Temperature effects on drift of suspended single-domain particles induced by the Magnus force

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We study the temperature dependence of the drift velocity of single-domain ferromagnetic particles induced by the Magnus force in a dilute suspension. A set of stochastic equations describing the translational and rotational dynamics of particles is derived, and the particle drift velocity that depends on components of the average particle magnetization is introduced. The Fokker-Planck equation for the probability density of magnetization orientations is solved analytically in the limit of strong thermal fluctuations for both the planar rotor and general models. Using these solutions, we calculate the drift velocity and show that the out-of-plane fluctuations of magnetization, which are not accounted for in the planar rotor model, play an important role. In the general case of arbitrary fluctuations, we investigate the temperature dependence of the drift velocity by numerically simulating a set of effective stochastic differential equations for the magnetization dynamics.

I. INTRODUCTION

Recently, the phenomenon of drift motion of single-domain ferromagnetic particles suspended in a viscous fluid has been analytically predicted and numerically confirmed [1,2]. This phenomenon arises from the Magnus effect, and drift of particles occurs if the external driving force, which induces their oscillatory motion, is properly synchronized with the rotating magnetic field, which induces their nonuniform rotation. The main characteristic of the drift motion, the particle drift velocity, has been calculated both analytically and numerically. Because the magnitude and direction of this velocity are easy to operate, the drift phenomenon has been proposed to separate large single-domain particles from suspensions.

The approach used in [1,2] was purely deterministic, i.e., thermal fluctuations in the translational and rotational dynamics of particles were completely ignored. According to [1], this is approximately true if the particle size exceeds a few tens or even hundreds of nanometers. At the same time, to be single domain, the particle size must be less than a critical one, which usually does not exceed hundreds of nanometers (see, e.g., Refs. [3,4]). As a consequence, the applicability of the deterministic approach is restricted only to relatively large single-domain particles whose size is about 100 nm. Therefore, in order to describe the drift motion of smaller particles, thermal fluctuations must be taken into account.

One of the most powerful tools to study the influence of thermal fluctuations on the behavior of dynamical systems is a set of stochastic differential equations (e.g., Langevin equations) in which the action of these fluctuations is modeled by Gaussian white noises [5–7]. The main advantage of these equations is that their solution is a Markov process, whose probability density function obeys the Fokker-Planck equation (see also Refs. [8,9]). Within this framework, the stochastic magnetization dynamics in fixed single-domain particles was first investigated by Brown [10]. To describe this dynamics, he introduced the stochastic Landau-Lifshitz-Gilbert equation, in which thermal fluctuations are accounted for by a random magnetic field with Gaussian white-noise components, and derived the corresponding Fokker-Planck equation. This approach (based on the stochastic Landau-Lifshitz-Gilbert or Landau-Lifshitz equations) has become a standard tool for studying the magnetization dynamics in nanosystems [11]. In particular, it was employed to study the influence of uniformly rotating magnetic field on the nanoparticle magnetization [12,13], magnetic relaxation [14,15], and precessional states of the magnetization [16].

A set of stochastic differential equations for the rotational motion of ferromagnetic particles in a viscous fluid depends on whether the magnetization vector is frozen into the particle body (see Ref. [7], and references therein) or not [17–20]. In the former case, which holds if the anisotropy magnetic field noticeably exceeds the external one, the rotational dynamics of particles satisfies Newton's second law for rotation with total torque containing Gaussian white-noise contributions. This approach is widely used in studying the role of thermal fluctuations in the rotational dynamics of suspended particles [7] (about the constructive role of these fluctuations see, e.g., Refs. [21–23]). But the role of temperature in the drift motion of single-domain ferromagnetic particles induced by the Magnus force has never been investigated before. Therefore, to fill this gap, in this work we develop a statistical theory of drift of ferromagnetic particles and determine the dependence of the particle drift velocity on temperature.

The outline of the paper is as follows. In Sec. II, we derive a set of stochastic equations that describes the translational and rotational dynamics of single-domain ferromagnetic particles in a viscous fluid and accounts for the influence of thermal fluctuations. The particle drift velocity, which is induced by the Magnus force and depends on the average particle magnetization, is introduced in Sec. III. To find the average magnetization, in Sec. IV we obtain the Fokker-Planck equation for the probability density function of magnetization orientations. In the same section, using this equation we

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propose a set of effective stochastic differential equations describing the magnetization dynamics, which is much simpler than the original one. In Sec. V, in the case of strong thermal fluctuations we find analytically the steady-state solutions of the Fokker-Planck equations, which correspond to the planar rotor and general models, and on this basis calculate the drift velocity as a function of temperature for these models. In Sec. VI, we examine our theoretical predictions numerically and present numerical results on the temperature dependence of the drift velocity of Co nanoparticles in water, obtained by solving a set of effective stochastic equations. Finally, our main findings are summarized in Sec. VII.

II. SET OF STOCHASTIC EQUATIONS OF MOTION

In our model, the ferromagnetic particles moving in a viscous fluid are considered to be spherical and smooth. Their radius a is assumed to be so small that the single-domain state is realized, and thus the particle magnetization $\mathbf{M}$ is a function of time only: $\mathbf{M} = \mathbf{M}(t)$. If the anisotropy magnetic field is large enough, then the vector $\mathbf{M}$ is approximately parallel to the anisotropy axis. In this approximation the magnetization is frozen into the particle body, and hence its dynamics is governed by the kinematic differential equation $d\mathbf{M}/dt = \boldsymbol{\omega} \times \mathbf{M}$, where $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ is the angular particle velocity. We also assume that the translational and rotational Reynolds numbers, Re_t and Re_r, are small compared to 1 (since max a usually does not exceed a few hundreds of nanometers [3,4], this condition is not very restrictive for suspended particles). In this case, the inertial effects in the particle dynamics can be neglected and, as a consequence, the equations for the translational and rotational motions of particles are reduced to the force and torque balance equations, $\mathbf{F} = 0$ and $\mathbf{T} = 0$, respectively.

