PARAMETRIC IDENTIFICATION OF STATE-SPACE DYNAMIC SYSTEMS: A TIME-DOMAIN PERSPECTIVE

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ABSTRACT. *In this paper we have presented a time-domain approach to parametric identification of state-space dynamic models comprised both an equation of motion and a system potential (a performance measure). The proposed techniques have been elaborated in order to obtain high simulation and forecast properties and applied to systems of non-stationary accelerator, gradient systems, and linear-quadratic stationary systems. We have also demonstrated a new concept of system potential specification in case of linear-quadratic stationary systems. It is based on the principle of its basis decomposition as an element of energy space. All models and algorithms have been approbated using real statistical data for models of macroeconomic dynamics.*
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1. **Introduction.** Dynamic systems design is one of the most urgent and difficult issues in different branches of science. Basically, it has three essential practical implications: simulation, forecast, and optimization. The first one is purely retrospective, while the others allow anticipating system dynamics. It is especially useful for analysts. This paper deals with techniques aimed at simulation and forecast. Thus, the quality of applied methodology will be considered in terms of simulation or forecast properties inherent in the resulted models.

The core issue of dynamic systems design is identification methodology and its practical efficiency. Basically, two types of concepts are common in the field of system identification [12]. The first one assumes full or partial specification of the relationship between systems inputs, states and outputs, while a number of unknown parameters should be estimated. Such an approach is called a grey box model. It uses parametric identification techniques (lest squares, general method of moments, maximum likelihood technique) [5, 1]. The other concept is a black box model, which assumes no prior specification and uses both parametric and non-parametric identification techniques (transient response analysis, Fourier analysis EFTE) [4, 21]. System identification can also be carried out in either the time (applying Markov Parameters [8], subspace system identification method [13]) or frequency domain (using singular value decomposition of Hankel matrix of Markov parameters [8]). This paper pertains only the time-domain identification within grey box model framework.

The main problems which arise in this area are computational difficulties connected with numerical realization of optimization algorithms; bias, inefficiency or inconsistency of estimators caused by failure to meet all preconditions of classical estimation techniques; insufficient precision and inadequacy of models. That results in low simulation

and forecast properties. Furthermore, standard time-domain identification methods deal exclusively with parameters of a state equation (a law of dynamic system motion). That allows one to skip the accompanying problem of a system potential (or a performance measure) identification. In this paper performance measure is specified as a complimentary internal characteristics of a dynamic system interconnected with a law of its motion. A performance measure can not be considered as additional state variable to avoid correlation problem which may apparently appear in such models. Moreover, presence of a performance measure provides the model with an extra advantage of using it in optimization problems.

Thus, we intend to demonstrate a new effective computational approach to time-domain parametric identification of state-space dynamic systems comprising both a law of motion and a system potential. Actually, the paper is organized as follows. Section 2 presents the general problem statement of state-space dynamic systems identification. The proposed identification techniques are elaborated in Section 3. Section 4 demonstrates results of approbation, based on real statistical data for models of macroeconomic dynamics. Eventually, we make a conclusion in Section 5.

2. **Problem Statement and Preliminaries.** We would like to confine ourselves to systems described by ordinary differential equations (in state variable form). Thus, let $\{x(t)\} = \{x(t) \in E^n \mid t_0 \leq t \leq t_1\}$ be a state trajectory of a dynamic system, a continuous vector-valued function. Its values are state column-vectors $x(t) = (x_i)^{i=1, n}$ from n-dimensional Euclidean space at time t. Similarly, let $\{u(t)\} = \{u(t) \in E^r \mid t_0 \leq t \leq t_1\}$ be an input trajectory, a piecewise continuous vector-valued function. Its values are input column-vectors $u(t) = (u_j)^{j=1, r}$ from r-dimensional Euclidean space at time t. Here and further on the simplifying assumption is made that the states are all available for measurement. Then the dynamic system may be described by n first-order differential equations in the matrix form:

$$\dot{x}(t) = f(x(t), u(t), t), \tag{1}$$

where $f(\cdots)$ is a vector of continuously differentiable functions. Its specification depends on physical interpretation of (1) [7].

Let G be a system potential, an empirical continuously differentiable function of the form:

$$G = G(x(t), \dot{x}(t), u(t), t). \tag{2}$$

Let us divide the continuous time domain $[t_0, t_1]$ into N discrete points of time. Supposing at every point of $t = 0, 1, \ldots, N-1$ there is a statistical information x_t about state column-vector $x(t)$ and statistical information G_t about system potential $G(t)$. If $f(\cdots)$ and G are specified with some functional forms, then the task is to identify system $(1), (2)$ so that the congruencies

$$x_t \cong x(t) \text{ and } G_t \cong G(t) \text{ for } t = 0, 1, \ldots, N-1$$

are performed with a certain precision.

3. **Time-domain Parametric Identification.** Generally speaking, time-domain parametric identification requires discretization of the examined system. Thus $(1), (2)$ should be transformed from the differential to difference form. As a result, for every t from $t = 0, 1, \ldots, N-1$

$$G = G(x(t), \Delta x(t), u(t), t), \tag{3}$$

$$\Delta x(t) = f(x(t), u(t), t), \tag{4}$$

where differences $x(t)$ may be either forward

$$\Delta x(t) = x(t+1) - x(t), \ t = 0, 1, \ldots, N-2 \tag{5}$$

or backward [17]

$$\Delta x(t) = x(t) - x(t-1), \ t = 1, 2, \ldots, N-1. \tag{6}$$

As investigation of forecast properties is essential in this paper, we will use backward differences (6) for further research.

The dynamic system presented by (3), (4) may be identified while using two different concepts. According to the first one (Section 3.1), a potential G identification is primary. According to the second concept (Section 3.2), a law of motion $f(\cdots)$ is proposed to be identified firstly. Section 3.1 is devoted to non-stationary accelerators and gradient systems identification. Stationary linear-quadratic (LQ) systems are examined in Section 3.2.

