Ministry of Education and Science of Ukraine
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DIFFERENTIAL EQUATIONS. STABILITY THEORY

Lecture notes

Sumy
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DIFFERENTIAL EQUATIONS.
STABILITY THEORY

Lecture notes

for foreign students of specialty *113 "Applied Mathematics"*
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1. Preliminaries

An ordinary differential equation is an equation containing the derivatives of an unknown function $x(t)$, with ta real variable, and possibly containing the unknown function itself, the independent variable t, and given functions. In addition, initial conditions, which the unknown function is required to satisfy, may be given. With such an equation, the object is two-fold:

(i) to find the unknown function or class of functions satisfying the equation,

(ii) whether (i) is possible or not, to gain some information about the behavior of any function satisfying the equation.

2. The Notion of Stability

A system is called *stable* if its long-termbehavior does not depend significantly on the initial conditions.

In terms of differential equations, the simplest system is represented by an ODE of the form

$$a_0 y'' + a_1 y' + a_2 y = f(t) \tag{1}$$

The general solution to (1) has the form

$$y = C_1 y_1 + C_2 y_2 + y_p, \tag{2}$$

where C_1 , C_2 – arbitrary constants, y_p is a *particular solution* to (1), and $C_1 y_1 + C_2 y_2$ is the *complementary function*, i. e., the general solution to the associated homogeneous equation (the one having $f(t) = 0$).

The initial conditions determine the exact values of C_1 and C_2. So from (2), the system modeled by (1) is stable if and only if for every choice of $C_1,\ C_2$ $C_1 y_1 + C_2 y_2 \to 0$ as $t \to \infty$.

If the ODE (1) is stable, the two parts of the solution (2) are named: y_p – *steady-state solution*; $C_1 y_1 + C_2 y_2$–*transient*.

Let we have the system of n differential equations

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \dots, x_n), \qquad i = 1, 2, \dots, n$$

with initial conditions

$$x_i(t_0) = x_{i0}, \qquad i = 1, 2, \dots, n.$$

We assume that the functions $f_i(t, x_1, x_2, \dots, x_n)$ are defined and continuous together with its partial derivatives on the set $\{t \in [t_0, +\infty), x_i \in R^n\}$. Then without loss of generality we may assume that the initial time is zero: $t_0 = 0$.

It is convenient to write the system of differential equations in vector form:

$$\boldsymbol{X}' = \boldsymbol{f}(t, \boldsymbol{X}),$$

where $\boldsymbol{X} = (x_1, x_2, \dots, x_n)$, $\boldsymbol{f} = (f_1, f_2, \dots, f_n)$.

In real systems, the initial conditions are specified with some precision. This raises the obvious question: how small changes in initial conditions affect the behavior of solutions for large time – in the extreme case when $t \to \infty$?

If the trajectory of the system varies little under small perturbations of the initial position, we say that the motion of the system is *stable*.

A mathematically rigorous definition of stability using $\varepsilon - \delta$-notation was proposed in 1892 by the Russian mathematician A. M. Lyapunov (1857−1918). Let us consider in more detail the concept of stability introduced by Lyapunov.

2.1. Lyapunov Stability

The solution $\varphi(t)$ of the system of differential equations

$$\boldsymbol{X}' = \boldsymbol{f}(t, \boldsymbol{X})$$

with initial conditions

$$\boldsymbol{X}(0) = \boldsymbol{X}_0$$

is *stable (in the sense of Lyapunov)* if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$, such that if

$$|\boldsymbol{X}(0) - \varphi(0)| < \delta, \quad \text{then} \quad |\boldsymbol{X}(t) - \varphi(t)| < \varepsilon$$

for all values $t \geq 0$. Otherwise, the solution $\varphi(t)$ is said to be *unstable*.

As the norm for measuring the distance between two points one can use, for example, the Euclidean metric $\|x_e\|$:

$$\|x_e\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}\ .$$

In the case $n = 2$, Lyapunov stability means that any trajectory $\boldsymbol{X}(t)$, which starts at $\delta(\varepsilon)$-neighborhood of the point $\varphi(0)$, remains inside the tube with a maximum radius ε for all $t \geq 0$ (Figure 1).

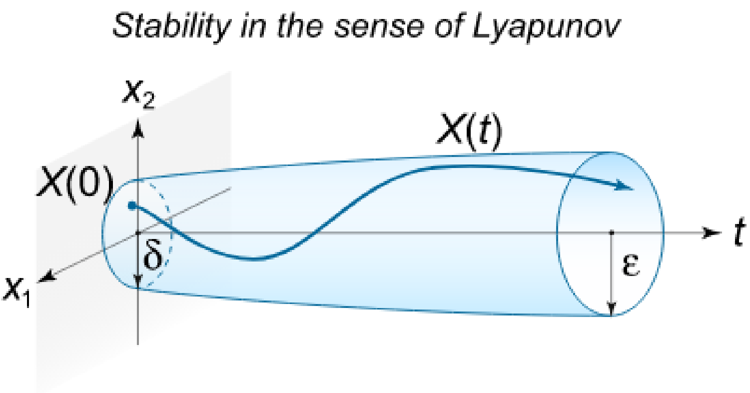


Figure 1

2.2. Asymptotic and Exponential Stability

If the solution $\varphi(t)$ of the system of differential equations is not only stable in the sense of Lyapunov, but also satisfies the relationship

$$\lim_{t\to\infty}|\boldsymbol{X}(t)-\varphi(t)|=0$$

provided that

$$|\boldsymbol{X}(0)-\varphi(0)|<\delta,$$

then we say that the solution $\varphi(t)$ is *asymptotically stable*.

In this case, all solutions that are sufficiently close to $\varphi(0)$ at the initial time, gradually converge to $\varphi(t)$ with increasing t. Schematically, this is shown in Figure 2.