Neglecting interactions between particles, we include in the total force $\mathbf{F}$ acting on the particle the external driving force $\mathbf{f}_d = \mathbf{f}_d(t)$, the friction force $\mathbf{f}_f = \mathbf{f}_f(t)$, the Magnus lift force $\mathbf{f}_l = \mathbf{f}_l(t)$, and the random force $\mathbf{f} = \mathbf{f}(t)$. We choose the driving force in the form $\mathbf{f}_d = f_m \sin(\Omega t - \phi)\mathbf{e}_x$, where f_m, Ω, and ϕ are, respectively, the amplitude, angular frequency, and initial phase of the force, and $\mathbf{e}_x$ is the unit vector along the x axis of the Cartesian coordinate system. Under the above assumptions, the friction force is determined by the Stokes law, $\mathbf{f}_f = -6\pi\eta a\mathbf{v}$ [η is the dynamic viscosity of the fluid and $\mathbf{v} = \mathbf{v}(t)$ is the linear particle velocity], and the Magnus force is given by [24] $\mathbf{f}_l = \pi\rho a^3\boldsymbol{\omega} \times \mathbf{v}$ (ρ is the fluid density and the sign $\times$ denotes the vector product). Finally, we approximate the random force that accounts for thermal fluctuations by a Gaussian white noise with zero mean and correlation function (see, e.g., Ref. [7])

$$\langle f_i(t)f_j(t')\rangle = 2\Delta_1\delta_{ij}\delta(t - t'). \quad (2.1)$$

Here, $f_i(t)$ ($i = x, y, z$) are the Cartesian components of the random force, the angular brackets denote averaging over all possible realizations of $\mathbf{f}(t)$, $\Delta_1 = 6\pi\eta ak_BT$ is the noise intensity, k_B is the Boltzmann constant, T is the absolute temperature, δ_{ij} is the Kronecker symbol, and $\delta(\tau)$ is the Dirac δ function. Thus, in the considered case the force balance equation $\mathbf{f}_d + \mathbf{f}_f + \mathbf{f}_l + \mathbf{f} = 0$ is stochastic and can be written in the form

$$\mathbf{v} - \frac{\mathrm{Re}_r}{6\omega_m}\boldsymbol{\omega} \times \mathbf{v} = v_m \sin(\Omega t - \phi)\mathbf{e}_x + \frac{1}{6\pi\eta a}\mathbf{f}, \quad (2.2)$$

where the rotational Reynolds number is defined as $\mathrm{Re}_r = \rho a^2\omega_m/\eta$, $\omega_m = \max|\boldsymbol{\omega}|$, and $v_m = f_m/6\pi\eta a$. (For completeness, the translational Reynolds number is defined as $\mathrm{Re}_t = \rho av_m/\eta$.)

Next, we assume that the suspended particles are subjected to a homogeneous external magnetic field $\mathbf{H} = \mathbf{H}(t)$. Since the torque exerted on the particle magnetic moment $V\mathbf{M}$ ($V = 4\pi a^3/3$ is the particle volume) by this field equals $V\mathbf{M} \times \mathbf{H}$ and the magnetization is considered to be frozen, the mechanical torque $\mathbf{t}_m$ induced by the magnetic field is given by $\mathbf{t}_m = V\mathbf{M} \times \mathbf{H}$. In addition to this contribution, the total torque $\mathbf{T}$ acting on the particle includes also the frictional torque $\mathbf{t}_f = -8\pi\eta a^3\boldsymbol{\omega}$ and the random torque $\mathbf{t} = \mathbf{t}(t)$, which is approximated by another Gaussian white noise with $\langle t_i(t)\rangle = 0$ and

$$\langle t_i(t)t_j(t')\rangle = 2\Delta_2\delta_{ij}\delta(t - t'), \quad (2.3)$$

where $\Delta_2 = 6\eta Vk_BT$ is the intensity of this noise [7]. Therefore, the torque balance equation takes the form $\mathbf{t}_m + \mathbf{t}_f + \mathbf{t} = 0$, and its solution with respect to the particle angular velocity yields

$$\boldsymbol{\omega} = \frac{1}{6\eta}\mathbf{M} \times \mathbf{H} + \frac{1}{6\eta V}\mathbf{t}. \quad (2.4)$$

Substituting this expression for the angular velocity into the kinematic equation, we obtain the following stochastic differential equation for the particle magnetization:

$$\frac{d}{dt}\mathbf{M} = -\frac{1}{6\eta}\mathbf{M} \times (\mathbf{M} \times \mathbf{H}) - \frac{1}{6\eta V}\mathbf{M} \times \mathbf{t}. \quad (2.5)$$

Thus, in our model, the translational and rotational motions of suspended ferromagnetic particles are described by the stochastic equations (2.2), (2.4), and (2.5). They show that, while the particle rotation influences (due to the Magnus force) the translational motion, the rotational motion does not depend on the translational one. Therefore, to find statistical characteristics of the translational motion of these particles, including their drift velocity, we must first determine the rotational properties that are described by Eqs. (2.4) and (2.5).

A. Set of stochastic equations for the magnetization dynamics

As a first step, let us rewrite Eq. (2.5) in the dimensionless form. Introducing the dimensionless time $\tau = \Omega t/2\pi$, magnetization $\mathbf{m} = \mathbf{m}(\tau) = \mathbf{M}(2\pi\tau/\Omega)/M$, external magnetic field $\mathbf{h} = \mathbf{h}(\tau) = \mathbf{H}(2\pi\tau/\Omega)/H_m$ ($H_m = \max|\mathbf{H}|$), and Gaussian white noise $\boldsymbol{\xi} = \boldsymbol{\xi}(\tau) = \mathbf{t}(2\pi\tau/\Omega)/\sqrt{4\eta a^3\Omega k_BT}$, one obtains

$$\dot{\mathbf{m}} = -\alpha\mathbf{m} \times (\mathbf{m} \times \mathbf{h}) - \beta\mathbf{m} \times \boldsymbol{\xi}. \quad (2.6)$$

Here, the overdot denotes the derivative with respect to τ, the mean and correlation function of $\boldsymbol{\xi}$ are given by

$$\langle \xi_i(\tau)\rangle = 0, \qquad \langle \xi_i(\tau)\xi_j(\tau')\rangle = 2\delta_{ij}\delta(\tau - \tau'), \quad (2.7)$$

and the dimensionless parameters α and β are as follows:

$$\alpha = \frac{\pi MH_m}{3\eta\Omega}, \qquad \beta = \left(\frac{k_BT}{4\eta a^3\Omega}\right)^{1/2}. \quad (2.8)$$

According to Eq. (2.6), the parameter α can be associated with the inverse rotational relaxation time, and the parameter β characterizes the magnitude of thermal fluctuations. Note also that in these variables the particle angular velocity (2.4) reads

$$\boldsymbol{\omega} = \frac{\Omega}{2\pi}(\alpha \mathbf{m} \times \mathbf{h} + \beta \boldsymbol{\xi}). \quad (2.9)$$

Next, we represent the unit vector $\mathbf{m}$ in the form

$$\mathbf{m} = \sin\theta \, \cos\varphi \, \mathbf{e}_x + \sin\theta \, \sin\varphi \, \mathbf{e}_y + \cos\theta \, \mathbf{e}_z, \quad (2.10)$$

where $\theta = \theta(\tau)$ and $\varphi = \varphi(\tau)$ are, respectively, the polar and azimuthal angles of $\mathbf{m}$. In addition, we assume that the external magnetic field has a fixed magnitude $|\mathbf{H}| = H_m$ ($|\mathbf{h}| = 1$), lies in the xy plane, and nonuniformly rotates around the x axis, i.e.,

$$\mathbf{h} = \cos\psi \, \mathbf{e}_x + \sin\psi \, \mathbf{e}_y, \quad (2.11)$$

where $\psi = \psi(\tau)$ is the azimuthal angle of $\mathbf{h}$. Now, substituting (2.10) and (2.11) into Eq. (2.6), we get the following set of stochastic equations for the polar and azimuthal angles of the magnetization vector:

$$\dot{\theta} = \alpha \, \cos\theta \, \cos(\psi - \varphi) - \beta(\xi_x \sin\varphi - \xi_y \cos\varphi),$$

$$\dot{\varphi} = \alpha \frac{\sin(\psi - \varphi)}{\sin\theta} - \beta \, \cot\theta(\xi_x \cos\varphi + \xi_y \sin\varphi) + \beta\xi_z. \quad (2.12)$$

Here, the angle ψ is a given function of the dimensionless time τ, which, similar to the driving force, is assumed to satisfy the antisymmetry condition $\psi(\tau + 1/2) = -\psi(\tau)$. It should also be stressed that this condition implies that $\psi(\tau)$ is a periodic function with period 1.