3.1. **Non-stationary accelerators and gradient systems.** In this section we will derive effective techniques for two classes of time-variant dynamic systems identification. They are frequently used in macroeconomic modeling [18] and differential games theory [11]. Following the works [15, 20], let it be that the law of motion (1) and the potential (2) form a circuited (closed) system. Besides, we will specify G only with state variables.

Then non-stationary accelerator is a system of the form:

$$\begin{aligned} G &= G(x(t)), \\ \dot{x}(t) &= u(t)G. \end{aligned} \tag{7}$$

Here the dimension r of the input column-vector $u(t)$ equals to n. Model (7) belongs to zero-order differentiating elements [10].

We will use the following model for gradient systems:

$$\begin{aligned} G &= G(x(t)), \\ \dot{x}(t) &= U(t)\left(\frac{\partial G}{\partial x}\right)^{\mathrm{tr}}, \end{aligned} \tag{8}$$

where U is time-variant $n \times n$ diagonal matrix with the coordinates $\{u_j\}_{j=1,n}$ on its diagonal. From here on we assume that a derivative of a scalar over a column (row) vector is a row (column) vector [7].

Although both non-stationary accelerators and gradient systems are quite different physically, technically, either of them has the same identification technique. It should be noted that the problem of systems (7) and (8) identification naturally gets transformed into inverse dynamic problem. Given the measurements of state variables and identified potential, it deals with input identification [16].

Let vector $g = (g_l)^{l=1,\,n}$ state for the n-dimensional column-vector $(G, G, \ldots, G)^{\mathrm{tr}}$ in case of non-stationary accelerator and for the n-dimensional column-vector $\left(\frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \ldots, \frac{\partial G}{\partial x_n}\right)^{\mathrm{tr}}$ in terms of gradient system. Then the general dynamic process will take the form of

$$\dot{x}(t) = U(t)g. \tag{9}$$

At the beginning the potential G should be specified. Firstly, it may be any smooth function to provide the existence of the first derivative. Secondly, it should be convenient for further research. Basically, log-linear form will be sufficient for the analysis of the first-order effects. The second order effects are usually examined by trans-log forms [6]. G is specified here as a multiplicative Cobb-Douglas function [19], i.e.

$$G(x_1, x_2, \ldots, x_n) = a_0 x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \tag{10}$$

or in the log-linear form

$$\ln G(x_1, x_2, \ldots, x_n) = \ln a_0 + a_1 \ln x_1 + a_2 \ln x_2 + \cdots + a_n \ln x_n. \tag{11}$$

Parameters from (10) may be easily identified. Ordinary least squares (OLS) estimation for the linearized model (11) can be applied here.

It is also convenient to use polynomial specification of inputs $u_j(t)$:

$$u_j(t) = b_{j0} + b_{j1}t + b_{j2}t^2 + \cdots b_{jk_j}t^{k_j}, \ j = 1, 2, \ldots, n. \tag{12}$$

Generally speaking, the order k_j may be chosen upon different considerations: to ensure high simulation or forecast properties, to increase adequacy of the model, etc. Nevertheless, the substantial restriction to achieve all this goals is that the higher the order k_j in (11) the lower the degree of freedom. The latter is undesirable for practical application. Further on this issue will be tackled with in details.

Given the identified system potential G and specified input $u(t)$, the equation of motion (9) identification is proceeded to. Using backward difference scheme (4), (6), we arrive at the following difference equations for $t = 1, 2, \ldots, N-1$ and $i = 1, 2, \ldots, n$:

$$x_i(t) = x_i(t-1) + b_{i0} \cdot g_i|_t + b_{i1} \cdot t \cdot g_i|_t + \cdots + b_{ik_i} \cdot t^{k_i} \cdot g_i|_t + \varepsilon_i(t), \tag{13}$$

where $\varepsilon_{k_i}(t)$ is a random disturbance, or if presented in the matrix form:

$$
\begin{pmatrix}
g_i|_{t=1} & g_i|_{t=1} & \cdots & g_i|_{t=1} \\
g_i|_{t=2} & 2\,g_i|_{t=2} & \cdots & 2^{k_i}g_i|_{t=2} \\
\vdots & \vdots & \ddots & \vdots \\
g_i|_{t=N-1} & (N-1)g_i|_{t=N-1} & \cdots & (N-1)^{k_i}g_i|_{t=N-1}
\end{pmatrix}
\begin{pmatrix}
b_{i0} \\
b_{i1} \\
\vdots \\
b_{ik_i}
\end{pmatrix}
+
$$

$$
+
\begin{pmatrix}
\varepsilon_i(1) \\
\varepsilon_i(2) \\
\vdots \\
\varepsilon_i(N-1)
\end{pmatrix}
=
\begin{pmatrix}
x_i(1) - x_i(0) \\
x_i(2) - x_i(1) \\
\vdots \\
x_i(N-1) - x_i(N-2)
\end{pmatrix}.
$$

The other identification methodology is based on integral specification of the equation of motion (9):

$$x(t) = x(t_0) + \int_{t_0}^{t} U(t)g\,dt. \tag{14}$$

Substituting (10), (12) in (14) and using left-hand Riemann sums approximation for integrating, we arrive at a discrete counterpart of the model (14) for $t = 0, 1, \ldots, N-1$ and $i = 1, 2, \ldots, n$:

$$x_i(t) = x_i^*(t_0) + b_{i0}\sum_{j=0}^{t-1} g_i|_{t=j} + b_{i1}\sum_{j=0}^{t-1} jg_i|_{t=j} + \cdots + b_{ik_i}\sum_{j=0}^{t-1} j^{k_i}g_i|_{t=j} + \nu_i(t), \tag{15}$$

where $\nu_i(t)$ is a random disturbance, or if presented in the matrix form:

$$
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & g_i|_{t=1} & g_i|_{t=1} & \cdots & g_i|_{t=1} \\
1 & \sum_{t=0}^{1} g_i|_t & \sum_{t=0}^{1} tg_i|_t & \cdots & \sum_{t=0}^{1} t^{k_i}g_i|_t \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \sum_{t=0}^{N-2} g_i|_t & \sum_{t=0}^{N-2} tg_i|_t & \cdots & \sum_{t=0}^{N-2} t^{k_i}g_i|_t
\end{pmatrix}
\begin{pmatrix}
x_i^* \\
b_{i0} \\
b_{i1} \\
\vdots \\
b_{ik_i}
\end{pmatrix}
+
\begin{pmatrix}
\nu_i^*(0) \\
\nu_i(1) \\
\nu_i(2) \\
\vdots \\
\nu_i(N-1)
\end{pmatrix}
=
\begin{pmatrix}
x_i(0) \\
x_i(1) \\
x_i(2) \\
\vdots \\
x_i(N-1)
\end{pmatrix}
$$

