EXTREMAL PROBLEMS IN THEORY OF CENTRAL WIMAN-VALIRON INDEX

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Abstract. We consider some properties of central index in Wiman-Valiron index. We introduce the notion of a determining sequence of a central index $\nu(r)$ corresponding to a fixed transcendental function f and the notion of a determining sequence for an arbitrary fixed central index $\nu(r)$. Let $\rho_1, \rho_2, \ldots, \rho_s, \ldots$ be the points of the jumps of the function $\nu(r)$ taken counting their multiplicities. This means that if at a point ρ_s the jump is equal to m_s, then the quantity ρ_s appears m_s times in this sequence. Such sequence is called determining sequence of the function $\nu(r)$. We introduce the notion of the regularization of the function $\nu(r)$, which is employed for proving main statements. We study two extremal problems in the class of functions with a prescribed central index. We obtain the expression for the maximum of the modulus of the extremal function in terms of its central index. The main obtained results are as follows. Let T_ν be the set of all transcendental functions f with a prescribed central index $\nu(r)$, $M(r, f) = \max\{|f(re^{i\theta})| : 0 \leqslant \theta \leqslant 2\pi\}$, and let $M(r, \nu) = \sup\{M(r, f) : f \in T_\nu\}$. Then for each $r > 0$, in the class of the functions T_ν, the quantity $M(r, \nu)$ is attained at the same function for all $r > 0$. We describe the form of such extremal function. We also prove that for each fixed $r_0 > 0$ and for each prescribed central index $\nu(r)$, in the class T_ν there exists a function $f_0(z)$ such that $M(r_0, f_0) = \inf\{M(r_0, f) : f \in T_\nu\}$.

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1. INTRODUCTION

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.1}$$

be an entire transcendental functional not coinciding with a polynomial, that is, this is an entire function such that infinitely many coefficients a_n in expansion (1.1) are non-zero. As it is known, see, for instance, [1], the maximal term of the function $f(z)$ is determined by the formula

$$\mu(r, f) = \max_n |a_n| r^n, \quad r \geqslant 0, \tag{1.2}$$

while the central index is defined as

$$\nu(r, f) = \max\{n : |a_n| r^n = \mu(r, f)\}. \tag{1.3}$$

In what follows, if it is clear from the context which function is discussed, we shall omit the symbol f in the notation of the maximal term of the function $f(z)$ and of its central index and we shall write $\mu(r)$, $\nu(r)$ instead of $\mu(r, f)$ and $\nu(r, f)$.

Thus, the central index $\nu(r, f)$ is characterized by two properties:

1) for a fixed r we have $|a_{\nu(r)}|r^{\nu(r)} = \max_n |a_n|r^n$;

2) $\nu(r)$ is the maximal index possessing property 1).

For polynomials $P_n(z) = a_n z^n + \cdots + a_0$ of degree n, starting from some $r > 0$, the role of the central index is played by the degree n: $\nu(r, P_n) = n$. For each entire transcendental function we have $\nu(r, f) \uparrow +\infty$ as $r \to +\infty$, see Lemma 2.1.

Since an arbitrary entire function $f(z)$ can be represented as $f(z) = cz^m f_0(z)$, where $f_0(0) = 1$, then without loss of generality we can assume that $a_0 = 1$ in representation (1.1). This yields that

$$\nu(0) = 0, \qquad \mu(0) = 1. \tag{1.4}$$

In what follows, without saying explicitly, condition (1.4) is supposed to hold.

The theory of maximal term and central index and its role in the theory of entire and meromoprhic functions and in various applications like ordinary differential equations, probability theory and others, was well described in the aforementioned classical monograph by H. Wittich [1]. The relations between the maximal term $\mu(r, f)$ and the maximal absolute value

$$M(r, f) = \max_{0 \leqslant \theta \leqslant 2\pi} |f(re^{i\theta})|$$

were studied in details in the theory of central index of Wiman [2], [3] and Valiron [4].