For convenience, let us introduce a new variable, the lag angle $\chi = \psi - \varphi$, instead of the azimuthal angle φ. Then Eqs. (2.12) can be rewritten in the form

$$\dot{\theta} = \alpha \, \cos\theta \, \cos\chi - \beta[\xi_x \sin(\psi - \chi) - \xi_y \cos(\psi - \chi)],$$

$$\dot{\chi} = \dot{\psi} - \alpha \frac{\sin\chi}{\sin\theta} + \beta \, \cot\theta$$
$$\times [\xi_x \cos(\psi - \chi) + \xi_y \sin(\psi - \chi)] - \beta\xi_z. \quad (2.13)$$

It is these stochastic differential equations that we will use to study the effect of thermal fluctuations on the drift of particles.

B. Random particle velocity

In principle, Eq. (2.2) can be solved exactly with respect to the dimensionless particle velocity $\mathbf{u} = \mathbf{u}(\tau) = \mathbf{v}(2\pi\tau/\Omega)/v_m$. However, since the Magnus force is calculated with linear accuracy in Reynolds numbers [24], the solution of this equation should be determined with the same accuracy. Therefore, representing $\mathbf{u}$ in the form $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1$, where $|\mathbf{u}_0| \sim 1$ and $|\mathbf{u}_1| \sim \mathrm{Re}_r$, from Eq. (2.2) we immediately find

$$\mathbf{u}_0 = \sin(2\pi\tau - \phi)\mathbf{e}_x + \sigma \boldsymbol{\nu} \quad (2.14)$$

and

$$\mathbf{u}_1 = \frac{\mathrm{Re}_r}{6\omega_m}\boldsymbol{\omega} \times \mathbf{u}_0. \quad (2.15)$$

Here, $\sigma = \sqrt{\Omega k_B T/2\pi f_m v_m}$ is the dimensionless parameter characterizing the magnitude of the random force fluctuations,

and $\boldsymbol{\nu} = \boldsymbol{\nu}(\tau) = \mathbf{f}(2\pi\tau/\Omega)/\sqrt{3\eta a\Omega k_B T}$ is a dimensionless Gaussian white noise with

$$\langle \nu_i(\tau) \rangle = 0, \qquad \langle \nu_i(\tau)\nu_j(\tau') \rangle = 2\delta_{ij}\delta(\tau - \tau'). \quad (2.16)$$

Thus, according to (2.14) and (2.15), the random particle velocity can be written as

$$\mathbf{u} = \left(\mathbf{e}_x + \frac{\mathrm{Re}_r}{6\omega_m}\boldsymbol{\omega} \times \mathbf{e}_x\right)\sin(2\pi\tau - \phi)$$
$$+ \sigma\left(\boldsymbol{\nu} + \frac{\mathrm{Re}_r}{6\omega_m}\boldsymbol{\omega} \times \boldsymbol{\nu}\right). \quad (2.17)$$

It is important to note that expression (2.17) is valid only if the particle angular velocity $\boldsymbol{\omega}$ does not contain a white-noise contribution. The reason is that the multiplication of generalized functions, including Gaussian white noises, is not mathematically defined (for more details see, e.g., Ref. [5]). Therefore, if $\boldsymbol{\omega}$ is given by (2.9), then the particle velocity (2.17) is, in general, not appropriate for studying the stochastic translational motion of suspended particles (since $\mathbf{u}$ contains the term proportional to $\boldsymbol{\xi} \times \boldsymbol{\nu}$). However, expression (2.17) can be used to find the drift velocity of these particles (see just below).

III. PARTICLE DRIFT VELOCITY

In order to derive the expression for the particle drift velocity, let us assume for the moment that the angular velocity is governed by the stochastic differential equation $J d\boldsymbol{\omega}/dt = \mathbf{T}$ (J is the moment of inertia of the particle). In this case, the above problem does not occur and the particle velocity (2.17) can be averaged with respect to realizations of white noises $\boldsymbol{\xi}$ and $\boldsymbol{\nu}$. Taking into account that $\langle \boldsymbol{\nu} \rangle = 0$ and, since components of these noises are statistically independent, the condition $\langle \boldsymbol{\omega} \times \boldsymbol{\nu} \rangle = \langle \boldsymbol{\omega} \rangle \times \langle \boldsymbol{\nu} \rangle = 0$ holds, we obtain

$$\langle \mathbf{u} \rangle = \left(\mathbf{e}_x + \frac{\mathrm{Re}_r}{6\omega_m}\langle \boldsymbol{\omega} \rangle \times \mathbf{e}_x\right)\sin(2\pi\tau - \phi). \quad (3.1)$$

Now, using the inertialess approximation ($J \to 0$) that leads to the torque balance equation $\mathbf{T} = 0$, from (3.1) and (2.9) one finds the following expression for the instantaneous mean particle velocity:

$$\langle \mathbf{u} \rangle = [\mathbf{e}_x - \gamma \mathbf{e}_x \times (\langle \mathbf{m} \rangle \times \mathbf{h})]\sin(2\pi\tau - \phi), \quad (3.2)$$

where the dimensionless parameter

$$\gamma = \frac{\mathrm{Re}_r\alpha\Omega}{12\pi\omega_m} = \frac{\rho a^2 M H_m}{36\eta^2} \quad (3.3)$$

($\gamma \ll 1$) characterizes the contribution of the Magnus force to $\langle \mathbf{u} \rangle$. Expression (3.2) shows that thermal fluctuations influence the mean particle velocity only through the dimensionless mean magnetization $\langle \mathbf{m} \rangle$ of the particle.

