Dynamics of price-level, national income and cost of money interaction

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Dynamics of a macroeconomic system in which national income, cost of money and price-level are in close interaction is studied. Such an interaction is simulated with the help of discrete dynamic system in R^3 [3]. The system has a curve formed by fixed points, which describe a balanced state of money, goods and service markets. It has been shown that there is a foliation which is transversal to the curve, each layer being invariant for the system. There are layers where balanced state can be both stable and unstable. The system dynamics is changing from layer to layer. There are two routes of bifurcations. The first one runs on the following scheme: a fixed point looses its stability resulting in appearance of stable invariant ellipse (Neimark-Sacker bifurcations). In the ellipse periodic hyperbolic orbits appear and cause chaos through transverse intersection of stable and unstable manifolds. The other way leads to chaos through bifurcation of period doubling. Minor random perturbations of the system which simulate exposure naturally have been viewed. In this case the system does not preserve fixed points and layer invariance which cause more complicated dynamics.

1 Introduction

Let's consider the dynamics of a macro economical system "national income" - "interest rate" - "price level". Such a dynamics is described by "IS-LM" model which is basically used for description of a current market economy [5, 9, 13, 15, 16]. For modeling of macroeconomic dynamics we will use a discrete system

$$u_{n+1} = F(u_n), \tag{1}$$

where u_n denotes a state of economy at time $t = n$, $n \in Z$. In the monograph [3] a discrete dynamic system for the above mentioned macro economical model has been developed and a form of the mapping F was given. Let us denote $u = (x, y, z)$ where $x = P/P_e$, P - price level, P_e - balanced value of a price level $y = (r_e/r)^s$, r is an interest rate, r_e is a balanced value of an interest rate, $z = Y/Y_e$, Y is national income, Y_e is a balanced value of a national income. So all variable data are pure numbers and their changes reflect deviations in a certain balanced state. The system (1) in the coordinates (x, y, z) according to [3] takes the form

$$\begin{aligned} x_{n+1} &= x_n \exp(a(1 - x_n y_n^{m/s} z_n)), \\ y_{n+1} &= y_n \exp(b(1 - x_n y_n^{m/s} z_n)), \\ z_{n+1} &= z_n \exp(c(y_n - z_n)), \end{aligned} \tag{2}$$

where $x, y, z > 0$. All parameters $a,\ b,\ c,\ m,\ s$ are positive ones, s being a marginal propensity to save (MPS), $0 < s < 1$. We can say that the variable x is proportional to the price level, the variable z is proportional to the national income, and the variable y is inversely proportional to s-degree of the interest rate. The system has been studied in [3]. The authors have considered the case when the system can be viewed as one-dimensional. The periodic orbits are shown numerically. In our paper general theoretical results have been obtained, the detailed research of global dynamics of the system and numerical calculations for specific system parameters have been carried out.

Lets first consider a simple case when $m = 0$. Equality $m = 0$ means that the demand does not depend on interest rate. A balanced state of the economics corresponds to a fixed point of system (1). It has been shown that the system (2) has a curve K formed by fixed points. Dynamics of the system near the curve K is studied in Section 2. In Section 3 it shows that transversally to the curve there is foliation with invariant layers, i.e. an invariant surface passes through each fixed point, these surfaces are mutually disjoint and fill out full space. The foliation is specified as level surfaces of the function $U = \frac{x^b}{y^a}$, i. e. a surface has the form

$$W = \{(x, y, z) : \frac{x^b}{y^a} = const\}.$$

The bifurcation of the topological structure of the system orbits takes place from layer to layer with the layers preserving their balanced state. There are layers with stable balanced states and layers with chaotic dynamics near balanced states. In Section 4 it shows such a topological structure exists at any $m > 0$. More than in Section 5 it has been proved that the system (2) for arbitrary $m > 0$ is topologically equivalent to the system with $m = 0$ and a changed a parameter.

Environment effect. The environment in which the microeconomics is developing affects the system. The external influence can be considered as uncontrolled perturbation. To simulate the external influence let's assume that the perturbation depends on time n, it is small and added to the mapping as a whole. So we have an equation

$$
\begin{aligned}
x_{n+1} &= x_n \exp(a(1 - x_n y_n^{m/s} z_n)) + \varepsilon q_1(n), \\
y_{n+1} &= y_n \exp(b(1 - x_n y_n^{m/s} z_n)) + \\
\varepsilon q_2(n),\ z_{n+1} &= z_n \exp(c(y_n - z_n)) + \\
\varepsilon q_3(n), &
\end{aligned}
\tag{3}
$$

where ε is a small positive number, $q_i(n)$ takes random variables on a segment [-1, 1] and chaotically depends on time n. To simulate perturbation $q_i(n)$ we use relationship

$$
q_i(n+1) = 1 - 2q_i^2(n), \quad q_i \in [-1, 1],
$$

where initial value $q_i(0)$ is separately specified for each $i = 1, 2, 3$. It is known [4, 12], that for almost every initial value $q_i(0)$ (according to Lebesgue measure) on the orbit $\{q_i(0)n\}$ is chaotic and distributed along the segment [-1,1] with a density

$$
\rho = \frac{1}{\pi(1 - x^2)^{1/2}}.
$$

Computer software. The investigation is attended with numerical experiments. Numerical calculations have been carried out according to the algorithms developed and substantiated by the author [7]. A computer program for the given algorithms and visualization has been developed by the St. Petersburg University alumnus Michael Senkov.