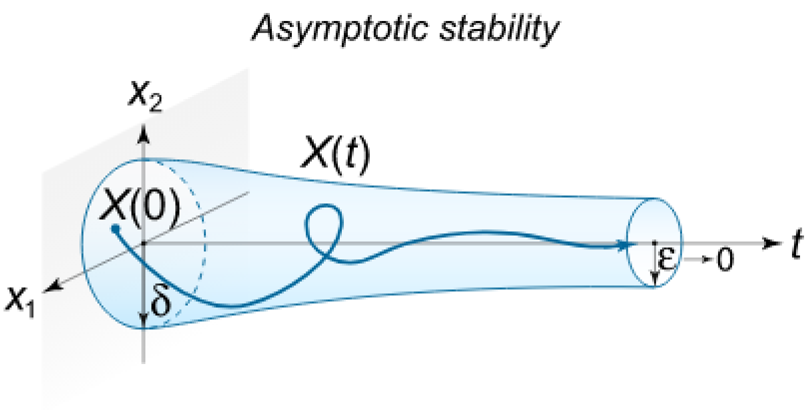


Figure 2

If the solution $\varphi(t)$ is asymptotically stable and, in addition, from the condition

$$|\boldsymbol{X}(0) - \varphi(0)| < \delta$$

it follows that

$$|\boldsymbol{X}(t) - \varphi(t)| < \alpha|\boldsymbol{X}(0) - \varphi(0)|e^{-\beta t}$$

for all $t \geq 0$, we say that the solution $\varphi(t)$ is *exponentially stable*. In this case all solutions that are close to $\varphi(0)$ at the initial time converge to $\varphi(t)$ with the rate (greater than or equal), which is determined by an exponential function with parameters α, β (Fig. 3).

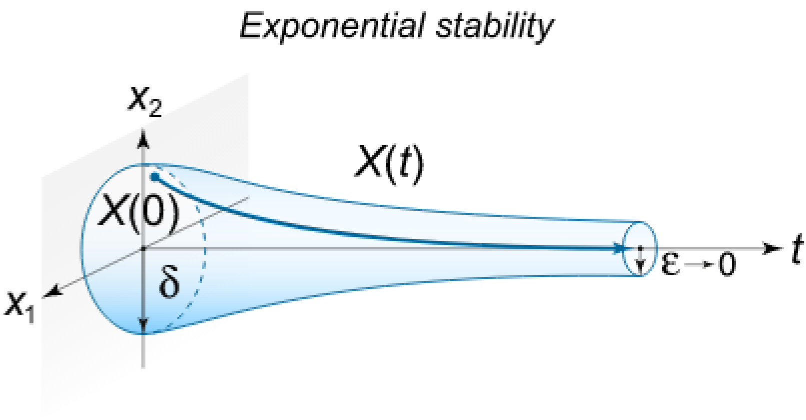


Figure 3

The general theory of stability, in addition to stability in the sense of Lyapunov, contains many other concepts and definitions of stable movement. In particular, the concepts of *orbital* and *structural* stability are important.

2.3. Orbital Stability

Orbital stability describes the behavior of a closed trajectory (orbit) under the action of small external perturbations.

Consider the autonomous system

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \dots, x_n), \qquad x_i(t_0) = x_{i0}, \qquad i = 1, 2, \dots, n,$$

that is the system of equations, the right hand side of which does not contain the independent variable t. In vector form, the autonomous system is written as

$$\boldsymbol{X}' = \boldsymbol{f}(\boldsymbol{X}), \quad \text{where} \quad \boldsymbol{X} = (x_1, x_2, \dots, x_n), \quad \boldsymbol{f} = (f_1, f_2, \dots, f_n)$$

Let $\varphi(t)$ be a periodic solution of the given autonomous system, that is has the form of a closed trajectory (orbit).

If for any $\varepsilon > 0$ there is a constant $\delta = \delta(\varepsilon) > 0$ such that the trajectory of any solution $\boldsymbol{X}(t)$ starting at the δ-neighborhood of the trajectory $\varphi(t)$ remains in the ε-neighborhood of the trajectory $\varphi(t)$ forall $t \geq 0$, then the trajectory $\varphi(t)$ is called *orbitally stable* (Fig. 4).

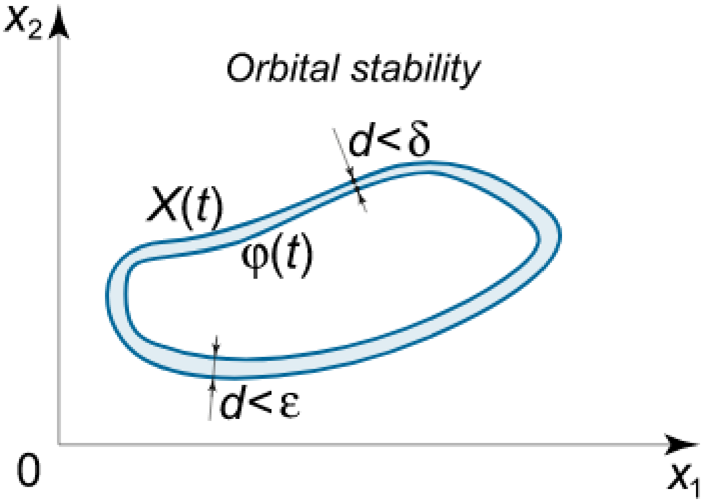


Figure 4

By analogy with the asymptotic stability in the sense of Lyapunov, one can also introduce the concept of *asymptotic orbital stability*. This type of motion occurs, for example, in systems with a *limit cycle*.

2.4. Structural Stability

Suppose that we have two autonomous systems with similar properties − in the sense that their phase portraits have the same singular points and geometrically similar trajectories. Such systems can be called *structurally stable*.

In the strict definition, it is required that these systems are *orbitally topologically equivalent*, i. e. there must be a *homeomorphism* (this means one-to-one continuous mapping), which converts the family of trajectories of the first system into the family of trajectories of the second system while preserving the direction of motion. In these terms, the structural stability is defined as follows.

Consider an autonomous system, which in the unperturbed and perturbed state is described, respectively, by two equations:

$$\boldsymbol{X}' = \boldsymbol{f}(\boldsymbol{X}),$$

$$\boldsymbol{X}' = \boldsymbol{f}(\boldsymbol{X}) + \varepsilon\boldsymbol{g}(\boldsymbol{X}).$$

If for any bounded and continuously differentiable vector function $\boldsymbol{g}(\boldsymbol{X})$ there exists a number $\varepsilon > 0$ such that the trajectories of the unperturbed and perturbed systems are *orbitally topologically equivalent*, then the system is called *structurally stable*.