Notice: $x(t_0)$ from (14) is not equal to $x^*(t_0)$ from (15), and $x^*(t_0)$ is assumed to be unknown. This allows obtaining regressions with constant terms represented by (15). Model (13) does not contain a constant term though. The latter requires some adjustments when applying classical techniques to parameters identification [22]. Further we will use both scheme (13) and (15) to compare them from the view-point of simulation and forecast properties.

But we still have to identify the order k_j from (12). The order k_i will be obtained under conditions of forecast confidence intervals minimization, while forecast properties of regression models are especially substantial for practical applications. The latter is defined as follows for the forecast value of state variable $x_i(N)$:

$$\hat{x}_i(N) - \delta t_\alpha < x_i(N) < x_i(N) + \delta t_\alpha, \tag{16}$$

where $\hat{x}_i(N)$ is a point forecast; t_α is a percentile of Student's distribution with α, level of significance; δ is a standard error of forecasting [14].

3.2. **Stationary LQ systems.** Let us consider the general form of a linear stationary process:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{17}$$

where A and B are fixed $n \times n$ and $n \times r$ matrices with unknown elements.

In this paper we suggest supposing $B = I$ and $A = \{a_{ij}\}_{i=1,\,n}^{j=1,\,n}$ as a symmetric matrix. This allows putting an interpretation on input $u(t)$ and considering A as a matrix of some quadratic form [7]. In this case we may also elaborate an energy approach to identification of A.

Let system (17) take a form of

$$\dot{x}(t) = Ax(t) + u(t). \tag{18}$$

Model (18) is an open lag model. It is overdetermined having r degrees of freedom ($u(t) \neq 0$). Similarly, the model

$$\dot{x}(t) = Ax(t) \tag{19}$$

is called a closed lag model [10]. Then vector-valued function $u(t)$ may be considered as a deviation of the open lag model (18) from the closed lag model (19). Obviously, $u(t)$ is characteristics of external impacts on the dynamic system. This allows treating input $u(t)$ as a control parameter.

Let us build up the following difference model using (6) to identify the symmetric matrix A for the open lag model (18), :

$$\Delta x(t) = Ax(t) + v(t), \tag{20}$$

where $v(t) = (v_i(t))^{i=1,\,n}$ is a discrete counterpart of $u(t)$. Then let (20) be transformed into regression model of the form

$$\begin{cases} \Delta x_1 = a_{10} + a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + w_1, \\ \Delta x_2 = a_{20} + a_{12}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + w_2, \\ \quad \cdots \\ \Delta x_n = a_{n0} + a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{nn}x_n + w_n, \end{cases} \tag{21}$$

where $a_0 = (a_{i0})^{i=1,\,n}$ is a vector of fixed coefficients, required from econometric considerations; $w = (w_i)^{i=1,\,n}$ is a vector of random disturbances.

The following parametric iterative procedure for matrix A identification has been established. Let us first estimate parameters of n regressions from (21) separately. OLS as

an estimation method can be used. It is proposed that all non-diagonal elements of A are averaged according to the formula

$$a_{ij}^{\mathrm{av}} = \frac{a_{ij}^* + a_{ji}^*}{2}, \ i,\, j = 1,\, 2,\, \ldots,\, n,$$

where $\left\{a_{ij}^*\right\}_{j=1,\,n}^{i=1,\,n}$ are OLS-estimators of $\left\{a_{ij}\right\}_{j=1,\,n}^{i=1,\,n}$. Then diagonal elements of A can be corrected. The models

$$\begin{cases} \Delta x_1 - a_{12}^{\mathrm{av}} x_2 - \cdots - a_{1n}^{\mathrm{av}} x_n = a_{10}^* + a_{11}^* x_1 + \varepsilon_1, \\ \Delta x_2 - a_{12}^{\mathrm{av}} x_1 - \cdots - a_{2n}^{\mathrm{av}} x_n = a_{20}^* + a_{22}^* x_1 + \varepsilon_2, \\ \qquad\qquad\qquad\qquad \ldots \\ \Delta x_n - a_{1n}^{\mathrm{av}} x_1 - \cdots - a_{n-1,\,n}^{\mathrm{av}} x_{n-1} = a_{n0}^* + a_{nn}^* x_1 + \varepsilon_n, \end{cases} \qquad (22)$$

may be OLS estimated. Here $\{\varepsilon_i\}_{i=1,\,n}$ are random disturbances.

Thus the estimated model corresponding to (4) takes the form of

$$\begin{cases} \Delta x_1 = a_{11} x_1 + a_{12}^{\mathrm{av}} x_2 + \cdots + a_{1n}^{\mathrm{av}} x_n + v_1, \\ \Delta x_2 = a_{12}^{\mathrm{av}} x_1 + a_{22} x_2 + \cdots + a_{2n}^{\mathrm{av}} x_n + v_2, \\ \qquad\qquad \ldots \\ \Delta x_n = a_{1n}^{\mathrm{av}} x_1 + a_{2n}^{\mathrm{av}} x_2 + \cdots + a_{nn} x_n + v_n, \end{cases} \qquad (23)$$

where $\{v_i | v_i = \varepsilon_i + a_{i0}\}$ are the coordinates of the vector-valued function $u(t)$; $\{a_{i0}\}_{i=1,\,n}$ are the OLS-estimators of $\{a_{i0}^*\}$.

Coefficient of determination R^2 should be computed for every regression from (21) and (22), as well as significance of estimated coefficients should be verified. If regression analysis [22] indicates essential insignificancy of a state variable, then it may be excluded from the model with further recomputation.