In the theory of ordinary differential equations, a special role is played by the fact that the derivative of a transcendental function $f(z)$ can be expressed via $f(z)$ and its central index $\nu(r, f)$. In particular, if ζ is a point of the maximum of the function $f(z)$ on the circumference $|\zeta| = r$, that is, $|f(\zeta)| = M(|\zeta|, f)$, then the relation

$$\frac{f'(\zeta)}{f(\zeta)} = \frac{\nu(r, f)}{\zeta}(1 + o(\zeta)), \qquad \zeta \to \infty,$$

holds as well as corresponding formulae for higher derivatives.

By means of the theory of central index one can also prove a little Picard theorem [1]:

An entire transcendental function $f(z)$ admits at most one finite value.

A lot of attention was paid to studying the relations between the maximal term and the maximal absolute value of the transcendental function, we mention only some works by P.C. Rosenbloom [5], R. London [6], P. Lockhart and E.G. Straus [7], M. Sheremeta [8], [9], [10], [11], P. Filevich [12], [13], [14] and many others. At the same time, in our opinion, essentially less attention was paid to the central index. We know only work by P. Filevich [15], in which it was studied how the growth of the maximum of the absolute value of an entire function depended on growth of its central index.

The present work is aimed also on covering partially this gap. We express our gratitude to A.F. Grishin , whose ideas stimulated this work.

Our main obtained results are as follows. Let T_ν be the set of all transcendental functions with a prescribed central index $\nu(r)$. We denote:

$$M(r, \nu) = \sup_{f \in T_\nu} M(r, f).$$

Then for each $r > 0$, the quantity $M(r, \nu)$ is attained in the class of functions T_ν on the same function for all $r > 0$. We provide the form of such extremal function, see Theorems 2.2, 2.3.

We also prove that for each fixed $r_0 > 0$ and each prescribed central index $\nu(r)$, there exists a function $f_0(z)$ in the class T_ν such that

$$M(r_0, f_0) = \inf_{f \in T_\nu} M(r_0, f),$$

see Theorem 3.1.

We provide two lemmata, namely, Lemma 3.1 and Lemma 3.2 on the existence of a function $f_0(z) \in T_\nu$ such that

$$M(r_0, f_0) = \inf_{f \in T_\nu(r_0)} M(r_0, f),$$
$$M(r_0, f_0) = \sup_{f \in T_\nu(r_0)} M(r_0, f).$$

In conclusion of this section let us briefly describe the structure of the paper. In the next section we introduce the main notations and consider the first extremal problem. In the third section we consider the second extremal work.

2. First extremal problem

First of all we formulate a lemma on characterization of the central index.

Lemma 2.1. *The central index is a non-decreasing, right continuous function on the interval* $(0; +\infty)$.

Proof. Since

$$\ln \mu(e^x) = \max_n (\ln |a_n| + nx),$$

the function $\ln \mu(e^x)$ is strictly increasing and convex.

Let $r_0 > 0$, $\nu = \nu(r_0)$ and $m < \nu$. Then, employing Properties 1) and 2) of the central index, for $r > r_0$ we obtain

$$\left| \frac{a_m}{a_\nu} \right| r^{m-\nu} < \left| \frac{a_m}{a_\nu} \right| r_0^{m-\nu} = \frac{|a_m| r_0^m}{|a_\nu| r_0^\nu} \leqslant 1$$

and this implies that

$$|a_m| r^m \leqslant |a_\nu| r^\nu.$$

This is why $\nu(r) \neq m$ for each $m < \nu$ and therefore,

$$\nu(r) \geqslant \nu = \nu(r_0).$$

This proves that $\nu(r)$ is a non-decreasing function. In particular, the inequality

$$\nu(r_0 + 0) \geqslant \nu(r_0)$$

holds true.