2 Dynamics near balance states.

Let's consider a discrete dynamic system (2). As it has been stated above first we'll see the case of $m = 0$, that will somehow simplify calculations, but reflects the gist of the dynamic process. This assumption will be true for the current and the following sections. The balance states are determined by fixed points of the system (1), i.e. by the equation

$$F(u) = u.$$

Passing to the coordinates (x, y, z) and to the system (2) we get the equalities $xy - 1 = 0$ and $y - z = 0$. Thus, the fixed points fill the curve $K = \{(x, y, z) : xy = 1, y = z\}$. Projection of K on (XY)-plane is a hyperbola $xy = 1$ and projection of K on (YZ)-plane is a straight $y = z$. Lets study economy dynamics near to balance states K. Topological type of a fixed point u^* of the system (1) is determined by eigenvalues $\{\lambda\}$ of the differential $DF(u^*)$ at u^*. The invariance of K leads to the tangent vector of K be an eigenvector to the differential. Moreover the restriction F on K is an identical mapping $F(u) = u$. This means that $\lambda = 1$ is an eigenvalue of $DF|_K$. On the K curve the right-hand part differential of (2) has the following form

$$D = \begin{pmatrix} 1-a & 0 & -ax^2 \\ -by^2 & 1 & -b \\ 0 & cz & 1-cz \end{pmatrix}. \tag{4}$$

The multipliers λ of the fixed points are determined by the equation

$$det(D - \lambda E) = (1 - \lambda)(\lambda^2 - \lambda(2 - a - cy) + 1 - a + (b + a - 1)cy).$$

It is clear that the multiplier $\lambda = 1$ corresponds to the fixed point curve. The rest of the multipliers $\lambda_{1,2}$ are determined by the equation

$$\lambda^2 - \lambda(2 - a - cy) + 1 - a + (b + a - 1)cy = 0. \tag{5}$$

Hence we have

$$\lambda_{1,2} = \frac{2 - a \pm \sqrt{\Delta}}{2},$$

where the discriminant $\Delta = c^2y^2 - 2cy(a+2b) + a^2$. Easy to check that $\lambda_{1,2} \neq 1$ as $y > 0$. From this it follows that the eigenspace of $\lambda_{1,2}$ is transversal to K. If $|\lambda_{1,2}| < 1$, the differential contracts to K. If $|\lambda_{1,2}| > 1$, the differential expands from K. If $|\lambda_1| < 1$ and $|\lambda_2| > 1$, we have hyperbolic dynamics near to K. If the discriminant $\Delta < 0$, λ_1 and λ_2 are complex conjugate and the differential rotates around K. The discriminant will be negative when y has the meanings between the roots of y_1 and y_2 of the equation

$$c^2y^2 - 2cy(a + 2b) + a^2 = 0, \tag{6}$$

where

$$y_{12}$$

$$= (a + 2b) \pm 2\sqrt{b(a + b)} \, . \, c$$

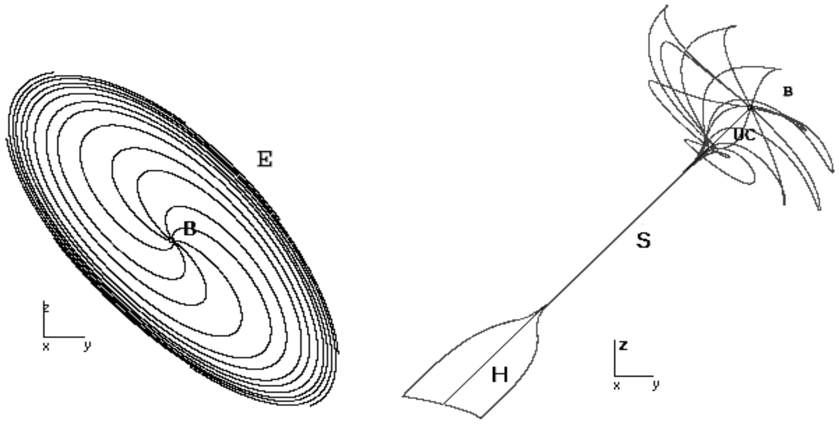

Figure 1: Dynamics near the fixed points K of the system (2).

For $a = 2.4,\ b = 0.87,\ c = 0.9$ there is $y_1 = 0.852,\ y_2 = 7.51$. Thus at $0.852 < y < 7.51$ the multipliers λ_{12} are complex conjugate. In this case a free member of the equation (5) is a square of multiplier modulus