3. Reduction to the Problem of Stability of the Zero Solution

Let an arbitrary non-autonomous system

$$\boldsymbol{X}' = \boldsymbol{f}(t, \boldsymbol{X})$$

be given with the initial condition $\boldsymbol{X}(0) = \boldsymbol{X}_0$ (Cauchy problem).

Here the vector-valued function $\boldsymbol{f}$ is defined on the set $\{t \in$

$\in [t_0, +\infty), x_i \in R^n\}$.

Suppose that the system has a solution $\varphi(t)$, the stability of which is to be examined. The stability analysis is simplified if we consider perturbations

$$\boldsymbol{Z}(t) = \boldsymbol{X}(t) - \varphi(t),$$

for which we obtain the differential equation

$$\boldsymbol{Z}'(t) = \boldsymbol{f}(t, \boldsymbol{Z}).$$

Obviously, the last equation is satisfied by the *trivial solution*

$$\boldsymbol{Z}(t, 0) \equiv 0,$$

which corresponds to the identity

$$\boldsymbol{X}(t) \equiv \varphi(t).$$

Thus, the study of stability of the solution $\varphi(t)$ can be replaced by the study of stability of the function $\boldsymbol{Z}(t)$ near the point $\boldsymbol{Z} = 0$.

4. Stability of Linear Systems

The linear system

$$\boldsymbol{X}' = A(t)\boldsymbol{X} + \boldsymbol{f}(t)$$

is said to be stable if all its solutions are stable in the sense of Lyapunov.

It turns out that the non-homogeneous linear system is stable with any free term $\boldsymbol{f}(t)$if the zero solution of the associated homogeneous system

$$\boldsymbol{X}' = A(t)\boldsymbol{X}$$

is stable. Therefore, when investigating stability in the class of linear systems, it is sufficient to analyze the *homogeneous differential systems*. In the simplest case, when the coefficient matrix A is constant, the stability conditions are formulated in terms of the *eigenvalues* of the matrix A.

Consider the homogeneous linear system

$$\boldsymbol{X}' = A\boldsymbol{X},$$

where A is a constant matrix of size $n \times n$. Such a system (which is also *autonomous*) has the zero solution $\boldsymbol{X}(t) = 0$. The stability of this solution is determined by the following theorems.

Let λ_i be the eigenvalues of A.

Theorem 1. A linear homogeneous system with constant coefficients is stable in the sense of Lyapunov if and only if all eigenvalues $\boldsymbol{\lambda_i}$ of $\boldsymbol{A}$ satisfy the condition

$$\boldsymbol{Re}[\boldsymbol{\lambda_i}] \leq \mathbf{0} \quad (\boldsymbol{i} = \mathbf{1}, \mathbf{2}, \dots, \boldsymbol{n}).$$

If the real part of an eigenvalue is equal to zero, the algebraic and geometric multiplicity of the eigenvalue must be the same (i.e. the corresponding *Jordan block* must be of size 1×1).

Theorem 2. A linear homogeneous system with constant coefficients is asymptotically stable if and only if all eigenvalues $\boldsymbol{\lambda_i}$ have negative real parts:

$$\boldsymbol{Re}[\boldsymbol{\lambda_i}] < \mathbf{0} \quad (\boldsymbol{i} = \mathbf{1}, \mathbf{2}, \dots, \boldsymbol{n}).$$

Theorem 3. A linear homogeneous system with constant coefficients is unstable if at least one of the conditions is satisfied:

- The matrix $\boldsymbol{A}$ has an eigenvalue $\boldsymbol{\lambda_i}$ with a positive real part;
- The matrix A has an eigenvalue λ_i with zero real part, and the geometric multiplicity of the eigenvalue λ_i is less than its algebraic multiplicity.

5. Equilibrium Points of Linear Autonomous Systems

Let a second order linear homogeneous system with constant coefficients be given:

$$\begin{cases} \dfrac{dx}{dt} = a_{11}x + a_{12}y \\ \dfrac{dy}{dt} = a_{21}x + a_{22}y \end{cases}$$

This system of equations is autonomous since the right hand sides of the equations do not explicitly contain the independent variable t.
In matrix form, the system of equations can be written as

$$\boldsymbol{X}' = A\boldsymbol{X}, \text{where} \boldsymbol{X} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} \end{bmatrix}.$$

The equilibrium positions can be found by solving the stationary equation

$$A\boldsymbol{X} = 0 .$$

This equation has the unique solution $\boldsymbol{X} = 0$ if the matrix A is nonsingular, i. e. provided that $\det A \neq 0$. In the case of a singular matrix, the system has an infinite number of equilibrium points.

Classification of equilibrium points is determined by the eigenvalues $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ of the matrix A. The numbers $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ can be found by solving the auxiliary equation

$$\boldsymbol{\lambda}^2 - (a_{11} + a_{22})\boldsymbol{\lambda} + a_{11}a_{22} - a_{12}a_{21} = 0.$$

In general, when the matrix A is nonsingular, there are 4 different types of equilibrium points:

№	Equilibrium Point	Eigenvalues $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$
1	Node	$\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ are real numbers of the same sign ($\boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2 > 0$)
2	Saddle	$\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ are real numbers and non-zero of opposite sign ($\boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2 < 0$)
3	Focus	$\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ are complex numbers, the real parts are equal and non-zero ($Re\, \boldsymbol{\lambda}_1 = Re\, \boldsymbol{\lambda}_2 \neq 0$)
4	Center	$\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ are purely imaginary numbers ($Re\, \boldsymbol{\lambda}_1 = Re\, \boldsymbol{\lambda}_2 = 0$)

The stability of equilibrium points is determined by the general theorems on stability (see chapter 1). So, if the real eigenvalues (or real parts of complex eigenvalues) are negative, then the equilibrium point is *asymptotically stable*. Examples of such equilibrium positions are *stable node* and *stable focus*.