Let us show that this relation is in fact an identity. Since the function $\ln \mu(r)$ is convex and this is why it is continuous, we have

$$\ln |a_{\nu(r_0+0)}| + \nu(r_0 + 0) \ln r_0 = \ln |a_{\nu(r_0)}| + \nu(r_0) \ln r_0$$

and this is why

$$|a_{\nu(r_0+0)}| r_0^{\nu(r_0+0)} = |a_{\nu(r_0)}| r_0^{\nu(r_0)}.$$

The inequality

$$\nu(r_0 + 0) > \nu(r_0)$$

The properties formulated in Lemma 2.1 are well-known, see, for instance, [16, Ch. IV], and are often cited. However, we have never seen their detailed proof. This is why, not pretending for the authorship, we decide to formulate them as a lemma.

contradicts the definition of $\nu(r_0)$. Thus, the function $\nu(r)$ is right continuous. The proof is complete. $\qquad\square$

Let us find the right derivative of the function $\ln \mu(r)$. Let $h > 0$ be a sufficiently small number. We have:

$$\frac{1}{h}(\ln \mu(r + h) - \ln \mu(r)) = \frac{1}{h}\nu(r)(\ln(r + h) - \ln r).$$

Then

$$(\ln \mu(r))'_+ = \frac{\nu(r)}{r}. \tag{2.1}$$

Now it follows from (1.4) and (2.1) that

$$\ln \mu(r) = \int_0^r \frac{\nu(t)}{t}\, dt = \int_0^r \nu(t)\, d(\ln t).$$

Integrating by parts, we then get:

$$\ln \mu(r) = (\nu(t) \ln t)\, |_0^r - \int_0^r \ln t\, d\nu(t) = \nu(r) \ln r - \int_0^r \ln t\, d\nu(t). \tag{2.2}$$

Let us introduce the notion of the determining sequence of the function $\nu(r)$. Let ρ_1, ρ_2, ..., ρ_s, ... be points of the jumps of the function $\nu(r)$ taken counting their multiplicities. This means that if at a point ρ_s the jump is equal to m_s, then in the above sequence the quantity ρ_s appears m_s times. Such sequence will be called a *determining sequence of the function* $\nu(r)$. Such definition is justified by the fact the function $\nu(r)$ is determined uniquely by its determining sequence. Indeed, for each $r \geqslant \rho_1$, there exists an index s such that $\rho_s \leqslant r < \rho_{s+1}$. Then $\nu(r) = m_s$ if $r \in [\rho_s; \rho_{s+1})$. But if $r \in [0; \rho_1)$, then $\nu(r) = 0$.

We also note that since for the transcendental function $f(z)$ the following limit is infinite [16, Ch. IV]

$$\lim_{r \to \infty} \nu(r, f) = +\infty,$$

then

$$\lim_{s \to \infty} \rho_s = +\infty.$$

We also note the identity

$$\int_0^r \ln t\, d\nu(t) = \ln \prod_{s=1}^{\nu(r)} \rho_s. \tag{2.3}$$

Since it follows from formulae (1.2) and (1.3) that

$$\ln \mu(r) = \ln |a_{\nu(r)}| + \nu(r) \ln r,$$

together with formulae (2.2) and (2.3) this gives

$$|a_{\nu(r)}| = \prod_{s=1}^{\nu(r)} \frac{1}{\rho_s}. \tag{2.4}$$

Let $n \in (\nu(r-0); \nu(r))$. Then taking into consideration identity (2.4), we have:

$$|a_n|r^n \leqslant |a_{\nu(r)}|r^{\nu(r)},$$

$$|a_n| \leqslant |a_{\nu(r)}|r^{\nu(r)-n} = \prod_{s=1}^{\nu(r)} \frac{1}{\rho_s} r^{\nu(r)-n}$$

$$= \prod_{s=1}^{\nu(r-0)} \frac{1}{\rho_s} \left(\frac{1}{r}\right)^{\nu(r)-\nu(r-0)} r^{\nu(r)-n}$$

$$= \prod_{s=1}^{\nu(r-0)} \frac{1}{\rho_s} \left(\frac{1}{r}\right)^{n-\nu(r-0)} = \prod_{s=1}^{n} \frac{1}{\rho_s}.$$

Thus, we obtain the inequality

$$|a_n| \leqslant \prod_{s=1}^{n} \frac{1}{\rho_s}. \tag{2.5}$$

We note that if $n = \nu(r)$ for some $r > 0$, then inequality (2.5) becomes an identity.