$$1 - a + (b + a - 1)cy = \lambda_1\lambda_2 = |\lambda|^2,$$

that gives the opportunity to determine the range of stability and instability of the fixed points. For $a = 2.4,\ b = 0.87,\ c = 0.9$ and $y = 0.852$ there is $|\lambda|^2 = 0.3406$, i.e. in the plane transverse to the curve of the fixed points we have a stable focus. Numerical experiments show that when a term $1 - a + (b + a - 1)cy$ is equal to 1, the Neimark-Sacker bifurcation takes place i.e. the balance state loses stability and stable invariant ellipse arises from the balance point. For $a = 2.4,\ b = 0.87,\ c = 0.9$ the bifurcation occurs at $y = 1.174743$. On invariant ellipse the orbits may be periodic or recurrent. Note that with practical point of view a recurrent orbit looks as periodic with big period. So we can consider the orbits in the ellipse as periodic. On the K curve between points $0.852 < y < 1.174743$ there are stable balanced states, but unstable balanced states emerge for $y > 1.174743$. By the Pliss's reduction principle [10, 11, 8] near the K curve through each fixed point there is an invariant disk, which is a stable manifold W^s when $|\lambda_{1,2}| < 1$ ($0.852 < y < 1.174743$), or an unstable manifold W^u when $|\lambda_{1,2}| > 1$ ($1.174743 < y < 7.51$). In the left-hand Fig. 1 an invariant disk $W^u(B)$ of balanced state B $(0.851, 1.175, 1.175)$ is shown. On the $W^u(B)$ there is an unstable balance

B and an invariant stable ellipse E. An orbit starts in the B point as $n = -\infty$ and ends in E for $n = +\infty$ On the K curve between two points in the interval $0.700 < y < 0.852$ there are balanced states with two negative multipliers which are modulo less than 1. When $0.685 < y < 0.700$ a hyperbolic state occurs i.e. one multiplier is modulo more than 1, but the other is modulo less than 1. When $y = 0.685$, one multiplier becomes equal to 0. It means that Jacobian matrix of the right-hand part of the system (2) is $detD = 0$. When $y < 0.685$, one multiplier is positive, while the other is negative. In this case the differential changes orientation (in an unstable manifold). The Jacobian matrix sign determines if the dynamic system preserves its orientation in this particular point. Thus the equation $detD(x, y, z) = 0$ specifies the Π surface where the system changes its orientation. So every balanced state in the plane transversal to K may be stable, unstable (with complex multipliers), and hyperbolic. In the last case one multiplier is negative and the other changes the sign on Π. In the right-hand Fig. 1 one can see intervals on K: H = {hyperbolic fixed points}; S = {stable fixed points }; $U\,C$ = {unstable fixed points with complex multipliers}.

3 Foliation with invariant layers

In this part we are going to prove that there is a function $U(x, y, z)$, the level surfaces $U(x, y, z) = const$ of which are invariant for a discrete system

$$\begin{aligned} x_{n+1} &= x_n \exp(a(1 - x_n z_n)), \\ y_{n+1} &= y_n \exp(b(1 - x_n z_n)), \\ z_{n+1} &= z_n \exp(c(y_n - z_n)). \end{aligned} \tag{7}$$

Function $U\,(x, y, z)$ is an analog of the integral for autonomous systems of differential equations. To obtain the function U we note that the first and the seconde equation of (7) differ in powers a and b. If the first equation is raised to power b and the second equation is raised to power a, we obtain uniformly expression $\exp(ab(1-x_n z_n))$. This gives an opportunity to construct the function U.

Proposition 1. Level surfaces of the function

$$U = \frac{x^b}{y^a}$$

are invariant for the system (7).

Proof. A level surface H is determined by the equation $U\,(x,\,y,\,z) = const$. Invariance of H means as an orbit of (7) starts at H, it remains in H all time. In other words a value of the function U is unchanged by iteration. So, to prove the invariance we have to check the following equality

$$U\,(x_n,\,y_n,\,x_n) = U\,(x_{n+1},\,y_{n+1},\,z_{n+1}).$$

Indeed,

$$U\,(x_{n+1},\,y_{n+1},\,z_{n+1}) = \frac{x^b_{n+1}}{y^a_{n+1}} = \frac{x^b_n \exp(ba(1-x_nz_n))}{y^a_n \exp(ab(1-x_nz_n))} = \frac{x^b_n}{y^a_n} = U\,(x_n,\,y_n,\,z_n).$$

Proving has been completed.

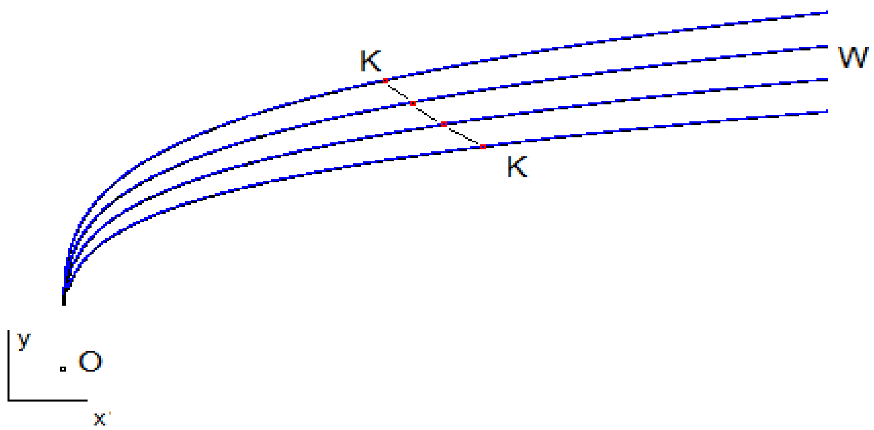

Figure 2: Projection of the fixed point curve $K = \{(x, y, z) : xy = 1, y = z\}$ and invariant layers $W = \{(x, y, z) : x = hy^{a/b}, h = const\}$ on XY-plane.