We define the dimensionless drift velocity $\langle \mathbf{s} \rangle$ of the particle as its steady-state displacement during one magnetic field period, i.e., $\langle \mathbf{s} \rangle = \lim_{n\to\infty} \int_n^{n+1} \langle \mathbf{u}(\tau) \rangle d\tau$ (n is a whole number). In order to find an explicit expression for $\langle \mathbf{s} \rangle$, we first note that, according to (2.10) and (2.11),

$$\mathbf{e}_x \times (\langle \mathbf{m} \rangle \times \mathbf{h}) = -\langle \sin\theta \, \sin\chi \rangle \mathbf{e}_y + \langle \cos\theta \rangle \cos\psi \, \mathbf{e}_z. \quad (3.4)$$

Then, putting $\tau = n + \xi$ with $\xi \in (0, 1)$, one can write $\lim_{n\to\infty}\langle \mathbf{u}(n + \xi) \rangle = \langle \mathbf{u}_{\mathrm{st}}(\xi) \rangle$, where the subscript "st" refers

to the steady state. Finally, using the symmetry-caused condition $\langle \cos \theta_{\rm st}(\xi) \rangle = 0$ (for its proof see the next section) and the above results, one obtains $\langle \mathbf{s} \rangle = \langle s_y \rangle \mathbf{e}_y$ and

$$\langle s_y \rangle = \gamma \int_0^1 \langle \sin \theta_{\rm st}(\xi) \sin \chi_{\rm st}(\xi) \rangle \sin (2\pi \xi - \phi) \, d\xi. \quad (3.5)$$

Thus, the drift motion of particles induced by the Magnus force occurs along the y axis with the time-independent dimensionless drift velocity (3.5). Note, this velocity is the dimensionless Magnus force averaged over the time period and thermal fluctuations. In general, it can be calculated by numerically solving a set of coupled stochastic equations (2.13) describing the magnetization dynamics. But to determine the drift velocity analytically, it is convenient to introduce the Fokker-Planck equation associated with these stochastic equations.

IV. FOKKER-PLANCK EQUATION FOR THE MAGNETIZATION DYNAMICS

The Fokker-Planck equation, which corresponds to a set of stochastic differential equations (2.13), can be obtained by standard techniques [5–9]. An important feature of these equations is that Gaussian white noises are multiplicative and, as a consequence, the statistical properties of their solution, $\theta(\tau)$ and $\varphi(\tau)$, depend on the interpretation of these noises. Using an arbitrary interpretation, in Ref. [23] we have shown that noises $\xi_i(\tau)$ must be interpreted in the Stratonovich sense [25] (to get Boltzmann distribution in equilibrium) and have derived the corresponding Fokker-Planck equation in the case of uniformly rotating magnetic field. A simple generalization of our approach to the case of nonuniformly rotating magnetic field leads to the Fokker-Planck equation which, in the operator form, can be written as

$$\frac{\partial}{\partial \tau} P + \dot{\psi} \frac{\partial}{\partial \chi} P = \hat{L} P. \quad (4.1)$$

Here, $P = P(\theta, \chi, \tau)$ is the probability density that $\theta(\tau) = \theta$ ($0 \leqslant \theta \leqslant \pi$) and $\chi(\tau) = \chi$ ($0 \leqslant \chi < 2\pi$), i.e.,

$$P = \langle \delta[\theta - \theta(\tau)] \, \delta[\chi - \chi(\tau)] \rangle, \quad (4.2)$$

and

$$\hat{L} P = - \alpha \cos \chi \frac{\partial}{\partial \theta} \cos \theta P + \alpha \frac{1}{\sin \theta} \frac{\partial}{\partial \chi} \sin \chi P$$

$$- \beta^2 \frac{\partial}{\partial \theta} \cot \theta P + \beta^2 \frac{\partial^2}{\partial \theta^2} P + \beta^2 \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \chi^2} P. \quad (4.3)$$

As usually, P should satisfy the normalization condition $\int_0^\pi \int_0^{2\pi} P \, d\theta \, d\chi = 1$ and the initial condition $P(\theta, \chi, 0) = \delta(\theta - \theta_0) \, \delta(\chi - \chi_0)$, where $\theta_0 = \theta(0)$ and $\chi_0 = \chi(0)$.

Now, let us prove that $\langle \cos \theta_{\rm st}(\xi) \rangle = 0$. To this end, we first define the steady-state probability density as $P_{\rm st}(\theta, \chi, \xi) = \lim_{n \to \infty} P(\theta, \chi, n + \xi)$ and represent $\langle \cos \theta_{\rm st}(\xi) \rangle$ in the form

$$\langle \cos \theta_{\rm st}(\xi) \rangle = \int_0^\pi \int_0^{2\pi} \cos \theta P_{\rm st}(\theta, \chi, \xi) \, d\theta \, d\chi. \quad (4.4)$$

Then, using the symmetry condition $P_{\rm st}(\theta, \chi, \xi) = P_{\rm st}(\pi - \theta, \chi, \xi)$, which follows from the Fokker-Planck equation (4.1), one can easily make sure that $\int_0^\pi \cos \theta P_{\rm st}(\theta, \chi, \xi) \, d\theta = 0$ and, hence, the condition $\langle \cos \theta_{\rm st}(\xi) \rangle = 0$ indeed holds.

Set of effective stochastic equations for the magnetization dynamics

Recall, the Fokker-Planck equation (4.1) corresponds to a set of stochastic differential equations (2.13), in which Gaussian white noises $\xi_i(\tau)$ are interpreted in the Stratonovich sense. It is not difficult to verify (see, e.g., Ref. [23]) that the same Fokker-Planck equation corresponds also to the following set of effective stochastic differential equations:

$$\dot{\theta} = \alpha \cos \theta \cos \chi + \beta^2 \cot \theta + \sqrt{2} \, \beta \zeta_1,$$

$$\dot{\chi} = \dot{\psi} - \alpha \frac{\sin \chi}{\sin \theta} - \sqrt{2} \, \beta \frac{1}{\sin \theta} \zeta_2, \quad (4.5)$$

where Gaussian white noises $\zeta_i = \zeta_i(\tau)$ ($i = 1, 2$) are interpreted in the Itô sense [26] and are characterized by

$$\langle \zeta_i(\tau) \rangle = 0, \qquad \langle \zeta_i(\tau) \zeta_j(\tau') \rangle = \delta_{ij} \delta(\tau - \tau'). \quad (4.6)$$

This means that solutions of Eqs. (2.13) and (4.5) with respect to the polar and lag angles $\theta(\tau)$ and $\chi(\tau)$ have the same probability density $P(\theta, \chi, \tau)$. The word "effective" is used here to emphasize that a set of Eqs. (4.5) (i) is not a set of ordinary stochastic differential equations (because the noiseless term $\beta^2 \cot \theta$ arises, nevertheless, from thermal fluctuations) and (ii) is much simpler than a set of basic equations (2.13).

An interesting property of Eqs. (4.5) that follows from the results of Ref. [27] is that, in spite of the multiplicative character of the noise ζ_2, the statistical characteristics of their solution do not depend on interpretation of noises ζ_i [23]. Note also that sets of effective stochastic differential equations describing the rotational dynamics of ferromagnetic particles in a viscous fluid and the magnetization dynamics in ferromagnetic particles embedded into a solid matrix have been proposed in Refs. [28] and [15], respectively.

