamma_1^{(1)}a_{1n}\sin(\alpha_n x_3/\mu_1) - \gamma_1^{(2)}a_{2n}\sin(\alpha_n x_3/\mu_2) - \bar{\gamma}_1^{(2)}a_{3n}\sin(\alpha_n x_3/\bar{\mu}_2)\}\varphi_n$$
$$+B_4\alpha_{11}x_3\nabla^2\Phi_0,$$
$$\varphi = h^{-1}\mathrm{Re}\sum_n \alpha_n^{-2}\{\gamma_2^{(1)}a_{1n}\sin(\alpha_n x_3/\mu_1) - \gamma_2^{(2)}a_{2n}\sin(\alpha_n x_3/\mu_2) - \bar{\gamma}_2^{(2)}a_{3n}\sin(\alpha_n x_3/\bar{\mu}_2)\}\varphi_n,$$

$$a_{1n} = 8[a\bar{b}\cos(\alpha_n h/\mu_2)\sin(\alpha_n h/\bar{\mu}_2) - \bar{a}b\sin(\alpha_n h/\mu_2)\cos(\alpha_n h/\bar{\mu}_2)],$$
$$a_{2n} = 8[\bar{b}\cos(\alpha_n h/\mu_1)\sin(\alpha_n h/\bar{\mu}_2) - \bar{a}\sin(\alpha_n h/\mu_1)\cos(\alpha_n h/\bar{\mu}_2)],$$
$$a_{3n} = 8[b\cos(\alpha_n h/\mu_1)\sin(\alpha_n h/\mu_2) - a\sin(\alpha_n h/\mu_1)\cos(\alpha_n h/\mu_2)],$$

$$B_1 = 8A\mathrm{Im}\left\{\frac{\bar{a}b}{\mu_2} + \frac{\bar{b}}{\bar{\mu}_2} - \frac{\bar{a}}{\mu_1}\right\}, \quad B_2 = 4A\mathrm{Im}\left\{\frac{\bar{a}b}{\mu_1^2\mu_2} + \frac{\bar{b}}{\mu_2^2\bar{\mu}_2} - \frac{\bar{a}}{\mu_2^2\mu_1}\right\},$$

$$B_3 = 4A\mathrm{Im}\left\{\frac{\bar{a}b}{\mu_2}\left[\frac{1}{\bar{\mu}_2^2} + \frac{1}{3\mu_2^2}\right] + \frac{\bar{b}}{\bar{\mu}_2}\left[\frac{1}{\mu_1^2} + \frac{1}{3\bar{\mu}_2^2}\right] - \frac{\bar{a}}{\mu_1}\left[\frac{1}{\bar{\mu}_2^2} + \frac{1}{3\mu_1^2}\right]\right\},$$

$$B_4 = -8A\mathrm{Im}\left\{\frac{\bar{a}b}{\mu_1\mu_2} + \frac{\bar{b}g}{\mu_2\bar{\mu}_2} - \frac{\bar{a}g}{\mu_1\bar{\mu}_2}\right\}, \quad g = \frac{\gamma_1^{(2)}}{\gamma_1^{(1)}},$$

$(\nabla^2 - \delta_n^2)\psi_n = 0$, $h\delta_n = \pi n\mu_3$, $(\nabla^2 - \alpha_n^2)\varphi_n = 0$, α_n are the roots of Eq. (34).

From these relationships we conclude that the biharmonic parts appear in the same form both in the potential and in the vortex part of the solution.

It can be proved that on the base of the layer the biharmonic parts satisfy the homogeneous boundary conditions.

5 Piezoceramic layers with side to side elliptic cavities: oblique (skew)-symmetric problems

Here, we examine the potential practical applications of the above derived homogeneous solutions. The solution of the boundary-value problems of electroelasticity for layers with tunnel-heterogeneity is usually obtained using the method of integral equations. The integral expressions of the solutions are constructed on the basis of the respective homogeneous solutions, which result in a system of integro-differential equations. The derivation of the integral expressions of the solutions can be obtained on the basis of the fundamental solutions. In this case, the boundary-value problem is reduced to a system of integral equations.

At this point, we examine the oblique-symmetric (skew-symmetric) boundary-value problem of electroelasticity for a piezoceramic layer defined in $-\infty < x_1, x_2 < \infty$, $|x_3| \leq h$, weakened by the existence of a side to side cavity along Ox_3 for which the transversal section is defined by the ellipse

$$\zeta = \xi_1 + i\xi_2, \quad \xi_1 = R_1\cos\varphi, \quad \xi_2 = R_2\sin\varphi, \quad 0 \leq \varphi \leq 2\pi. \tag{35}$$

We assume that the axis Ox_3 has the direction of the potential lines of the electric field of the preliminary polarized ceramic. We also assume that the bases of the layer are covered by a diaphragm, considered to be rigid at its plane at the perpendicular (normal) direction.

On the side, the surface of the cavity acts on the stress vector $\{X_1, X_2, X_3\}$. The bases of the layer and the cavity are tangential to the air.