Each surface

$$W = \{(x, y, z) : x = hy^{a/b}\},\ h = const$$

is transversal to the fixed point K curve. So, invariant foliation caused by stable W^s and unstable W^u discs exists not only around K curve but it is determined globally. Since the variable z does not appear implicit in U, each invariant layer $W = \{(x, y, z) : x = hy^{a/b}\}$, $h = const$ is a linear surface with straights parallel to a Z-axis. The system on the surface $W = \{(x, y, z) : x = hy^{a/b}\}$ is specified as

$$y_{n+1} = y_n \exp(b(1 - hy_n^{a/b} z_n)),$$
$$z_{n+1} = z_n \exp(c(y_n - z_n)), \tag{8}$$

where $h > 0$ specifies an invariant layer. Thus $h = 1$ specifies the surface passing through a balanced state (1,1,1). For any $W(h)$ surface a balanced state is determined by the equality

$$xy = 1,\ y = z,\ x = hy^{a/b}.$$

Whence it happens that a balanced state has

$$(h^{\frac{b}{a+b}},\ h^{-\frac{b}{a+b}},\ h^{-\frac{b}{a+b}})$$

coordinates for each $W(h)$ layer.

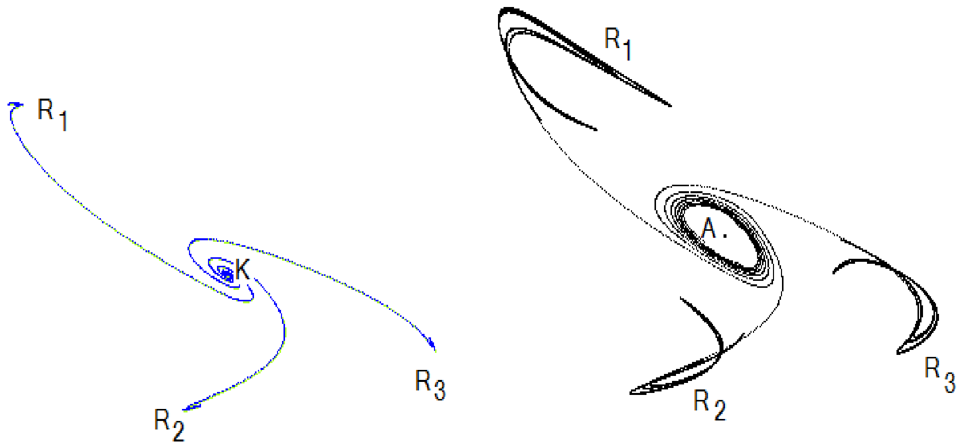

Figure 3: Appearance of a stable invariant ellipse at h changing in the interval 0.8 - 0.58.

Let us see the changing of the system dynamics in the $W(h) = \{(x, y, z) : x = hy^{a/b}\}$ layers in dependence on h values. In other words, we will observe the system (8) bifurcation at h changing. In each $W(h)$ layer there is a point $W(h)$ K which is fixed for the system (8). If the layer is fixed, we will denote the fixed point as K^*. We can consider the fixed point K^* as origin of the layer. Let us consider the system at (8) $a = 2.4$; $b = 0.9$; $c = 0.9$. On the $W(h = 1)$ layer there is a balanced state $K^*(1, 1)$ with complex multipliers. If the value of h decreases, the type of stability starts to change. So, on the $W(h = 0.8)$ layer (see a left-hand Fig. 3) there is a 3-periodic invariant set $R = \{R_1, R_2, R_3\}$, from which orbits pass to the stable balanced state K^*. With changing of h parameters from 0.8 to 0.58 a stable invariant ellipse A appears due to stable balanced state, while balanced state K^* loses its stability, i.e. the Neimark-Sacker bifurcation occurs (see a right-hand Fig. 3). At the same time the invariant set R increases in size. When h parameters tend to decrease, 3-periodic hyperbolic orbit P appears in the stable A ellipse (see a left-hand Fig. 4). When h reaches the value of 0.515970, an unstable manifold $W^u(P_3)$ crosses transversally a stable manifold $W^s(P_1)$ of the P_1 (0.8037147 1.33638665) orbit (see a right-hand Fig. 4). Simulation of these manifolds and their estimation have been carried out in accordance with [**?**, 7]. By the Smale's theorem [2] a transversal intersection generates chaos near the intersection points. The A set loses its stability and merges with the R set forming one invariant set Ω, which is the closure of the $W^u(P)$ unstable manifold and the P orbit (see a left-hand Fig. 4). It should be