In case when the proposed technique gives unsatisfied results concerning R^2, some adjustments are necessary. Particularly, (22) should be transformed into

$$\begin{cases} \Delta x_1 - \lambda_{12} x_2 - \cdots - \lambda_{1n} x_n = a_{10}^* + a_{11}^* x_1 + \varepsilon_1, \\ \Delta x_2 - \lambda_{21} x_1 - \cdots - \lambda_{2n} x_n = a_{10}^* + a_{11}^* x_1 + \varepsilon_1, \\ \qquad\qquad\qquad\qquad \ldots \\ \Delta x_n - \lambda_{n1} x_1 - \cdots - \lambda_{n,\,n-1} x_{n-1} = a_{n0}^* + a_{nn}^* x_1 + \varepsilon_n, \end{cases} \qquad (24)$$

where adjustment parameters $\{\lambda_{ij}\}_{j=1,\,n}^{i=1,\,n}$ are iteratively chosen from the interval $[a_{ij},\, a_{ji}]$. This arbitrariness in (24) is aimed at improving the resulted models (for example, increasing of coefficient of determination R^2).

Once high values of coefficients of determination R^2 for regressions (23) have been obtained, one can start considering the equation of motion (18), where the elements of A are the corresponding coefficients of model (23) and vector-valued function $v(t)$ states for input $u(t)$. As it is shown in [15, 20], such an approach provides sufficiently high simulation and forecast properties of dynamic models.

After matrix A has been fully identified, it enables one to find an analytical solution of (18). This issue belongs to the class of direct dynamic problems, i.e. to solve matrix differential equation (18) given A and inputs $u(t)$. Notice: $u(t)$ is defined at discrete points of time $t = 1,\, 2,\, \ldots,\, N - 1$. In order to obtain continuous control, $u(t)$ may be presented as a kinked line.

When considering real dynamic systems, it appears that state variables are monotonous over time. For instance, the main macroeconomic indices are increasing for developed countries. (Then $\dot{x}(t) > 0$, $t \in [t_0,\, t_1]$.) In such cases matrix A may be either positive or

negative definite. The latter allows our solving system of ODE (18) analytically. Specifically, if A is negative definite, then the left boundary condition $x(t_0) = x_0$ is given and the solution is found in direct time:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}u(\tau)\,d\tau; \tag{25}$$

if A is positive definite, then the right boundary condition $x(t_1) = x_1$ is given and the solution is found in inverse time:

$$x(t) = e^{A(t-t_1)}x_1 + \int_{t_1}^t e^{A(t-\tau)}u(\tau)\,d\tau. \tag{26}$$

Solutions (25) and (26) are stable [3] and they may be employed for simulation and forecast.

Now we proceed to system potential (3) specification and identification. For this we elaborate and employ an energy approach. It is based on potential G basis decomposition as an element of energy space. The following assumption and theorem are essential for further discussion.

Let model (2) be an autonomous system and the impact of all components on $G(\cdots)$ be characterized with the corresponding quadratic forms:

$$Q_{P_1}(x) = \frac{1}{2}x^{\mathrm{tr}}P_1 x, \ \ Q_{P_2}(\dot{x}) = \frac{1}{2}\dot{x}^{\mathrm{tr}}P_2\dot{x}, \ \ Q_{P_3}(u) = \frac{1}{2}u^{\mathrm{tr}}P_3 u.$$

The quadratic form $Q_{P_1}(x)$ states for potential energy of the open lag model (18) at time t, $Q_{P_2}(\dot{x})$ for kinetic energy of the open lag model (18) at time t, and, at last, $Q_{P_3}(u)$ for energy of external inputs.

Then considering G as an element of energy space and energies $Q_{P_1}(x)$, $Q_{P_2}(\dot{x})$, $Q_{P_3}(u)$ as components of the basis of this space, one can arrive at the following basis decomposition of potential G:

$$G = C_0 + C_1\frac{1}{2}Q_{P_1}(x) + C_2\frac{1}{2}Q_{P_2}(\dot{x}) + C_3\frac{1}{2}Q_{P_3}(u), \tag{27}$$

where C_0, C_1, C_2 and C_3 are unknown constants.

Generally, the matrices P_1, P_2 and P_3 are hard to identify. Nevertheless, further considerations will substantially simplify this issue.

Theorem 3.1. *The identity*

$$\frac{1}{2}x^{tr}Ax + \frac{1}{2}\dot{x}^{tr}A^{-1}\dot{x} - \frac{1}{2}u^{tr}A^{-1}u \equiv \frac{1}{2}\frac{d}{dt}(x^{tr}x) \tag{28}$$

holds true, if and only if the law of motion is described by (18).

Proof: It follows from (18) that

$$\frac{1}{2}u^{\mathrm{tr}}A^{-1}u \equiv \frac{1}{2}(\dot{x}^{\mathrm{tr}} - x^{\mathrm{tr}}A)A^{-1}(\dot{x} - Ax) \equiv \frac{1}{2}x^{\mathrm{tr}}Ax + \frac{1}{2}\dot{x}^{\mathrm{tr}}A^{-1}\dot{x} - \frac{1}{2}\dot{x}^{\mathrm{tr}}x - \frac{1}{2}x^{\mathrm{tr}}\dot{x}.$$

Notice: $\dot{x}^{\mathrm{tr}}x \equiv x^{\mathrm{tr}}\dot{x} \equiv \frac{d}{dt}(x^{\mathrm{tr}}x)$. Then formula (28) obviously arises. Apparently, the equation of motion (18) is also derived from (28).