V. ANALYTICAL RESULTS: STRONG THERMAL FLUCTUATIONS

Because small thermal fluctuations do not affect strongly the deterministic drift of particles, which was studied in detail in Refs. [1,2], below we analytically study the role of large thermal fluctuations only. For this purpose, we consider the stochastic particle dynamics within the framework of the planar rotor model and general model.

A. Planar rotor model

Within this model, the magnetization rotation is assumed to occur only in the xy plane. This rotation is described by the second equation in (4.5) with $\theta = \pi/2$, i.e.,

$$\dot{\chi} = \dot{\psi} - \alpha \sin \chi - \sqrt{2} \, \beta \zeta_2. \quad (5.1)$$

The Fokker-Planck equation associated with Eq. (5.1) can be written as follows [cf. Eq. (4.1)]:

$$\frac{\partial}{\partial \tau} P + \dot{\psi} \frac{\partial}{\partial \chi} P = \hat{L}_\perp P, \quad (5.2)$$

where $P = P(\chi, \tau)$ is the probability density that $\chi(\tau) = \chi$ and

$$\hat{L}_\perp P = \alpha \frac{\partial}{\partial \chi} \sin \chi P + \beta^2 \frac{\partial^2}{\partial \chi^2} P. \quad (5.3)$$

In equilibrium, when $\dot{\psi} = 0$ [$\psi(\tau) = $ const] and $\tau \to \infty$, i.e., when the left-hand side of the Fokker-Planck equation (5.2) equals zero, the equilibrium probability density $P_{\text{eq}}(\chi)$ satisfies the equation $\hat{L}_{\perp} P_{\text{eq}}(\chi) = 0$. It can easily be verified that its normalized solution on any interval of length 2π [e.g., $\chi \in [0, 2\pi]$] is the von Mises probability density function

$$P_{\text{eq}}(\chi) = \frac{e^{\epsilon \cos \chi}}{2\pi I_0(\epsilon)}, \qquad (5.4)$$

where the dimensionless parameter

$$\epsilon = \frac{\alpha}{\beta^2} = \frac{M H_m V}{k_B T} \qquad (5.5)$$

is the ratio of magnetic to thermal energy and $I_0(\epsilon)$ is the modified Bessel function of the first kind and order zero. Note, since the von Mises distribution is the circular analog of the normal distribution, it is widely used in various applications (see, e.g., Refs. [29,30]).

As follows from (3.5), in the planar rotor model the drift velocity of ferromagnetic particles is given by

$$\langle s_y \rangle = \gamma \int_0^1 \langle \sin \chi_{\text{st}}(\xi) \rangle \sin (2\pi \xi - \phi) d\xi \qquad (5.6)$$

and the mean value of $\sin \chi_{\text{st}}(\xi)$ is defined as

$$\langle \sin \chi_{\text{st}}(\xi) \rangle = \int_0^{2\pi} \sin \chi P_{\text{st}}(\chi, \xi) d\chi \qquad (5.7)$$

with $P_{\text{st}}(\chi, \xi) = \lim_{n \to \infty} P(\chi, n + \xi)$. Our aim here is to determine the drift velocity (5.6) in the limit of large thermal fluctuations, when $\beta \to \infty$.

Since, according to (5.5), in this limit $\epsilon \to 0$, the steady-state solution of Eq. (5.2) can be represented in the form

$$P_{\text{st}}(\chi, \xi) = \sum_{l=0}^{\infty} P_l(\chi, \xi), \qquad (5.8)$$

where the functions $P_l(\chi, \xi) = P_l$ are of the order of ϵ^l, $P_l \sim \epsilon^l$, and satisfy the periodicity conditions $P_l(\chi + 2\pi, \xi) = P_l(\chi, \xi)$ and $P_l(\chi, \xi + 1) = P_l(\chi, \xi)$. In the zeroth-order approximation one obtains $P_0 = \lim_{\epsilon \to 0} P_{\text{eq}}(\chi) = 1/2\pi$, and the functions P_l ($l \geqslant 1$) are connected with the functions P_{l-1} as follows:

$$\frac{\partial^2}{\partial \chi^2} P_l = \frac{\epsilon}{\alpha} \left(\frac{\partial}{\partial \xi} P_{l-1} + \dot{\psi}(\xi) \frac{\partial}{\partial \chi} P_{l-1} - \alpha \frac{\partial}{\partial \chi} \sin \chi P_{l-1} \right). \qquad (5.9)$$

Using this recurrence equation, it is not difficult to show that the steady-state probability density (5.8) in the second-order approximation (when $l \leqslant 2$) is given by

$$P_{\text{st}}(\chi, \xi) = \frac{1}{2\pi} + \frac{\epsilon}{2\pi} \cos \chi + \frac{\epsilon^2}{8\pi} \cos 2\chi + \frac{\epsilon^2}{2\pi \alpha} \dot{\psi}(\xi) \sin \chi. \qquad (5.10)$$

The first three terms on the right-hand side of this expression represent the second-order expansion of $P_{\text{eq}}(\chi)$ and, according to (5.6) and (5.7), they do not contribute to the drift velocity $\langle s_y \rangle$. In contrast, the last term in (5.10), which accounts for the

particle rotation induced by the magnetic field, yields

$$\langle s_y \rangle = \frac{\gamma \epsilon^2}{2\alpha} \int_0^1 \dot{\psi}(\xi) \sin (2\pi \xi - \phi) d\xi. \qquad (5.11)$$

Let us assume for simplicity that the azimuthal angle $\psi(\tau)$ of the magnetic field (2.11) changes with time as

$$\psi(\tau) = \psi_m \cos (2\pi \tau), \qquad (5.12)$$

where ψ_m is the amplitude (in radians) of the periodic function $\psi(\tau)$. Then, taking into account that in this case

$$\int_0^1 \dot{\psi}(\xi) \sin (2\pi \xi - \phi) d\xi = -\pi \psi_m \cos \phi, \qquad (5.13)$$

the particle drift velocity (5.11) reduces to

$$\langle s_y \rangle = -\frac{\pi \psi_m \gamma \epsilon^2}{2\alpha} \cos \phi. \qquad (5.14)$$

According to this result, strong thermal fluctuations essentially decrease the drift velocity ($\langle s_y \rangle$ tends to zero quadratically as $\epsilon \to 0$). In addition, it shows that $\langle s_y \rangle$ depends linearly on the amplitude ψ_m of the magnetic field azimuthal angle and periodically on the initial phase ϕ of the external force.

B. General model

Because the deterministic dynamics of the particle magnetization occurs in the plane of magnetic field rotation [1,2], the planar rotor model is quite appropriate to describe the drift motion of ferromagnetic particles in the absence of thermal fluctuations. However, the presence of such fluctuations, especially strong ones, casts doubt on the applicability of this model. The reason is that the out-of-plane fluctuations of the magnetization vector, which always exist, are excluded from consideration in the planar rotor model. Therefore, to investigate the influence of these fluctuations on the drift velocity, we will use a general Fokker-Planck equation (4.1).