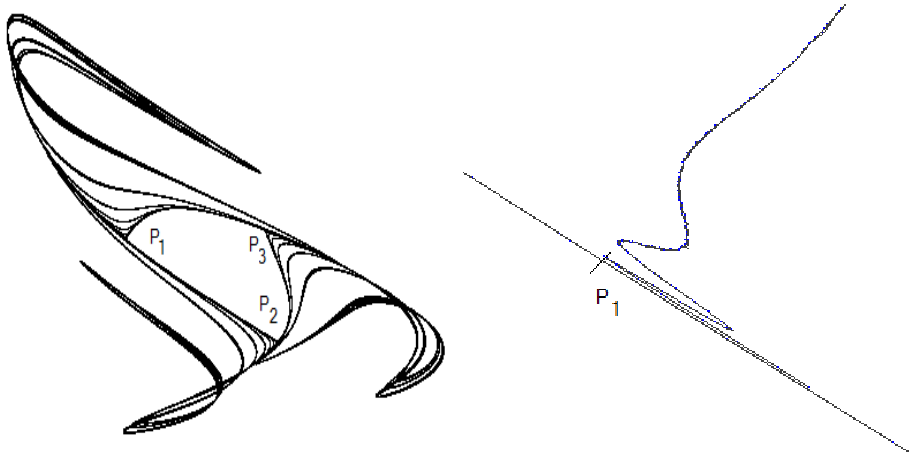

Figure 4: Chaos on the invariant layer $W(h = 0.515970)$, the closure of the unstable manifold $W^u(P)$. Intersection of unstable and stable manifolds of the 3-periodic orbit P .

noted that the orbit that starts near stable state $K^*(1.1977742, 1.1977742)$, reaches the Ω set and then wanders within it. The B (1.2, 1.2) point orbit is shown in a left-hand Fig. 5. The E entropy of the system in the Ω invariant set has been evaluated as an exponent (on base 2) of the curve length growth [6]. The estimate from below E = 0.69314. As the entropy is a measure of chaos, then we can say, that in the Ω set the system allows chaos. The orbit of the B point clearly shows this in the left-hand Fig.5.

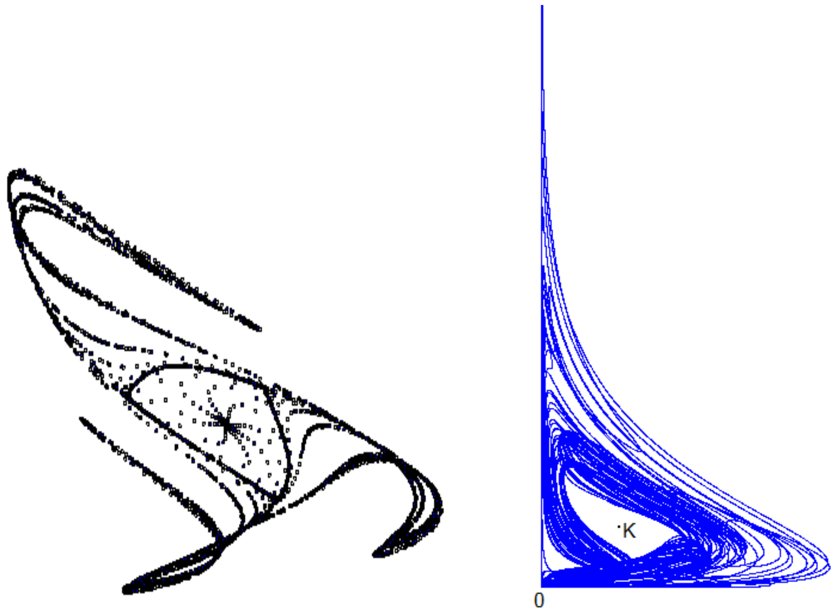

Figure 5: Orbit of the (1.2, 1.2) point on the invariant W (h = 0.515970) layer. Chaos in a W (h = 0.3) layer is in a scale of 1:10.

Further decrease of the h parameter leads to more chaos. Numerical experiments show that the system (2) has layers, where chaos in the attractors reaches the enormous size In the right-hand Fig. 5 a chaotic set in a W (h = 0.3) layer is presented in the scale 1:10. This set is ω-limited for any orbit, which starts near a K^* fixed point (1.3886835 1.3886835). Another picture of bifurcations is observed with the increase of h parameters.

The system (8) on a W (h) layer, h > 0.8 has a K^* stable balanced state up to h = 4.02787. At first the multipliers are complex, and then they become real. When h = 4.02787 the K^* balanced state loses its stability and becomes hyperbolic with one multiplier which is less than -1. Unstable manifold $W^u(K^*)$ of the hyperbolic point K^* (where the system changes its orientation) ends in 2-periodic stable orbit A. With the further increase of $h > 4$, the 2-periodic orbit A loses its stability and a period-doubling bifurcation occurs. The doubling period bifurcation is apparently repeated at sufficiently large h.

4 General case study

Let us consider the discrete system (2), where the m parameter takes positive values. The fixed points of the system are determined by the equalities $1 = xy^{m/s}z$, $y = z$. Thus, the curve of balanced states has the form

$$K = \{(x, y, z) : xy^{1+m/s} = 1, y = z\}.$$

It is not difficult to prove that the dynamical system with an m arbitrary has the same foliation with invariant layers, as for $m = 0$. This foliation is defined as the level surfaces of function