Therefore, it follows from the energy equality that P_1, P_2 and P_3 may be reasonably identified as A, A^{-1} and A^{-1} respectively. Finally, the unknown coefficients from (27) are to be estimated. Employing OLS, we arrive at the identified model of system potential:

$$\hat{G} = \hat{C}_0 + \hat{C}_1\frac{1}{2}(x^{\mathrm{tr}}Ax) + \hat{C}_2\frac{1}{2}(\Delta x^{\mathrm{tr}}A^{-1}\Delta x) + \hat{C}_3\frac{1}{2}(u^{\mathrm{tr}}A^{-1}u), \tag{29}$$

where $\hat{C}_0$, $\hat{C}_1$, $\hat{C}_2$ and $\hat{C}_3$ are estimated coefficients. They may be obtained by solving the following matrix equation:

$$
\begin{pmatrix}
1 & \frac{1}{2}Q_A(x)|_{t=1} & \frac{1}{2}Q_{A^{-1}}(\Delta x)|_{t=1} & \frac{1}{2}Q_{A^{-1}}(u)|_{t=1} \\
1 & \frac{1}{2}Q_A(x)|_{t=2} & \frac{1}{2}Q_{A^{-1}}(\Delta x)|_{t=2} & \frac{1}{2}Q_{A^{-1}}(u)|_{t=2} \\
\vdots & \vdots & \vdots & \vdots \\
1 & \frac{1}{2}Q_A(x)|_{t=N-1} & \frac{1}{2}Q_{A^{-1}}(\Delta x)|_{t=N-1} & \frac{1}{2}Q_{A^{-1}}(u)|_{t=N-1}
\end{pmatrix}
\begin{pmatrix}
\hat{C}_0 \\ \hat{C}_1 \\ \hat{C}_2 \\ \hat{C}_3
\end{pmatrix}
=
\begin{pmatrix}
G_1 \\ G_2 \\ \vdots \\ G_{N-1}
\end{pmatrix},
$$

where $u_t = x_t - x_{t-1} - Ax_t$ for $t = 1, 2, \ldots, N-1$.

4. **Numerical Experiment.** This section deals with numerical application of proposed identification techniques to macroeconomic modeling for USA, Japan and a number of EU countries.

In all models we use GDP as potential of macroeconomic system. This specification follows from the fact that GDP accumulates all information of the system's production capacity and is considered as one of the most essential macroeconomic indicator. As to state variables specification, it generally depends on the results of correlation and regression analysis. We will use fixed capital stock, property costs, labor force and final consumption expenditures as initial candidates for this role though. All statistical information is available on Internet (see, for example, http://epp.eurostat.ec.europa.eu).

4.1. **A model of investment development.** Let us consider an open macroeconomic system that consists of $n-1$ industries. As experiment reveals [15] state variables of such a system are fixed capital stocks $\{x_i\}_{i=1, n-1}$ per industry and foreign debt x_n.

The rate of change $\dot{x}_i(t)$ of fixed capital is the value of net investments into the ith industry for $i = 1, 2, \ldots, n-1$. The rate of change $\dot{x}_n$ of foreign debt is the value of its gross accumulation. These values are used for the analysis of investment activity in economic theory [10], but mainly as ratios to gross output (e.g. GDP) rather than in absolute numbers. Therefore, specifying vector-valued function $f(\cdots)$ as a product of dimensionless value $u(t)$ and GDP G, we obtain a model of non-stationary accelerator (7). In this case economic interpretation of control variables is also obvious. Functions $\{u_j(t)\}_{j=1, n-1}$ may be considered as indices of investment activity per industry and $u_n(t)$ as an index of export-import non balance.

Approbation of (7) is based on two-industrial Danish economy in 1966-1997. Let the first industry consist of manufacturing and agricultural branches and the second one of services.

Using (10) Denmark's GDP is specified with two regressors: fixed capital stock $x_1 + x_2$ and foreign debt x_3. The obtained OLS estimations are:

$$
\ln G = \underset{(0.3195)}{0.3859} + \underset{(0.0428)}{0.8145}\ln(x_1 + x_2) + \underset{(0.0252)}{0.1120}\ln x_3, \quad R^2 = 0.9968, \tag{30}
$$
$$
\text{(s.e.)}
$$

where numbers in brackets are standard errors of regression coefficients. All coefficients, except the first one, appear to be statistically significant (we use Student's test for the number of freedom $l = 29$ and level of significance $\alpha = 5\%$). Such results are quite natural in econometric literature [6] when using log-linear functional forms for the regression analysis and with high coefficient of determination R^2 imply that the model is quite fulfilled with specified factors. It is also evident that the impact of foreign debt on Denmark's GDP was not essential in the analyzed period of time (although it appeared to be statistically significant). Another conclusion is that the diminishing return to scale was present ($a_1 + a_2 = 0.9265 < 1$) at that time.

The exponential form of the estimated log-linear production function is

$$
G(x_1 + x_2, x_3) = 1.4710(x_1 + x_2)^{0.8145}x_3^{0.1120}.
$$

Let us now turn to the polynomial (12) identification. The results of OLS estimation are given below:

$$u_1(t) = \underset{(0.0118)}{0.1469} - \underset{(0.0005)}{0.0037} t, \quad u_2(t) = \underset{(0.0448)}{0.2219} - \underset{(0.0043)}{0.0066} t + \underset{(0.0001)}{0.000042} t^2$$
$$\underset{\text{(s.e.)}}{} \qquad \underset{\text{(s.e.)}}{}$$

for difference identification scheme (13) and

$$u_1(t) = \underset{(0.0033)}{0.1543} - \underset{(0.0002)}{0.0041} t, \quad u_2(t) = \underset{(0.0167)}{0.2278} - \underset{(0.0019)}{0.0062} t$$
$$\underset{\text{(s.e.)}}{} \qquad \underset{\text{(s.e.)}}{}$$

for integral identification scheme (15). All coefficients in estimated polynomials appear to be statistically significant and provide the high level of approximation for state variables x_1 and x_2. Particularly coefficients of determination R^2 are 0.9979 and 0.9966 for scheme (13) and 0.9986 and 0.9982 for scheme (15).

As experiments reveal, optimal order k_j for each of polynomials (12) appears to be equal to 1 or 2. Interval forecasts (16) for state variables x_1 and x_2 are 1197674.539 $\pm$ 20073.928 and 1800344.834 $\pm$ 36369.454 for scheme (13) versus 11883527.018 $\pm$ 34599.568 and 1764626.422 $\pm$ 56578.455 for scheme (15). Obviously confidence intervals for the forecasts based on difference identification algorithm are narrower: 1.68% and 2.02% versus 2.92% and 3.21% (as percentages of point forecasts). The results obtained in this subsection confirm that difference identification scheme (13) is more preferable for forecast purposes. Integral identification scheme (15) ensures better simulation properties.