In the case of static magnetic field [when $\psi(\tau) = $ const] the equilibrium solution of Eq. (4.1) has a well-known form

$$P_{\text{eq}}(\theta, \chi) = \frac{1}{4\pi} \frac{\epsilon}{\sinh \epsilon} \sin \theta e^{\epsilon \sin \theta \cos \chi}. \qquad (5.15)$$

At the same time, if $\psi(\tau) \neq 0$ and $\epsilon \to 0$, the steady-state solution of this equation can be represented as

$$P_{\text{st}}(\theta, \chi, \xi) = \sum_{l=0}^{\infty} P_l(\theta, \chi, \xi), \qquad (5.16)$$

where $P_l(\theta, \chi, \xi) = P_l \sim \epsilon^l$ and $\int_0^{\pi} \int_0^{2\pi} P_l \, d\theta \, d\chi = 0$ for $l \geqslant 1$. From (5.15) one immediately finds $P_0 = \lim_{\epsilon \to 0} P_{\text{eq}}(\theta, \chi) = (1/4\pi) \sin \theta$ and, using Eq. (4.1), it is not difficult to show that, if P_{l-1} (with $l \geqslant 1$) is known, then P_l can be determined by solving the following equation:

$$\frac{\partial^2}{\partial \theta^2} P_l + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \chi^2} P_l - \frac{\partial}{\partial \theta} \cot \theta P_l$$
$$= \frac{\epsilon}{\alpha} \left(\frac{\partial}{\partial \tau} P_{l-1} + \dot{\psi} \frac{\partial}{\partial \chi} P_{l-1} + \alpha \cos \chi \frac{\partial}{\partial \theta} \cos \theta P_{l-1} \right.$$
$$\left. - \alpha \frac{1}{\sin \theta} \frac{\partial}{\partial \chi} \sin \chi P_{l-1} \right). \qquad (5.17)$$

The sequential solution of this equation for $l = 1$ and 2 permitted us to determine the steady-state probability density $P_{st}(\theta, \chi, \xi)$ with the second-order accuracy

$$P_{st}(\theta, \chi, \xi) = \frac{1}{4\pi} \sin\theta + \frac{\epsilon}{4\pi} \sin^2\theta \cos\chi - \frac{\epsilon^2}{24\pi} \sin\theta$$

$$\times (1 - 3\sin^2\theta \cos^2\chi) + \frac{\epsilon^2}{8\pi\alpha} \dot\psi(\xi) \sin^2\theta \sin\chi. \quad (5.18)$$

Like in the previous case, the first three terms on the right-hand side of this formula arise from the expansion of the equilibrium probability density (5.15) and, as follows from the definition

$$\langle \sin\theta_{st}(\xi) \sin\chi_{st}(\xi) \rangle = \int_0^\pi \int_0^{2\pi} \sin\theta \, \sin\chi \, P_{st}(\theta, \chi, \xi) d\theta \, d\chi, \quad (5.19)$$

they do not contribute to the drift velocity $\langle s_y \rangle$. In contrast, since $\int_0^\pi \sin^3\theta \, d\theta = 4/3$ and $\int_0^{2\pi} \sin^2\chi \, d\chi = \pi$, the last term in (5.18) does contribute to $\langle s_y \rangle$, yielding

$$\langle s_y \rangle = \frac{\gamma\epsilon^2}{6\alpha} \int_0^1 \dot\psi(\xi) \sin(2\pi\xi - \phi) d\xi \quad (5.20)$$

for arbitrary $\psi(\tau)$ and

$$\langle s_y \rangle = -\frac{\pi\psi_m\gamma\epsilon^2}{6\alpha} \cos\phi, \quad (5.21)$$

if $\psi(\tau)$ is defined by (5.12).

By comparing formulas (5.11) and (5.20), we see that the drift velocity predicted by the planar rotor model in the case of large thermal fluctuations is three times larger than that predicted by the general model. Because the only difference between these models is the out-of-plane fluctuations of the magnetization vector, we may conclude that these fluctuations are responsible for this decrease of the drift velocity.

VI. TEMPERATURE EFFECTS IN THE DRIFT MOTION: NUMERICAL RESULTS

In order to verify our analytical results (5.14) and (5.21) for the drift velocity, we first performed numerical simulations of the stochastic equations (5.1) and (4.5) and determined their solutions, $\chi^{(i)}(n + \xi)$ and $\theta^{(i)}(n + \xi)$, in the ith run. Then, choosing $n \gg 1$, we approximately calculated the means $\langle \sin\chi_{st}(\xi) \rangle$ and $\langle \sin\theta_{st}(\xi) \sin\chi_{st}(\xi) \rangle$ as

$$\langle \sin\chi_{st}(\xi) \rangle = \frac{1}{N} \sum_{i=1}^{N} \sin\chi^{(i)}(n + \xi) \quad (6.1)$$

and

$$\langle \sin\theta_{st}(\xi) \sin\chi_{st}(\xi) \rangle = \frac{1}{N} \sum_{i=1}^{N} \sin\theta^{(i)}(n + \xi) \sin\chi^{(i)}(n + \xi), \quad (6.2)$$

where $N(\gg 1)$ is the total number of runs (these formulas become exact if $N, n \to \infty$). Finally, using (6.1) and (6.2), we determined the drift velocity (5.6) within the planar rotor model and the drift velocity (3.5) within the general model.

The reduced drift velocity $\langle s_y \rangle / \gamma$ as a function of ϵ, obtained by the above procedure for two models, is shown in Fig. 1 (the
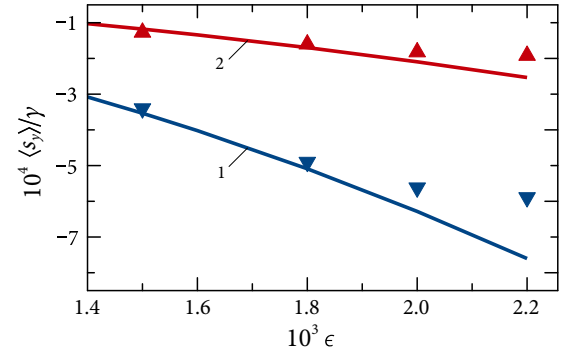

FIG. 1. The reduced drift velocity as a function of the parameter ϵ for $\epsilon \ll 1$. The inverted and upright triangles represent the simulation data for the planar rotor and general models, and the solid lines 1 and 2 represent the theoretical results from (5.14) and (5.21), respectively.

simulation parameters are chosen to be $\alpha = 0.01$, $\psi_m = 1$ rad, $\phi = 0$ rad, $n = 200$, and $N = 10^7$). As seen, the simulation data are in good agreement with analytical predictions, if the parameter ϵ is small enough. (The difference between the numerical and analytical results grows with increasing ϵ because of limitation of the second-order approximation.) This validates both our analytical results and simulation procedure. Moreover, the obtained numerical results confirm a strong influence of the out-of-plane fluctuations of the particle magnetization on the drift velocity. Therefore, to account for these fluctuations, in our further analysis we use a general set of effective stochastic differential equations (4.5).