$$U = \frac{x^b}{y^a}.$$

Each surface

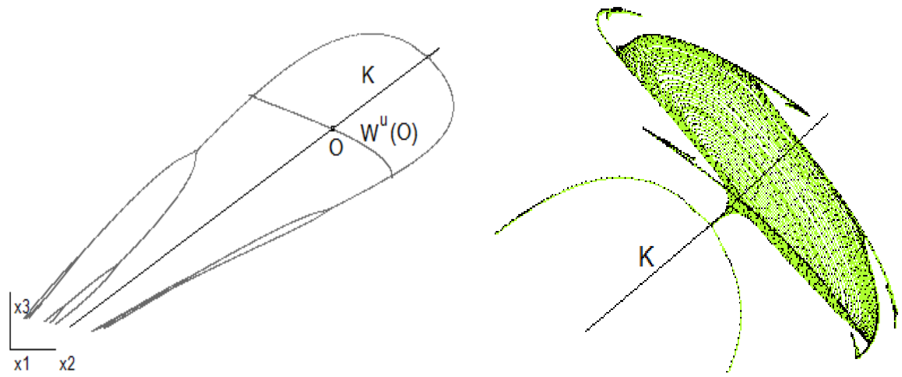

Figure 6: Dynamics near the balanced states K curve.

$$W = \{(x, y, z) : x = hy^{a/b}\}, h = const$$

is invariant for the system (2). The system on the W surface is specified in the form of

$$y_{n+1} = y_n \exp(b(1 - h y_n^{a/b+m/s} z_n)), \quad z_{n+1} = z_n \exp(c(y_n - z_n)), \tag{9}$$

where the value $h > 0$ specifies an invariant layer. Thus, the system (2) has the same foliation as in the case of $m = 0$, but the dynamics of the layers differs from the $m = 0$ case. Let us consider the differences that arise when $m > 0$. To be precise we will consider the system (2) when $a = 2.45$, $b = 0.6$, $c = 0.9$, $m = 0.25$, $s = 0.5$. The dynamics of a system near a curve of K balanced state is shown in Fig. 6. The $W = \{(x, y, z) : \frac{x^b}{y^a} = h\}$ invariant layer corresponds to each point on the K curve. The layers are transversal to the K curve. They are not shown in Fig. 6 but only the periodic orbits stable in the layers are. The O (1,1,1) balanced state is hyperbolic. The unstable manifold $W^u(O)$ which ends in 2-periodic stable orbit is shown in the left-hand Fig. 6. In $W^u(O)$ the system changes the orientation at each iteration. In the left-hand Fig. 6. it is seen that in the layers below the O point layer a period-doubling bifurcation arises. In the right-hand Fig. 6, the dynamics in layers above the layer passing through the O (1,1,1) balanced state is shown. The increase of y and z results in the bifurcation of balanced states (as well as in the case of $m = 0$). First, the balanced state becomes stable in the layer. This happens in $W(h = 0.3083)$, where the balanced state has (1.234603, 1.234603) coordinates. Then the complex multipliers in $W(h = 0.28)$ appear. Finally, the balanced state loses its stability in $W(h = 0.2311)$ and the the Neimark-Sacker bifurcation takes place. In the left-hand Fig. 7 the dynamics near the invariant stable ellipse in a W ($h = 0.18$) layer is shown. Here there is a K^* (1.359516, 1.359516) unstable balanced state, and a 5-hyperbolic periodic orbit H places near the ellipse, H_1 having (1.289692, 1.235095) coordinates. The unstable manifold $W^u(H)$ tends to a stable ellipse with one end and to the P 5-stable periodic orbit with the other, P_1 having (1.156926 1.338456) coordinates (see a left- hand Fig.7). The orbit, which starts near the K^* point, ends on the ellipse. Further reduction of h leads to the bifurcation of the invariant ellipse to the G set, which is shown in the right-hand Fig.7 for the W ($h = 0.155$) layer. The G invariant set is limiting for any orbit, starting near the K^* point. Estimating the entropy as an exponent of the curve length growth during iteration [6], the G chaotic invariant set can be shown. When h = 0.15075, the G limiting set of any orbit starting near the K^* point becomes large

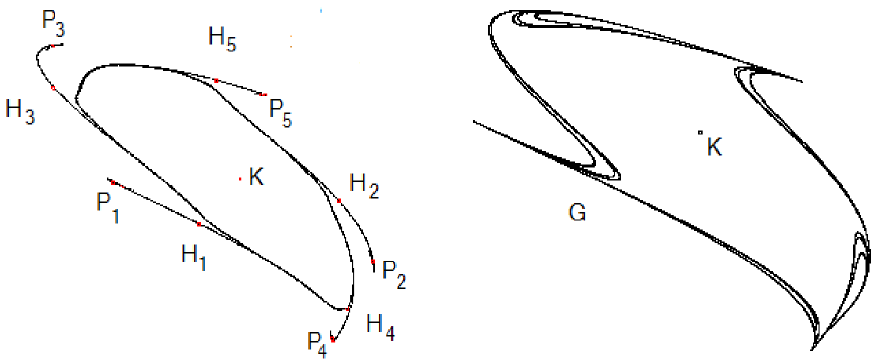

Figure 7: Dynamics near invariant ellipse in the $W(h = 0.18)$ layer. Invariant set G in the $W(h = 0.155)$ layer.

in size, the dynamics of the system being similar to the one shown in the right-hand Fig. 5.

5 Equivalence of systems with $m \geq 0$, and $m = 0$

Comparing the dynamics of the $m \geq 0$ general case, and the $m = 0$ particular case, one can notice a certain analogy. Indeed, there is a topological equivalence of these systems, however, only if the a parameter is changed.