4.2. A Gradient macroeconomic model.

Let macroeconomic system be an aggregate market of n participants (players), which impact a system potential G acting either within competitive or cooperative framework. Let $x(t)$ be a vector of n strategies (factors) available for each of n players. All strategies can be divided into those that impact G positively and those that impact it negatively. Thus a potential G accumulates aggregate results of players' interaction.

Following [20], the aforementioned situation may be described using gradient model (8). Therefore coordinates of $u(t)$ may be considered as rates of each players investing into development of the corresponding factor. Then $\frac{\partial G}{\partial x}$ is a vector of investment efficiencies.

In this sub-section we will also employ Danish economy as an example. As experiment reveals [20], it is sufficient to consider three aggregate players: producers (firms) operated with fixed capital consumption (x_1), households operated with labor force (x_2) and government operated with final consumption expenditures (x_3).

OLS-estimation of unknown parameters in linearized production function (11) gives:

$$\ln G = \underset{(1.538)}{-1.714} + \underset{(0.162)}{0.450} \ln x_1 + \underset{(0.183)}{0.434} \ln x_2 + \underset{(0.177)}{0.487} \ln x_3, \quad R^2 = 0.9986, \qquad (31)$$
$$\underset{\text{(s.e.)}}{}$$

or in the form of (10):

$$G(x_1, x_2, x_3) = 0.180 \, x_1^{0.450} \, x_2^{0.434} \, x_3^{0.487}.$$

Notice that all coefficients in (31) are statistically significant with $\alpha = 0.05$, the level of significance. Therefore it can be employed for modeling of Danish macroeconomic system. It is also derived from (31) that all factors impact GDP positively and more or less evenly provide increasing return to the scale ($a_1 + a_2 + a_3 = 1.371 > 1$).

Using difference scheme (13) for polynomials (12) identification, one can obtain:

$$u_1(t) = \underset{(25.735)}{389.526}, \quad u_2(t) = \underset{(0.201)}{0.466} - \underset{(0.0082)}{0.018} t,$$
$$\underset{\text{(s.e.)}}{} \qquad \underset{\text{(s.e.)}}{}$$

$$u_3(t) = \underset{(1471.323)}{4937.955} + \underset{(1107.170)}{2504.817} t - \underset{(218.916)}{503.835} t^2 + \underset{(11.831)}{26.465} t^3.$$
$$\underset{\text{(s.e.)}}{}$$

TABLE 1. The results of modeling for USA, Japan, France, Italy, Netherlands, and Finland

Country	Fixed capital consumption		Labor force		Property costs	
	R^2	2004	R^2	2004	R^2	2004
For difference identification scheme						
USA	0.937	1036859.5	0.924	13808	0.937	8323649.7
		-1.89; 2.74		1.41; 2.68		2.86; 4.18
Japan	0.930	910333.3	-	-	0.902	2709948.7
		-3.04; 3.12				3.76; 4.39
France	0.960	209874.3	0.889	25418	0.987	1339665.2
		1.30; 2.31		-1.79; 2.05		-0.58; 1.53
Italy	-	-	0.940	24583	0.985	1079854.7
		1.30; 2.31		-1.43; 2.35		0.93; 2.22
Netherlands	0.945	74377.4	-	-	0.965	364853.0
		-1.26; 2.92				-1.26; 2.12
Finland	0.932	22237.0	0.864	2398	-	-
		-0.73; 1.78		-1.38; 2.03		
For integral identification scheme						
USA	0.962	1043251.9	0.966	137283.8	0.970	8356617.2
		-2.49; 3.38		2.00; 3.03		-3.24; 4.67
Japan	0.953	921095.2	-	-	0.923	2917384.3
		-4.17; 4.55				-4.60; 5.21
France	0.987	211839.9	0.927	25009.5	0.994	1310524.7
		0.36; 2.44		-0.19; 2.21		1.63; 2.03
Italy	-	-	0.963	23495.9	0.991	1059082.1
		1.30; 2.31		3.13; 3.74		2.90; 3.45
Netherlands	0.969	72299.1	-	-	0.982	360944.2
		1.57; 3.43				-0.19; 2.63
Finland	0.981	22567.1	0.946	2419.1	-	-
		2.19; 3.52		2.23; 3.65		

Interval forecasts (16) of state variables are 31757.1 ± 594.2 for fixed capital consumption, 2749 ± 53 for labor force and 147740.5 ± 3670.1 for final consumption expenditures. In its turn, integral identification scheme (15) results in:

$$u_1(t) = \underset{(s.e.)}{355.147} + \underset{(17.379)}{8.169}\, t \quad u_2(t) = \underset{(s.e.)}{-2.069} + \underset{(0.288)}{1.104}\, t, \underset{(0.107)}{} -\underset{(0.0083)}{0.0859}\, t^2$$

$$u_3(t) = \underset{(s.e.)}{7266.0158.} \underset{(146.178)}{}$$

In this case interval forecasts (16) of state variables are 32327.9 ± 934.7 for fixed capital consumption, 2675 ± 99 for labor force and 146293.8 ± 4581.1 for final consumption expenditures.

As experiments reveal, optimal order k_j for each of polynomials (12) appears to range from 0 to 3. Obviously, confidence intervals for the forecasts based on difference identification algorithm are narrower: 1.87%, 1.93% and 2.48% versus 4.52%, 5.89% and 6.31% (as percentages of point forecasts). All coefficients in estimated polynomials appear to be statistically significant, providing the high level of approximation for state variables x_1, x_2 and x_3. Computations of coefficients of determination R^2 give values 0.99414, 0.91348, 0.99841 for scheme (13) and 0.99832, 0.98068, 0.99517 for scheme (15).