The boundary-value problem of electroelasticity is reduced to a system of singular integral equations. The homogeneous solutions appearing in [37] have been used for the derivation of the fundamental solutions of special form [38] for layers with bases bounded by the air and covered by a diaphragm which is considered rigid at its plane and less rigid at the direction perpendicular at its plane. The distribution of the epicycloidal normal stress on the boundaries of a cavity in piezoceramic layers reads:

$$\sigma_{\theta\theta} = \sigma_{11}\sin^2\psi - \sigma_{12}\sin 2\psi + \sigma_{22}\cos^2\psi. \tag{36}$$

Below, we give the results of the analysis for the piezoceramic material PZT-5H [18]. They refer to the distribution of the electrostatic fields in layers with a cavity.

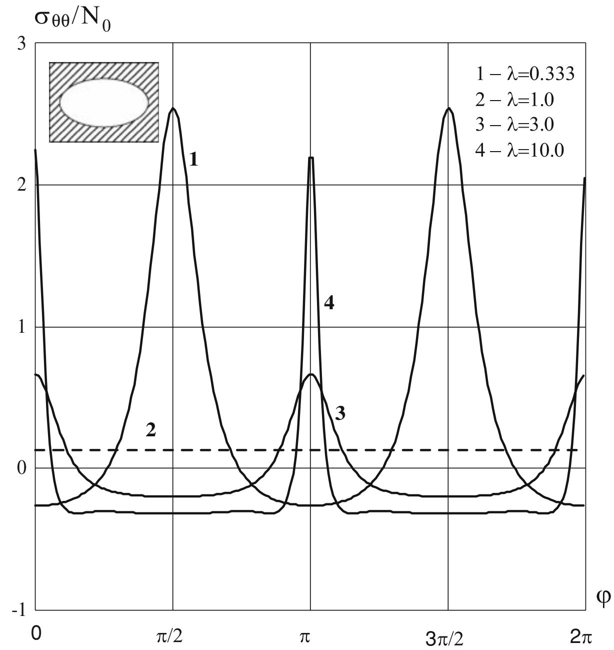

Fig. 1 Distribution of the normal stresses for the piezoceramic material PZT-5H with cavity

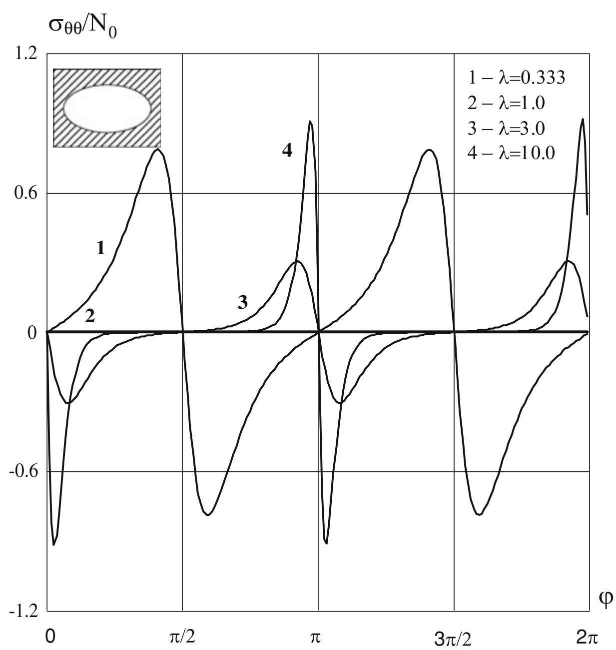

Fig. 2 Distribution of the shear stresses for the piezoceramic material PZT-5H with cavity

In Figs. 1 and 2, we give the distribution of the relative epicyclical normal stress $\sigma_{\theta\theta}/N_0$ ($N_0 = const$) on the boundary of a cavity with an elliptic transversal section (35) within piezoceramic layers for various values of the parameter $\lambda = R_1/R_2$ ($R_2 = 1$) at the section $x_3 = 0.8h$ ($h = 1$). Figure 1 shows the distribution $\sigma_{\theta\theta}/N_0$ when a normal axisymmetric loading $N = N_0 x_3/h$ acts on the boundaries of the cavity. Figure 2 shows the distribution of $\sigma_{\theta\theta}/N_0$ when a shear loading $T = N_0 x_3/h$ acts on the boundary of the cavity.

6 Conclusions

The procedure developed in this work gives the possibility to essentially simplify the derivation of the homogeneous solutions on the basis of the Lurié method. When the bases of the layers are built-in and grounded, the derived homogeneous solutions for the symmetric, to the mid-surface of the layer, electroelastic state do not contain any biharmonic terms. The spectrum $\{a_i\}$ of the characteristic equations corresponding to the various boundary conditions revealed some interesting properties: the spectrum is discrete, it is symmetrically placed at the complex plane and its accumulation point is at infinity. There are three asymptotic distributions $\{a_i\}$ one of which is the real axis and the other two are the lines $\arg(\mathrm{a}) = \pm\upsilon(\upsilon \neq 0)$, where the angle υ is determined by the electroelastic properties of the material. At infinity the difference between the consecutive roots of the characteristic equations is of the same magnitude and depends only on the electroelastic properties of the material (i.e. it is independent of the boundary conditions at the bases of the layer).

The proposed homogeneous solutions can be used for the investigation of boundary-value problems of piezoceramic cylinders and layers with non-homogeneous tunnels within the framework electroelasticity. Numerical results for the oblique-symmetric boundary-value problem for layers which are weakened due to an elliptic tunnel cavity are also given.

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