Proposition 2. A mapping F of the form

$$\begin{aligned} X &= xy^{m/s}, \\ Y &= y, \\ Z &= z \end{aligned} \tag{10}$$

converts a discrete system

$$x_{n+1} = x_n \exp(a(1 - x_n y^{m/s} z_n)), \; y_{n+1} = y_n \exp(b(1 - x_n y^{m/s} z_n)), \; z_{n+1} = z_n \exp(c(y_n - z_n)) \tag{11}$$

into the system of the following type

$$X_{n+1} = X_n \exp(d(1 - X_n Z_n)),\quad Y_{n+1} = Y_n \exp(b(1 - X_n Z_n)),\quad Z_{n+1} = Z_n \exp(c(Y_n - Z_n)), \tag{12}$$

where $d = a + \frac{bm}{s}$.

Proof. It is necessary to show a commutative character of the diagram

$$\begin{tikzcd}
(x_n, y_n, z_n) \arrow[r] \arrow[d, "F"] & (x_{n+1}, y_{n+1}, z_{n+1}) \arrow[d, "F"] \\
(X_n, Y_n, Z_n) \arrow[r] & (X_{n+1}, Y_{n+1}, Z_{n+1})
\end{tikzcd}$$

or equality

$$\begin{aligned}
X_{n+1}(F(x_n, y_n, z_n)) &= F_x(x_{n+1}, y_{n+1}, z_{n+1}),\\
Y_{n+1}(F(x_n, y_n, z_n)) &= F_y(x_{n+1}, y_{n+1}, z_{n+1}),\\
Z_{n+1}(F(x_n, y_n, z_n)) &= F_z(x_{n+1}, y_{n+1}, z_{n+1}).
\end{aligned} \tag{13}$$

For the first equation we have

$$X_{n+1} = X_n \exp(d(1 - X_n Z_n)) = x_n y_n^{m/s} \exp((a + bm/s)(1 - x_n y_n^{m/s} z_n)) =$$

$$x_n \exp(a(1 - x_n y_n^{m/s} z_n)) y_n^{m/s} \exp((bm/s)(1 - x_n y_n^{m/s} z_n)) = x_{n+1} y_{n+1}^{m/s}.$$

For the second equation we have

$$Y_{n+1} = Y_n \exp(b(1 - X_n Z_n)) = y_n \exp(b(1 - x_n y_n^{m/s} z_n)) = y_{n+1}.$$

The last equation is trivial, as the F mapping is identical in y and z. The proving has been completed.

Thus, the systems (11) and (12) are topologically equivalent. That is why the system (11) for $a = 2.45$, $b = 0.6$, $c = 0.9$, $m = 0.25$, $s = 0.5$ is equivalent to the system (12), where $d = 2.75$, $b = 0.6$, $c = 0.9$. The system (11) has foliation with the invariant layers of the form

$$W(h) = \{(x, y, z) : h = \frac{x^b}{y^a}\},$$

but the system (12) has foliation with the invariant layers of the form

$$W(H) = \{(X, Y, Z) : H = \frac{X^b}{Y^d}\}.$$

We can show that the F mapping converts $W(h)$ layer i n the $W(H)$ one, both systems coinciding on these layers. Taking into account that $x = P/P_e$, $y = (r_e/r^s)$, P are price levels, r is the interest rate, $X = P/P_e((r_e/r)^s)^{m/s} = P/r^m\ r_e^m/P_e$, i.e. from economic point of view the X coordinate is propor- tional to the price level and inversely proportional to the interest rate in m power.

6 Uncontrolled perturbation of the system

Usually an economic system is subject to uncontrolled and random perturbation. In this section we will examine the dynamics of a perturbed system of the (3) type, where ε is a small positive number

$$q_i(n + 1) = 1 - 2q_i^2(n), \tag{14}$$

Initial values $q_i(0) \notin \cancel{1}, 1]$ are specified for each $i = 1, 2, 3$ at random. Thus the 6-dimensional system consisting of the system equations (3) and equations (14), $i = 1, 2, 3$ is analysed. It should be expected that the described perturbations do not preserve invariant foliation. In each invariant layer of (2) there is an attractor with a certain area of attraction. The above described results show that the attractors are formed out of the stable balanced states of the K curve, and the loss of stability results in appearance of the attractors which are changing continuously from layer to layer. Layer-by-layer integration of such attractors creates a set which does not disappear at perturbation. This can be seen in the right-hand Fig. 8, where the (1,1,1) point orbit of the perturbed system (3) is shown for the $a = 2.45$, $b = 0.6$, $c = 0.9$, $m = 0$ parameters and with the $\varepsilon = 0.01$ chaotic disturbing values.