If the real part of at least one eigenvalue is positive, the corresponding equilibrium point is *unstable*. For example, it may be a *saddle*.

Finally, in the case of purely imaginary roots (when the equilibrium point is a *center*), we are dealing with the classical *stability in the sense of Lyapunov*.

Our next goal is to study the behavior of solutions near the equilibrium positions. For second order systems, it is convenient to do this graphically using the *phase portrait*, which is a set of *phase trajectories* in the coordinate plane. The arrows on the phase trajectories show the direction of movement of the point (i.e., a particular state of the system) over time.

Let's consider each type of equilibrium point.

Stable and Unstable Node

The eigenvalues $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ of the points of type "node" satisfy the conditions:

$$\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in R, \boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2 > 0 .$$

The following particular cases may arise here.

- The roots $\boldsymbol{\lambda_1}$, $\boldsymbol{\lambda_2}$ are distinct $(\boldsymbol{\lambda_1} \neq \boldsymbol{\lambda_2})$ and negative $(\boldsymbol{\lambda_1} < \mathbf{0},\ \boldsymbol{\lambda_2} < \mathbf{0})$.

Draw a schematic phase portrait for this system. Suppose for definiteness that $|\boldsymbol{\lambda}_1| < |\boldsymbol{\lambda}_2|$. The general solution has the form

$$\mathbf{X}(t) = C_1 e^{\boldsymbol{\lambda}_1 t}\mathbf{V}_1 + C_2 e^{\boldsymbol{\lambda}_2 t}\mathbf{V}_2 ,$$

where $\mathbf{V}_1 = (V_{11}, V_{21})^T$, $\mathbf{V}_2 = (V_{12}, V_{22})^T$ are eigenvectors corresponding to the eigenvalues $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ and C_1, C_2 are arbitrary constants.

Since both eigenvalues are negative, then the solution $\mathbf{X} = 0$ is *asymptotically stable*. Such an equilibrium point is called *stable node*. As $t \to \infty$, the phase curves tend to the origin $\mathbf{X} = 0$.

Specify the direction of the phase trajectories. Since

$$x(t) = C_1 V_{11} e^{\boldsymbol{\lambda}_1 t} + C_2 V_{12} e^{\boldsymbol{\lambda}_2 t}, \quad y(t) = C_1 V_{21} e^{\boldsymbol{\lambda}_1 t} + C_2 V_{22} e^{\boldsymbol{\lambda}_2 t} ,$$

the derivative $\frac{dy}{dx}$ is

$$\frac{dy}{dx} = \frac{C_1 V_{21} \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_1 t} + C_2 V_{22} \boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_2 t}}{C_1 V_{11} \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_1 t} + C_2 V_{12} \boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_2 t}} .$$

Divide the numerator and denominator by $e^{\boldsymbol{\lambda}_1 t}$:

$$\frac{dy}{dx} = \frac{C_1 V_{21} \boldsymbol{\lambda}_1 + C_2 V_{22} \boldsymbol{\lambda}_2 e^{(\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1)t}}{C_1 V_{11} \boldsymbol{\lambda}_1 + C_2 V_{12} \boldsymbol{\lambda}_2 e^{(\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1)t}}$$

In this case, $\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1 < 0$. Therefore, the terms with the exponential function tend to zero as $t \to \infty$. As a result, at $C_1 \neq 0$, we obtain:

$$\lim_{t\to\infty} \frac{dy}{dx} = \frac{V_{21}}{V_{11}}$$

that is the phase trajectories become parallel to the eigenvector $\mathbf{V}_1$ as $t \to \infty$.

If $C_1 = 0$, the derivative at any t equals

$$\frac{dy}{dx} = \frac{V_{22}}{V_{12}},$$

i. e. the phase trajectory lies on a line directed along the eigenvector $\mathbf{V}_2$.

Now we consider the behavior of the phase trajectories as $t \to -\infty$. Obviously, the coordinates $x(t),\ y(t)$tend to infinity, and the derivative $\frac{dy}{dx}$ at $C_2 \neq 0$ takes the following form:

$$\frac{dy}{dx} = \frac{C_1 V_{21} \boldsymbol{\lambda}_1 e^{(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2)t} + C_2 V_{22} \boldsymbol{\lambda}_2}{C_1 V_{11} \boldsymbol{\lambda}_1 e^{(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2)t} + C_2 V_{12} \boldsymbol{\lambda}_2} = \frac{V_{22}}{V_{12}} \quad ,$$

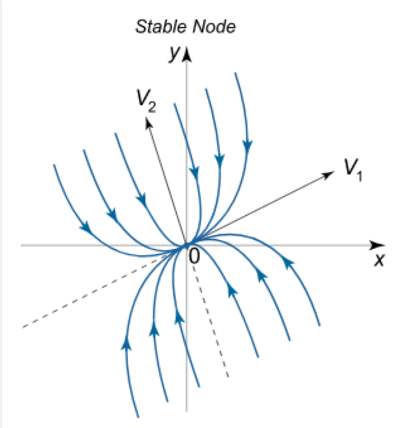
that is the phase curves at the points at infinity become parallel to the vector $\mathbf{V}_2$.

Accordingly, when $C_2 = 0$, the derivative is

$$\frac{dy}{dx} = \frac{V_{21}}{V_{11}}.$$

In this case, the phase trajectory is determined by the direction of the eigenvector $\mathbf{V}_1$.

Given the above properties of the phase trajectories, the phase portrait of a *stable node* is shown schematically in Figure 5.



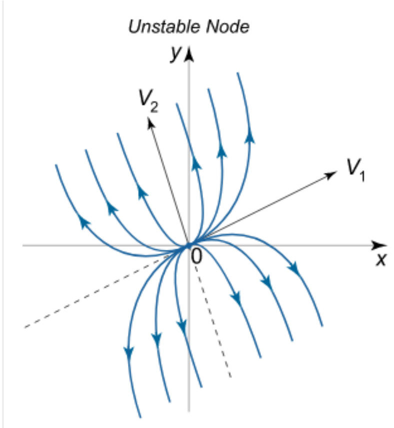


Figure 5 Figure 6

Similarly, we can study the behavior of the phase trajectories for other types of equilibrium points. Furthermore, omitting the detailed analysis, we consider basic qualitative characteristics of the other equilibrium points.