It should be noted that, according to Eqs. (4.5), the influence of temperature on the rotational properties of particles may occur not only because the parameter β explicitly depends on T [see (2.8)], but also due to the temperature dependence of the magnetization M and dynamic viscosity η. For illustration, we consider Co nanoparticles suspended in water whose temperature varies in the interval ΔT ranging from 273 K (near-freezing temperature) to 373 K (near-boiling temperature). Since the Curie temperature of cobalt is 1.398×10^3 K (see Ref. [4], Table 2.3), the Co magnetization slowly varies in the interval ΔT. Therefore, its value in this interval can be chosen to be constant, e.g., $M = 1.422 \times 10^3$ emu/cm^3, which is the Co saturation magnetization at room temperature (see Ref. [4], Table 2.9). In contrast, the dynamic viscosity of water strongly decreases with increasing temperature. For example, in accordance with the experimental data (see Ref. [31], Chap. 6), one obtains $\eta|_{T=273.64\,K}/\eta|_{T=371.24\,K} = 6.095$. As the parameters α and β depend on η, this means that the temperature dependence of the dynamic viscosity must be taken into account to correctly describe the drift motion of these nanoparticles. For this purpose, we use the approximate formula

$$\eta = 10^{A+B/(T-C)}, \quad (6.3)$$

which well reproduces the experimental data for the dynamic viscosity of water in the reference interval ΔT (see Ref. [31], Table 4.9). Here, $A = -3.5318$, $B = 220.57$, $C = 149.39$, T is in kelvin, and η is in poise (P).

Using the above described procedure, we numerically determined the reduced drift velocity $\langle s_y \rangle / \gamma$ as a function of
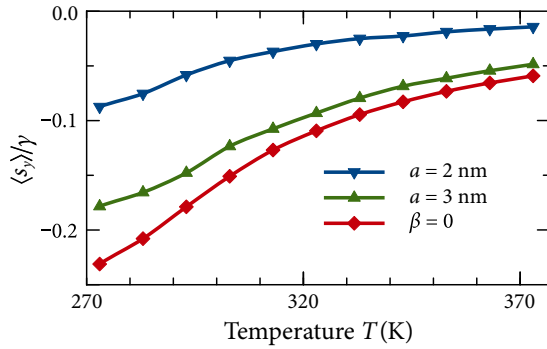
FIG. 2. The temperature dependence of the reduced drift velocity for Co nanoparticles of different radii suspended in water. The dependence at $\beta = 0$ corresponds to the large radius limit.

temperature for Co nanoparticles suspended in water (in all simulations we take $n = 50$ and $N = 10^5$). Figure 2 illustrates this function in the cases when $H_m = 100$ Oe, $\psi_m = 1$ rad, $\Omega = 10^6$ rad/s, $\phi = 0$ rad, $a = 2$ nm (line with inverted triangles), and $a = 3$ nm (line with upright triangles). With increasing the particle radius a, the temperature dependence of the drift velocity very rapidly approaches the limiting function (represented by the line with diamonds), which corresponds to the noiseless case $\beta = 0$. In particular, the difference between these functions can safely be neglected already at $a > (5–10)$ nm (this radius slowly increases with decreasing H_m). This means that thermal fluctuations reduce the drift velocity of rather small particles (when $a < 5$ nm), while in the case of larger particles their drift velocity depends on temperature solely due to the temperature dependence of the dynamic viscosity of water. In this context, it is important to note that, according to the above theory, the deterministic approach to the description of the drift motion of ferromagnetic particles is valid if $a \gtrsim 10$ nm. This fact significantly expands the range of applicability of the deterministic theory, which was roughly estimated as $a \sim 100$ nm [1].

According to (3.5), the drift velocity $\langle s_y \rangle$ is a periodic function of the initial phase ϕ satisfying the condition $\langle s_y \rangle|_{\pi+\phi} = -\langle s_y \rangle|_{\phi}$. As a consequence, the sign of $\langle s_y \rangle$ at a fixed temperature T can always be changed (i.e., the drift direction can be reversed) by varying ϕ. One may also expect that inversion of the drift direction can be caused by changing T, if ϕ is

properly chosen. Indeed, assuming for simplicity that thermal fluctuations are absent (i.e., $\beta = 0$), from Eq. (5.1) it follows that its steady-state solution $\chi_{st}(\tau)$ and hence the drift velocity (5.6) depend on the parameter α (for more details, see Ref. [2]) and, since $\eta = \eta(T)$, on temperature. This means that if the initial phase ϕ is chosen to be close to ϕ_0, where ϕ_0 is a solution of the equation $\langle s_y \rangle = 0$, then the inversion of the sign of $\langle s_y \rangle$ can always be achieved by changing T. These expectations are fully confirmed by our simulation results presented in Fig. 3.

In order to gain more insight into the behavior of the reduced drift velocity of Co nanoparticles, we calculated also its dependence on the magnetic field characteristics, namely, on the magnitude H_m, frequency Ω, and angle ψ_m. The dependence of $\langle s_y \rangle/\gamma$ on H_m for a nanoparticle of radius $a = 5$ nm determined at room temperature $T = 295$ K, $\Omega = 5 \times 10^5$ rad/s, $\psi_m = 1$ rad, and different values of ϕ is shown in Fig. 4. Since the change of H_m is equivalent to changing the parameter α, the obtained results are similar to those derived in Ref. [2] in the deterministic approach. Figure 5 illustrates the dependence of $\langle s_y \rangle/\gamma$ on Ω calculated for $T = 295$ K, $H_m = 10$ Oe, $\psi_m = 1$ rad, $\phi = 0$ rad, and different particle radii. As seen, decreasing the particle size decreases the absolute value of the reduced drift velocity and shifts the minimum of this velocity to the right. This occurs because the magnitude of thermal fluctuations, which is characterized by the parameter β, increases as the particle size decreases [see (2.8)]. In

FIG. 4. The reduced drift velocity of Co nanoparticles as a function of the magnetic field magnitude H_m for different values of the initial phase ϕ. Line 1 corresponds to $\phi = 0.4$ rad, line 2 to $\phi = 2.0$ rad, and line 3 to $\phi = (\pi + 0.4)$ rad.
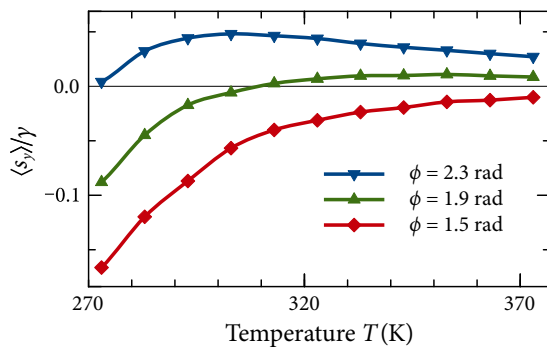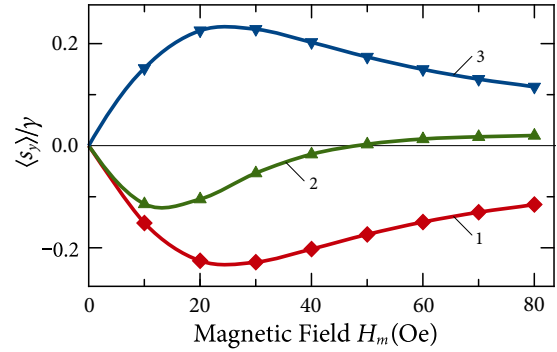