It should be noted that the chaotic perturbation causes the orbit to move up and down near the attractors of the unperturbed system. The perturbation can not only transport the orbit into the chaotic region (see the upper part of the right-hand Fig.8), but also turns it back from this area. From the economic point of view there are perturbations, which can be reduced to

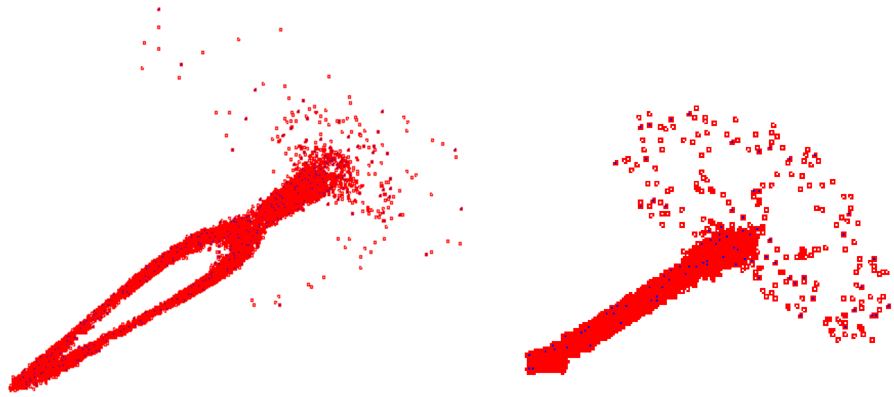

Figure 8: Orbit of (1,1,1) point of the perturbed system (3) $\varepsilon = 0.01$. Orbit of (1,1,1) point of the system(15) with $\varepsilon_1 = 0.01,\ \varepsilon_{2,3} = 0$.

zero or be made insignificant, but there are perturbations, which we can not reduce to a significant extent. For example, the Central Bank can control the interest rate to avoid chaotic perturbations. But we can not get rid of chaotic perturbations of the price level. Thus, it is desirable to find out, what perturbations influence the system dynamics greatly and which are not. For this purpose let us consider the system of equations of the following type

$$
\begin{aligned}
x_{n+1} &= x_n \exp((1 - x_n y_n^{m/s} z_n)) + \varepsilon_1 q_1(n), \\
y_{n+1} &= y_n \exp(-b(1 - x_n y_n^{m/s} z_n)) + \\
&\varepsilon_2 q_2(n),\ z_{n+1} = z_n \exp(c(y_n - z_n)) + \\
&\varepsilon_3 q_3(n),
\end{aligned}
\tag{15}
$$

where ε_1, ε_2 and ε_3 are different, and q_i obey system (14). It should be noted that the perturbation of the last equation preserves invariant layers, because each layer is a ruled surface parallel to Z axis. Thus, the perturbations of the $\varepsilon_1 = 0,\ \varepsilon_2 = 0$ type and $\varepsilon_3 = 0$ preserve layers of the unperturbed system and perturb it in a layer. This means that the perturbation of the national income (production) does not change significantly the dynamics near the attractors on the layers. Weak monitoring of the price levels leads to the perturbation of the first equation. In the right-hand Fig. 8 the

(1,1,1) point orbit of the system (15) with the ε_1 = 0.01, $\varepsilon_{2,3}$ = 0 is shown. The results of the numerical calculation show that the price level perturbation leads to the growth of y and z (national income). The increase of y means the interest rates reduction. It should be noted, that the strongest bifurcation toward chaos occurs at the weak control of the interest rate. Numerical experiments for the system (15) with $\varepsilon_2 = 0.01$ $\varepsilon_1 = \varepsilon_3 = 0$, show that the behavior of the solutions of such a system practically does not differ from the perturbed system of a general type (3) with $\varepsilon = 0.01$ (see the left-hand Fig. 8).

Small perturbations of a general form leads to the fact, that the orbit starts moving along balanced states and first gets into an unstable balanced state, and then into the layer with chaos. The magnitude of the chaos may both increase and decrease while perturbing.

7 Conclusion

The study of a discrete macroeconomic model (2) has been conducted. It has been shown that the system (2) with $m > 0$ is topologically equivalent to the system (2) with $m = 0$. It has a curve filled with a balanced state, and transversally to the curve there are invariant level surfaces of the $U = \frac{x^b}{u^a}$ function which form foliation.

We can assume that a balanced state is the center of each invariant surface. There is an attractor on each layer which almost all orbits tend to. The attractor can be a balanced state or has a rather complicated (chaotic) structure. When a level surface changes, bifurcation of the system dynamics from a steady state to chaos takes place. It should be noted that chaos in the macroeconomic model is an intrinsic characteristic of the system, and it does not always result in economic crisis. In this case, chaos means the impossibility of long-term forecasting. The numerical results have shown that there are layers, where chaos reaches the enormous degree. Then the imbalance of the economic system takes place and a crisis breaks out. Minor external perturbation may destroy the above described topological structure of the system orbits. Numerical experiments and economic practice shows that not all perturbations influence the dynamics of the system equally. So, the national income perturbation (z) does not change the invariant foliation, the level of prices perturbation (from x) leads to weak changes in dynamics preserving the attractors of the layers. The perturbation of the interest rate influences the system dynamics most significantly. Minor perturbations lead to the fact that the orbit starts moving along balanced states and gets first in the unstable balanced state and then into the layer, where there is chaos which can assume enormous proportions.

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Osipenko G.S. Dynamics of price-level, national income and cost of money interaction [Текст] / G.S. Osipenko, T.N. Korzh, E.K. Ershov //Моделирование, управление и устойчивость (MCS-2012): межд. конф.(10-14 сентября 2012 г.). - Симферополь: ДИАЙПИ, 2012. - С. 158-159.