- The roots λ_1, λ_2 are distinct ($\lambda_1 \neq \lambda_2$) and positive ($\lambda_1 > 0,\ \lambda_2 > 0$).

In this case, the point $\mathbf{X} = 0$ is an *unstable node*. Its phase portrait is shown in Figure 6.

Note that in the case of both stable and unstable node, the phase trajectories touch the line, which is directed along the eigenvector corresponding to the smallest (in absolute value) eigenvalue $\boldsymbol{\lambda}$.

Dicritical Node

Let the auxiliary equation have one zero root of multiplicity 2, i. e. consider the case $\lambda_1 = \lambda_2 = \lambda \neq 0$. The system has a basis of two eigenvectors, i. e. the geometric multiplicity of the eigenvalue λ is 2. In terms of the linear algebra, this means that the dimension of the eigenspace of A is equal to 2: $\dim\ker A = 2$. This situation occurs in systems of the form

$$\frac{dx}{dt} = \lambda x, \quad \frac{dy}{dt} = \lambda y .$$

The direction of the phase trajectories depends on the sign of λ. Here the following two cases can arise:

- Case $\boldsymbol{\lambda_1 = \lambda_2 = \lambda < 0}$.

Such an equilibrium position is called a *stable dicritical node* (Fig. 7).

- Case $\boldsymbol{\lambda_1 = \lambda_2 = \lambda > 0}$.

This combination of eigenvalues corresponds to an *unstable dicritical node* (Fig. 8).

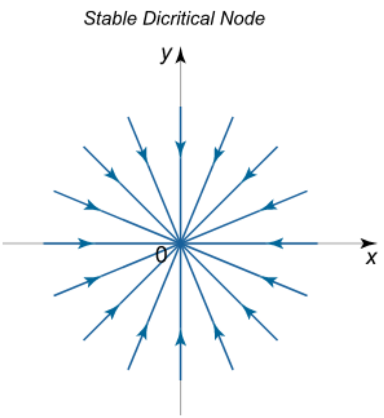


Figure 7

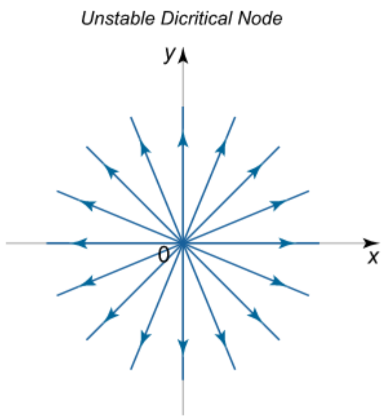


Figure 8

Singular Node

Let the eigenvalues of A be again coincident: $\lambda_1 = \lambda_2 = \lambda \neq 0$. Unlike the previous case, we assume that the geometric multiplicity of the eigenvalue (or in other words, the dimension of the eigenspace) is now 1. This means that the matrix A has only one eigenvector $\mathbf{V}_1$. The second linearly independent vector required for the basis is defined as a generalized eigenvector $\mathbf{W}_1$ connected to $\mathbf{V}_1$.

- Case $\boldsymbol{\lambda_1 = \lambda_2 = \lambda < 0}$.

The equilibrium point is called *stable singular node* (Fig. 9).

- Case $\boldsymbol{\lambda_1 = \lambda_2 = \lambda > 0}$.

The equilibrium position is called *unstable singular node* (Fig. 10).

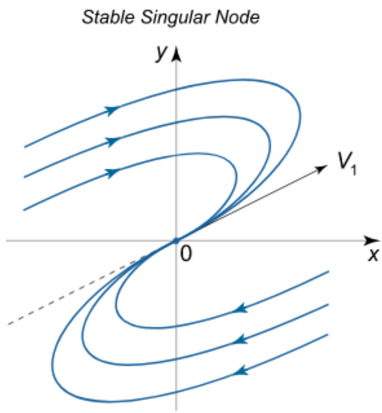


Figure 9

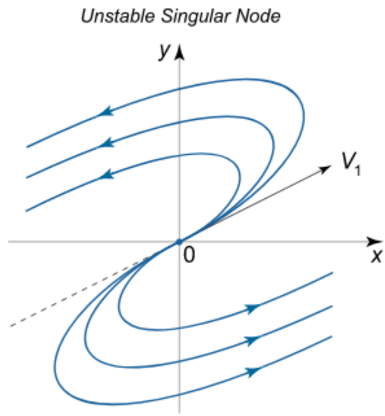


Figure 10

Saddle

The equilibrium point is a *saddle* under the following condition:

$$\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in R, \boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2 < 0.$$

Since one of the eigenvalues is positive, the saddle is an unstable equilibrium point. Suppose, for example, $\lambda_1 < 0, \lambda_2 > 0$. The eigenvalues λ_1 and $\boldsymbol{\lambda}_2$ are associated with the corresponding eigenvectors $\mathbf{V}_1$ and $\mathbf{V}_2$. The straight lines directed along the eigenvectors $\mathbf{V}_1, \mathbf{V}_2$, are called *separatrices*. These are the asymptotes of other phase trajectories that have the form of a hyperbola. Each of the separatrices can be associated with a certain direction of motion.

If the separatrix is associated with a negative eigenvalue $\lambda_1 < 0$, i. e. in our case is directed along the vector $\mathbf{V}_1$, the movement along it occurs towards the equilibrium point $\mathbf{X} = 0$. And conversely, at $\lambda_2 > 0$, i. e. for the separatrix associated with the vector $\mathbf{V}_2$, the movement is directed from the origin. The phase portrait of the saddle is shown schematically in Figure 11.