The results of similar modeling for USA, Japan, France, Italy, Netherlands and Finland are presented in Table 1. It consists of forecast values (numbers from above) and confidence intervals as percentages of point forecasts (numbers from below). System potential G was initially specified with three factors. If it appeared that G contained insignificant coefficients, then the recomputations would be held with corresponding two factors. Table 1 as well as aforementioned results coincided with those from Subsection 4.1: difference identification scheme (13) is more preferable for prospective analysis. Integral identification scheme (15) has better descriptive properties though.

The adequacy of the results from Subsections 4.1 and 4.2 was verified using Gauss-Markov assumptions for classical regressions [22]. As computations [15, 20] reveal, only the assumption of non-autocorrelation failed. The presence of autocorrelation in models (13) and (15) is considered admissible, while input $u(t)$ is approximated by polynomials depended on time [6, 14].

4.3. An energy macroeconomic model.

This subsection deals with approbation of stationary LQ model (18), (27) identification techniques. As before, it is proposed to investigate Denmark's economy. Let the dynamic system (18), (27) be specified with fixed capital stock (x_1), property costs (x_2), and final consumption expenditures (x_3). All data are considered as ratios to initial time.

We obtain acceptable results right after the first iteration, when applying the proposed iterative procedure of matrix A identification (see Subsection 3.2):

$$A = \begin{pmatrix} 0.2042 & -0.2974 & -0.0421 \\ -0.2974 & 0.7154 & -0.1601 \\ -0.0421 & -0.1601 & 0.2527 \end{pmatrix}.$$

All regressions from (23) appear to have high coefficients of determination R^2 (97-98%). This argues in favor of models correspondence with real trends in macroeconomic dynamics.

Then, we obtain identified system potential (29), using OLS estimation:

$$\hat{G} \underset{\text{(s.e.)}}{=} \underset{(0.11)}{0.74} + \underset{(0.14)}{0.53}\, Q_A(x) + \underset{(0.14)}{0.39}\, Q_{A^{-1}}(\dot{x}) - \underset{(0.22)}{0.64}\, Q_{A^{-1}}(u). \tag{32}$$

According to Student's criterion with the level of significance $\alpha = 0.05$, all components from specified energy space impact system potential (GDP) significantly. Coefficient of determination R^2 equal to 98% also argues in favor of proposed model. Figure 1(a) demonstrates real (designated with dots) and simulated (curves) values of potential G dynamics. We may also make a forecast. We obtain a point forecast 3.02 and its standard error $\delta = 0.05$, while extrapolating (32) for one period ahead.

Our investigation concerns dynamics of stable development. All eigenvalues of A are positive: $\lambda_1 = 0.8787$, $\lambda_2 = 0.0166$, $\lambda_3 = 0.2770$. Therefore A is defined positive. Therefore, in order to verify simulation and forecast properties, the formula (26) can be employed. Here are confidence intervals of forecasts for x_1, x_2 and x_3: 4.95 ± 0.20, 3.32 ± 0.09 and 3.04 ± 0.08.

Figure 1 demonstrates real and simulated vales of fixed capital stock (Figure 1(b)), property costs (Figure 1(c)), and final consumption expenditures (Figure 1(d)). It reveals high simulation quality of the models and thus efficiency of the proposed identification techniques. The sensitivity of output to the inputs has been tested using the analytical solution (26) and econometric model (20). Both lead to the same results and indicates the stability of obtained models.
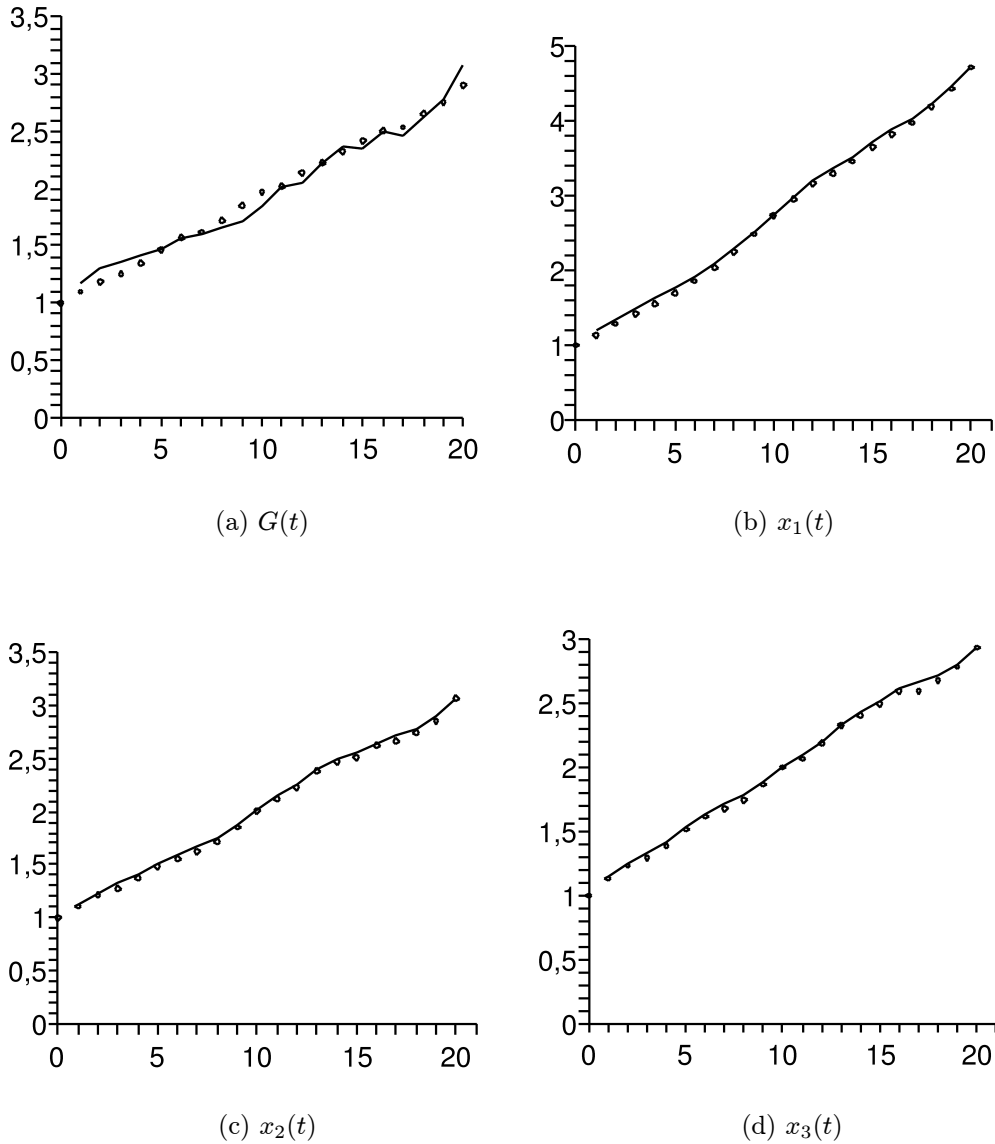
(a) $G(t)$