FIG. 3. The reduced drift velocity of Co nanoparticles as a function of temperature for different values of the initial phase ϕ. The particle radius is $a = 3$ nm and the other parameters are the same as in Fig. 2.
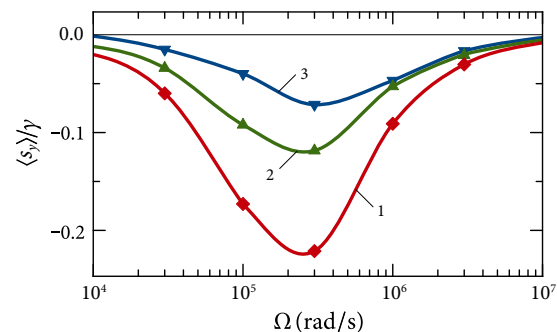

FIG. 5. The frequency dependence of the reduced drift velocity for different radii of Co nanoparticles. Line 1 corresponds to the large radius limit ($\beta = 0$), line 2 to $a = 5$ nm, and line 3 to $a = 4$ nm. The Ω axis is in logarithmic scale.
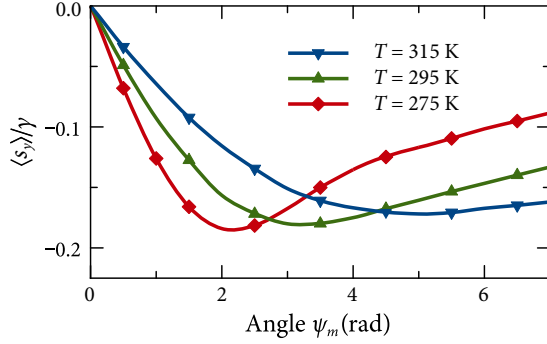
FIG. 6. The reduced drift velocity of Co nanoparticles as a function of the angle ψ_m for different temperatures.

Fig. 6, we show the dependence of $\langle s_y \rangle / \gamma$ on ψ_m calculated for $a = 5$ nm, $H_m = 10$ Oe, $\Omega = 5 \times 10^5$ rad/s, $\phi = 0$ rad, and different temperatures. In this case, the influence of temperature on the reduced drift velocity is caused by both the temperature dependence of the dynamic viscosity of water and thermal fluctuations. It should also be kept in mind that the dependencies of the dimensional drift velocity $v_{dr} = v_m \langle s_y \rangle$ on temperature differ from that obtained for the reduced drift velocity $\langle s_y \rangle / \gamma$. The reason is that the quantities v_m and γ, according to their definitions, depend on T through the dynamic viscosity of water.

Finally, let us evaluate the characteristic velocity $v_m = f_m / 6\pi\eta a$ and the drift velocity v_{dr} for Co nanoparticles suspended in water at room temperature. If the driving force $\mathbf{f}_d$ is generated by the gradient magnetic field $\mathbf{H}_g = gx \sin(\Omega t - \phi)\mathbf{e}_x$ (g is the magnetic field gradient), then its amplitude can be estimated as $f_m = VMg$, and so $v_m = 2a^2 Mg / 9\eta$. Since the characteristic velocity v_m and parameter γ strongly grow with increasing particle size, it is expected that the drift motion induced by the Magnus force can be used to separate large particles (because they drift with relatively large velocities). In particular, assuming that $a = 100$ nm (although this radius exceeds the critical one [4], the single-domain state is expected to be realized due to the large magnetic field), $g = 10^3$ Oe/cm, $H_m = 10^3$ Oe and taking into account that $\eta = 9.62 \times 10^{-3}$ P at $T = 295$ K, one obtains $v_m = 3.29 \times 10^{-3}$ cm/s and $\gamma = 4.27 \times 10^{-2}$. Therefore, taking $\langle s_y \rangle / \gamma = -0.17$ (see Fig. 2, the line with diamonds), we get $v_{dr} = -239$ nm/s. Note also that the magnitude of the drift velocity can be increased by an appropriate choice of the external parameters, at which the macroscopic displacement of nanoparticles can occur for rather short times [1].

single-domain ferromagnetic particles induced by the Magnus force. Our approach is based on a set of stochastic differential equations that describes the translational and rotational motions of particles with frozen magnetization in the limit of small Reynolds numbers. In this approximation, the rotational motion, which is caused by a rotating magnetic field, affects (due to the Magnus effect) the translational motion, which is caused by an external harmonic force. In contrast, the translational motion does not influence the rotational one. These features of particle dynamics were used to derive a general expression for the particle drift velocity in the steady state. It shows that, as in the deterministic case [1,2], drift occurs when the translational and rotational motions of particles are properly synchronized, and thermal fluctuations do not destroy the drift motion; they only decrease the drift velocity. Thus, the statistical theory developed here confirms the existence of the drift phenomenon in suspensions of ferromagnetic particles and has a much wider range of applicability compared to the deterministic one.

Using the planar rotor model and general model, we have solved the corresponding Fokker-Planck equations in the limit of strong thermal fluctuations and, on this basis, calculated the particle drift velocity for these models. It turned out that the out-of-plane fluctuations, which are not accounted for by the planar rotor model, play an important role, resulting in decreasing the drift velocity by three times.

To verify our analytical results obtained in the case of large thermal fluctuation and to analyze the dependence of the drift velocity on temperature and other parameters, we have solved numerically a set of effective stochastic differential equations describing the magnetization dynamics. In this way, it has been shown that the analytical and numerical results are in good agreement with each other, if the magnetic energy of a particle is very small compared to the thermal energy. Next, using the validated numerical procedure, we have calculated the temperature dependence of the drift velocity of Co nanoparticles suspended in water. An important feature of our approach is that the temperature dependence of the dynamic viscosity of water is explicitly taken into account. This permitted us to determine the drift velocity in a wide temperature interval and clarify the role of thermal fluctuations. It has been shown, in particular, that, in contrast to previous estimates, the deterministic approximation is valid for single-domain particles whose minimum size is of the order of 10 nm. Another remarkable observation is that the sign of the drift velocity (i.e., the drift direction) can be changed by changing temperature, if the initial phase of the external driving force is properly chosen. It seems that this property of the drift velocity may be of interest for particle separation based on the Magnus effect.

VII. CONCLUSIONS

We have studied analytically and numerically the temperature dependence of the drift velocity of suspended

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