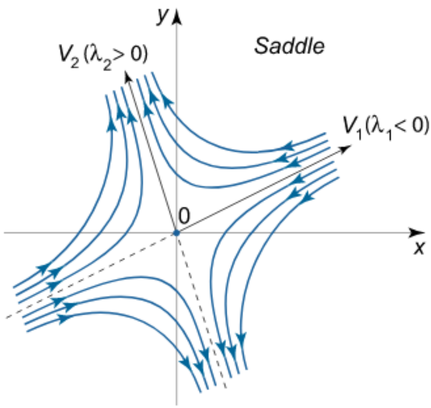


Figure 11

Stable and Unstable Focus

Now suppose that the eigenvalues $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ are complex numbers whose real parts are non-zero. If the matrix A is composed of real numbers, the complex roots will be presented in the form of complex conjugate numbers:

$$\boldsymbol{\lambda}_{1,2} = \alpha \pm i\beta.$$

Find out what kind of phase trajectories are in the neighborhood of the origin. Construct a complex solution $\mathbf{X}_1(t)$ corresponding to the eigenvalue $\boldsymbol{\lambda}_1 = \alpha + i\beta$:

$$\mathbf{X}_1(t) = e^{\lambda_1 t}\mathbf{V}_1 = e^{(\alpha+i\beta)t}(\mathbf{U} + i\mathbf{W}),$$

where $\mathbf{V_1} = \mathbf{U} + i\mathbf{W}$ is the complex-valued eigenvector associated with the eigenvalue $\boldsymbol{\lambda}_1$, **U** and **W** are real vector functions. As a result, we obtain:

$$\mathbf{X_1}(t) = e^{\alpha t}e^{i\beta t}(\mathbf{U} + i\mathbf{W}) = e^{\alpha t}(\cos\beta t + i\sin\beta t\,)(\mathbf{U} + i\mathbf{W}) =$$
$$= e^{\alpha t}(\mathbf{U}\cos\beta t + i\,\mathbf{U}\sin\beta t + i\,\mathbf{W}\cos\beta t - \mathbf{W}\sin\beta t\,) =$$
$$= e^{\alpha t}(\mathbf{U}\cos\beta t - \mathbf{W}\sin\beta t) + ie^{\alpha t}(\mathbf{U}\sin\beta t + \mathbf{W}\cos\beta t)\,.$$

The real and imaginary parts in the last expression form the general solution of the type

$$\mathbf{X}(t) = C_1 Re[\mathbf{X_1}(t)] + C_2 Im[\mathbf{X_1}(t)] =$$

$$= e^{\alpha t}[C_1\,(\mathbf{U}\cos\beta t - \mathbf{W}\sin\beta t) + C_2(\mathbf{U}\sin\beta t + \mathbf{W}\cos\beta t)] =$$

$$= e^{\alpha t}[\mathbf{U}\,(C_1\cos\beta t + C_2\sin\beta t) + \mathbf{W}\,(C_2\cos\beta t + C_1\sin\beta t)\,.$$

We represent the constant C_1,C_2 as

$$C_1 = C\sin\delta\,, \quad C_2 = C\cos\delta\,,$$

where δ is an auxiliary angle. Then the solution is written as

$$\mathbf{X}(t) = Ce^{\alpha t}[\mathbf{U}\,(\sin\delta\cos\beta t + \cos\delta\sin\beta t) + \mathbf{W}\,(\cos\delta\cos\beta t -$$

$$- \sin\delta\sin\beta t)] = Ce^{\alpha t}[\mathbf{U}\sin(\beta t + \delta) + \mathbf{W}\cos(\beta t + \delta)].$$

Thus, the solution $\mathbf{X}(t)$ can be expanded in the basis of the vectors **U** and **W**:

$$\mathbf{X}(t) = \mu(t)\mathbf{U} + \eta(t)\mathbf{W},$$

where the coefficients $\mu(t)$, $\eta(t)$ are given by

$$\mu(t) = Ce^{\alpha t}\sin(\beta t + \delta), \;\; \eta(t) = Ce^{\alpha t}\cos(\beta t + \delta).$$

This shows that the phase trajectories are spirals. When $\alpha < 0$, the spirals twist approaching the origin. Such an equilibrium position is called *stable focus*. Accordingly, when $\alpha > 0$, we have an *unstable focus*.

The direction of twist can be identified by the sign of the coefficient a_{21} in the original matrix A. Indeed, consider the derivative $\frac{dy}{dt}$, for example, at the point $(1,0)$:

$$\frac{dy}{dt}(1,0) = a_{21} \cdot 1 + a_{22} \cdot 0 = a_{21}.$$

The positive coefficient $a_{21} > 0$ corresponds to the twist counterclockwise as shown in Figure 12. When $a_{21} < 0$, the spirals will twist in a clockwise direction (Fig. 13).

Thus, taking into account the direction of twist, there are only 4 different types of focus. Schematically, they are shown in Figures 12−15.