(b) $x_1(t)$

(c) $x_2(t)$

(d) $x_3(t)$

FIGURE 1. Real and simulated values of (a) system potential G, (b) fixed capital consumption x_1, (c) property costs x_2 and (d) consumption expenditures x_3

5. **Conclusions.** A series of time-domain parametric identification techniques for a number of dynamic systems have been presented. On the one hand, two identification algorithms for models of non-stationary accelerator and gradient systems have been elaborated. Difference identification scheme has appeared to be more applicable for forecast analysis, while integral one has demonstrated better simulation properties. System potential for these models should be identified primarily. On the other hand, a new approach to specification and identification of stationary LQ systems has been established. In this case the equation of motion should be identified primarily. A parametric iterative procedure has been proposed for this purpose. It is based on analogues between overdetermined equations and regressions. System potential has been specified as an element of energy space with the components: potential, kinetic, and energy of inputs. The energy equality

has also been established. It has considerably simplified identification procedures. All identification techniques has been approbated using real statistical data for models of macroeconomic dynamics. As experiment reveals, all models have demonstrated good simulation, forecast properties and good correspondence with reality. Concerning identification time-span as a learning domain, further investigations may be connected with optimization problems in future periods.

REFERENCES

[1] S. I. Aihara and A. Bagchi, Filtering and identification of stochastic volatility for parabolic type factor models, *International Journal of Innovative Computing, Information and Control*, vol.2, no.5, pp.917-925, 2006.

[2] D. P. Bertsekas *Dynamic Programming and Optimal Control*, Athena Scientific, Belmont, Massachusetts, 1995.

[3] A. Bryson and Y. C. Ho, *Applied Optimal Control: Optimization, Estimation and Control*, Blaisdell Publishing Company, Waltham, Massachusetts, Toronto, London, 1969.

[4] G. F. Franklin, J. D. Powell and M. Workman, *Digital Control of Dynamic Systems*, Addison-Wesley, Menlo Park, California, 1998.

[5] M. Gevers and G. Li, *Parameterization in Control, Estimation and Filtering Problems*, Springer, London, 1993.

[6] W. H. Greene, *Econometric Analysis: Fifth edition*, Prentice Hall, Upper Saddle River, N.J., 2003.

[7] M. D. Intriligator, *Mathematical Optimization and Economic Theory*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2002.

[8] J. N. Juang, *Applied System Identification*, PTR Prentice Hall, Englewood Cliffs, NJ, 1994.

[9] D. E. Kirk, *Optimal Control Theory: An Introduction*, Prentice-Hall, Englewood Cliffs, N.J., 1998.

[10] V. A. Kolemaev, *Economical And Mathematical Modeling. Macroeconomic Processes and Systems Modeling*, Unity, Moscow, 2005 (in Russian).

[11] N. N. Krasovskii and A. I. Subbotin, *Positional Differential Games*, Nauka, Moskow, 1974.

[12] L. Ljung, *System Identification: Theory for the User*, PTR Prentice Hall, Upper Saddle River, N.J., 1999.

[13] M. Moonen, B. De Moor, L. Vandenberghe and J. Vandewalle, On- and off-line identification of linear state-space models, *International Journal of Control*, vol.49, no.1, pp.219-232, 1989.

[14] O. M. Nazarenko, *Basic Econometrics*, The Center of Academic Literature, Kyiv, 2005 (in Ukrainian).

[15] O. M. Nazarenko and D. V. Filchenko, Optimal investment flows distribution in a dynamic model of macroeconomic system, *Physical and Mathematical Modeling and Information Technologies*, no.5, pp.5-16, 2007 (in Ukrainian).

[16] Y. S. Osipov and A. V. Kryazhimskii, *Inverse Problem of Ordinary Differential Equations: Dynamical Solutions*, Gordon and Beach, London, 1995.

[17] A. Quarteroni, R. Sacco and F. Saleri, *Numerical Mathematics*, Springer, N.Y., 2000.

[18] D. Romer, *Advanced Macroeconomics*, McGraw-Hill, London, 1996.

[19] S. J. Turnovsky, *Methods of Macroeconomic Dynamics*, The MIT Press, Cambridge, 2000.

[20] A. A. Vasil'ev and O. M. Nazarenko, Macroeconmic dynamic differential-game models discretization and numerical identification, *Bulletin of Kharkiv National University*, vol.733, no.6, pp.35-46, 2006 (in Russian).

[21] F. Wang, H. Li, Y. Zhang and R. Li, Novel ICA algorithm with nonparametric estimation based on GGD kernel, *International Journal of Innovative Computing, Information and Control*, vol.2, no.2, pp.427-440, 2006.

[22] J. Wooldridge, *Introductory Econometrics: A Modern Approach*, Southwestern Publishers, N.Y., 2000.

[23] Z. Mao and B. Jiang, Fault identification and fault-tolerant control for a class of networked control systems, *Int. J. Innovative Computing, Information and Control*, vol.3, no.5, pp.1121-1130, 2007.

[24] A. Ohsumi, S. Komiyama, M. Kashiwagi, M. Watanabe and T. Takatsu, Detection of pollution load and identification of its discharged location and magnitude for polluted river, *Int. J. of Innovative Computing, Information and Control*, vol.4, no.1, pp.63-78, 2008.