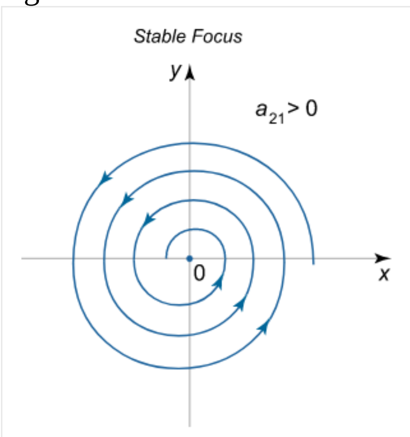


Figure 12

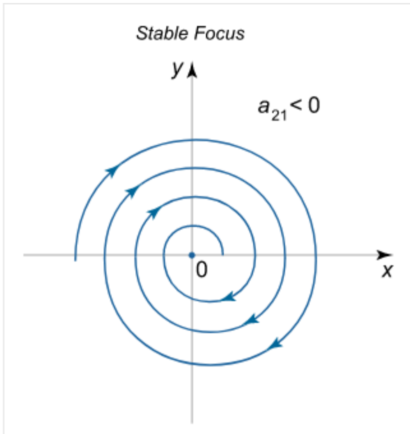


Figure 13

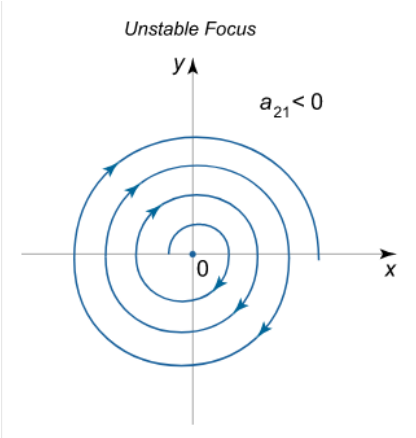


Figure 14

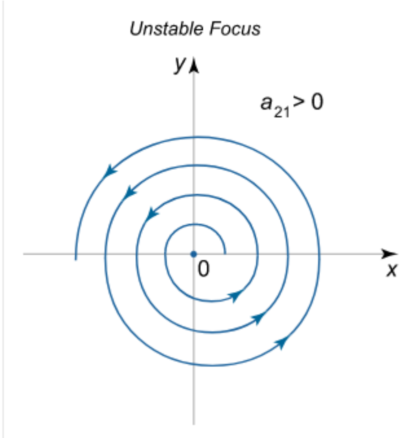


Figure 15

Center

If the eigenvalues of the matrix A are purely imaginary numbers, then this equilibrium point is called a *center*. For a matrix with real elements, the imaginary eigenvalues are complex conjugate pairs. In the case of a center, the phase trajectories are formally obtained from the equation of spirals at $\alpha = 0$ are *ellipses*, i. e. they describe periodic motion of a point in the phase space. A center equilibrium position is stable in the sense of Lyapunov.

There are two types of centers, which differ in the direction of movement of the points (Fig. 16, 17). As in the case of focus, the direction of movement can be determined by the sign of the derivative $\frac{dy}{dt}$ at some point. If we take the point (1.0), then

$$\frac{dy}{dt}(1.0) = a_{21}.$$

that is the direction of rotation is determined by the sign of the coefficient a_{21}.

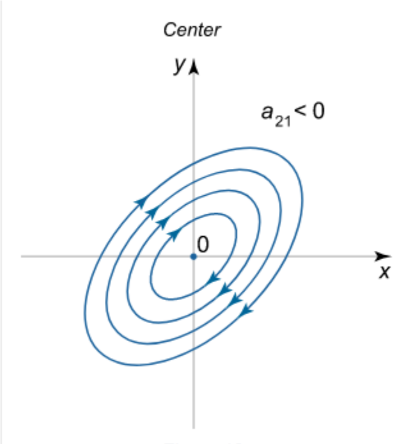


Figure 16

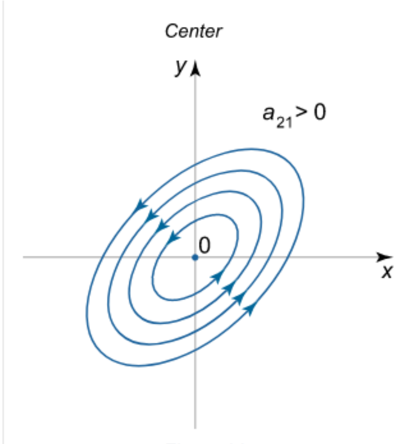


Figure 17

Thus, we have considered different types of equilibrium points in the case of a *non-singular matrix* A ($\det A \neq 0$). Taking into account the direction of phase trajectories, there are total 13 different phase portraits (shown, respectively, in Figures 2−17).

We now turn to the case of a *singular matrix* A.

Singular Matrix

If the matrix is singular, then it has one or both eigenvalues equal to zero. In this case, there are the following special cases:

- Case $\boldsymbol{\lambda_1 \neq 0, \lambda_2 = 0}$.

Here, the general solution has the form

$$\mathbf{X}(t) = C_1 e^{\lambda_1 t}\mathbf{V}_1 + C_2\mathbf{V}_2\,,$$

where $\mathbf{V}_1 = (V_{11}, V_{21})^T$, $\mathbf{V}_2 = (V_{12}, V_{22})^T$, are the eigenvectors corresponding to the eigenvalues $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$. It turns out that in this case the whole line passing through the origin and directed along the vector $\mathbf{V}_2$ consists of the equilibrium points (these points do not have a special name). The phase trajectories are rays parallel to the other eigenvector $\mathbf{V}_1$. Depending on the sign of $\boldsymbol{\lambda}_1$, the motion at $t \to \infty$ occurs either in the direction of the line $\mathbf{V}_2$ (Figure 18), or away from it (Fig. 19).

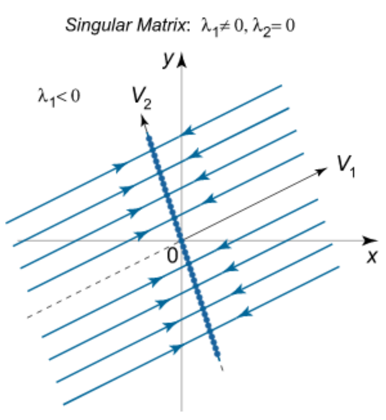


Figure 18

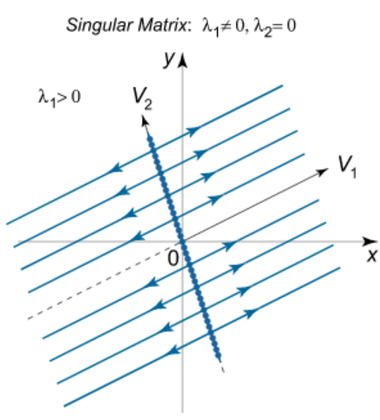


Figure 19

- Case $\boldsymbol{\lambda_1 = \lambda_2 = 0}$, $\mathbf{dim\,ker}\,\boldsymbol{A} = \mathbf{2}$.

In this case, the dimension of the eigenspace of the matrix is equal to 2 and, therefore, there are two eigenvectors $\mathbf{V}_1$ and $\mathbf{V}_2$. This may happen when A is the *zero matrix*. The general solution is given by

$$\mathbf{X}(t) = C_1\mathbf{V}_1 + C_2\mathbf{V}_2 .$$

It follows that every point in the plane is an equilibrium position of the system.

- Case $\boldsymbol{\lambda_1 = \lambda_2 = 0}, \mathbf{dim\,ker}\,\boldsymbol{A} = \mathbf{1}$.

This case is different from the previous one in that there is only one eigenvector (the matrix *A* will then be *non-zero*). To construct a basis, we can take the generalized eigenvector $\mathbf{W}_1$ connected to $\mathbf{V}_1$ as a second linearly independent vector. The general solution can be written as

$$\mathbf{X}(t) = (C_1 + C_2 t)\mathbf{V}_1 + C_2 \mathbf{W}_1 .$$

Here, all points of the straight line passing through the origin and directed along the eigenvector $\mathbf{V}_1$ are unstable equilibrium positions. The phase trajectories are straight lines parallel to $\mathbf{V}_1$. The direction of movement along these lines as $t \to \infty$ depends on the constant C_2: with $C_2 < 0$, the motion is from left to right, and with $C_2 > 0$ – in the opposite direction (Fig. 20).

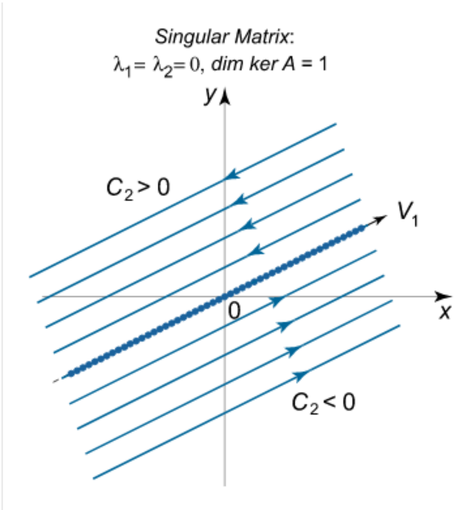


Figure 20

As seen, there are 4 different phase portraits in the case of a singular matrix. Therefore, the linear second order autonomous system allows total 17 different phase portraits.

Bifurcation Diagram

In the above, we have reviewed the classification of equilibrium points of a linear system based on the eigenvalues. However, the type of an equilibrium point can be determined without computing the eigenvalues $\boldsymbol{\lambda_1}, \lambda_2$, knowing only the determinant of the matrix $\det A$ and its trace $\operatorname{tr} A$.

Recall that the *trace of the matrix* is the number equal to the sum of the diagonal elements:

$$A = \begin{pmatrix} a_{11} a_{12} \\ a_{21} a_{22} \end{pmatrix}, \quad \operatorname{tr} A = a_{11} + a_{22} ,$$

$$\det A = a_{11}a_{22} - a_{12}a_{21} .$$

Indeed, the auxiliary equation of the matrix is

$$\lambda^2 - (a_{11} + a_{22})\boldsymbol{\lambda} + a_{11}a_{22} - a_{12}a_{21} = 0 .$$

It can be written in terms of the determinant and the trace of the matrix:

$$\lambda^2 - \operatorname{tr} A \cdot \boldsymbol{\lambda} + \det A = 0 .$$

The discriminant of this quadratic equation is given by

$$D = (\operatorname{tr} A)^2 - 4\det A.$$

Thus, the bifurcation curve delineating the different stability modes is a parabola in the plane $(\operatorname{tr} A, \det A)$(Fig. 21):

$$\det A = \left(\frac{\operatorname{tr} A}{2}\right)^2 .$$

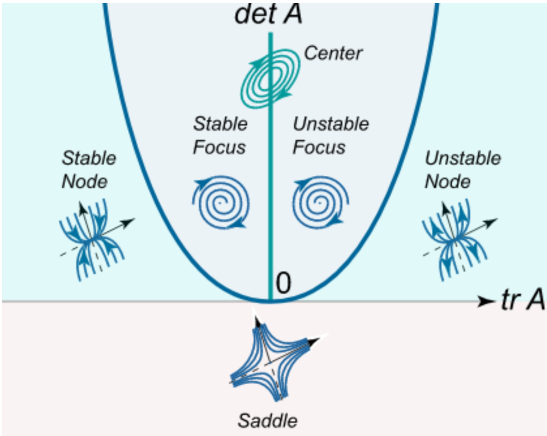


Figure 21

The equilibrium points of the type “focus” and “center” are above the parabola. The points of the type “center” are located on the positive y-axis, i. e. provided that tr $A = 0$. The “nodes” and “saddles” are below the parabola. The parabola itself contains dicritical or singular nodes.

Stable modes of motion exist in the upper left quadrant of the bifurcation diagram. The other three quadrants correspond to unstable equilibrium positions.

6. How to Sketch a Phase Portrait

To draw the phase portrait of a second order linear autonomous system with constant coefficients

$$\boldsymbol{X}' = A\boldsymbol{X}, A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} x \\ y \end{bmatrix},$$

it is necessary to do the following steps:

1. Find the eigenvalues of the matrix by solving the auxiliary equation
$$\lambda^2 - (a_{11} + a_{22})\boldsymbol{\lambda} + a_{11}a_{22} - a_{12}a_{21} = 0\,.$$

2. Determine the type of the equilibrium point and the character of stability.

Hint:

The type of the equilibrium position can also be determined based on the bifurcation diagram (Fig. 18), knowing the trace and the determinant of the matrix:

$$\operatorname{tr} A = a_{11} + a_{22}\ , \quad \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

3. Find the equations of the *isoclines*:
$$\frac{dx}{dt}=a_{11}x + a_{12}y \text{(vertical isocline)},$$
$$\frac{dy}{dt}=a_{21}x + a_{22}y \text{(horizontal isocline)}.$$
4. If the equilibrium position is a *node* or a *saddle*, it is necessary to compute the eigenvectors and draw the asymptotes parallel to the eigenvectors and passing through the origin.
5. Schematically draw the phase portrait.
6. Show the direction of motion along the phase trajectories (this depends on the stability or instability of the equilibrium point). In the case of a *focus*, one should determine the direction of trajectories twisting. This can be done by calculating the velocity vector $\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$ at any point, for example, at the point (1.0). Similarly, we can determine the direction of movement if the equilibrium position is a *center*.

The algorithm described here is not a rigid scheme. In the study of a particular system, other tricks and techniques are acceptable in order to draw up the phase portrait.

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Навчальне видання

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