We denote

$$\tilde{a}_n = \prod_{s=1}^{n} \frac{1}{\rho_s}. \tag{2.6}$$

Definition 2.1. *A sequence $\{\tilde{a}_n\}_{n=1}^{\infty}$ determined by identity (2.6) is called a convex regularization of the sequence $\{a_n\}_{n=1}^{\infty}$.*

It is clear that

$$|a_n| \leqslant \tilde{a}_n, \tag{2.7}$$

and if $n = \nu(r)$ for some $r > 0$, then inequality (2.7) becomes an identity. Generally speaking, inequality (2.7) can become an identity also for other n.

The above arguing implies the following theorem.

Theorem 2.1. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad f_1(z) = \sum_{n=0}^{\infty} b_n z^n,$$

be transcendental functions, $\nu(r, f)$ be the central index of the function $f(z)$ and $\nu(r, f_1)$ be the central index of the function $f_1(z)$. Let the inequality $|b_n| \leqslant \tilde{a}_n$ hold, where $\{\tilde{a}_n\}_{n=1}^{\infty}$ is a convex regularization of the sequence $\{a_n\}_{n=1}^{\infty}$ and if $n = \nu(r, f)$ for some $r > 0$, then $|b_n| = \tilde{a}_n$. Then $\nu(r, f_1) = \nu(r, f)$.

We give the following definition.

Definition 2.2. *Suppose that we are given a central index $\nu(r)$ and let $\{\rho_s\}_{s=1}^{\infty}$ be the determining sequence of the function $\nu(r)$. The sequence $\{\tilde{a}_n\}_{n=1}^{\infty}$ defined by identity (2.6) is called a convex regularization of the function $\nu(r)$.*

We note that Definition 2.1 corresponds to a given transcendental function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

while Definition 2.2 corresponds to the given central index $\nu(r)$.

We denote by T_ν the set of all transcendental functions with a prescribed central index $\nu(r)$ and let $M(r, \nu) = \sup_{f \in T_\nu} M(r, f)$. Theorem 2.1 provides a solution to the following extremal problem.

Theorem 2.2. *Suppose that we are given a central index $\nu(r)$. Then for each $r > 0$ the quantity $M(r, \nu)$ is attained in the class of functions T_ν at the same function for all $r > 0$:*

$$\tilde{f}(z) = \sum_{n=0}^{\infty} \tilde{a}_n z^n,$$

where $\{\tilde{a}_n\}_{n=1}^{\infty}$ is a convex regularization of the function $\nu(r)$.

Proof. It is clear that $\tilde{f}(z) \in T_\nu$. By inequality (2.7) we obtain that for each function $f \in T_\nu$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the relation

$$M(r, f) \leqslant \sum_{n=0}^{\infty} |a|_n r^n \leqslant \sum_{n=0}^{\infty} \tilde{a}_n r^n = M(r, \tilde{f})$$

holds. The proof is complete. $\qquad\square$

The next theorem provides a representation for quantity $M(r, \tilde{f})$.

Theorem 2.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a transcendental function, $\nu(r) = \nu(r, f)$ be its central index, $\{\rho_n\}_{n=1}^{\infty}$ be the determining sequence of the jump points of the functions $\nu(r)$. Then*

$$M(r, \tilde{f}) = \sum_{n=0}^{\infty} \tilde{a}_n r^n = 1 + \sum_{n=1}^{\infty} \prod_{m=1}^{n} \left(\frac{r}{\rho_m} \right). \tag{2.8}$$

In the case, when the jumps of the function $\nu(r)$ are equal to 1, we have

$$M(r, \tilde{f}) = \sum_{n=0}^{\infty} \tilde{a}_n r^n = 1 + \int_0^{\infty} \exp\left(\int_{[0,s]} \ln\frac{r}{t} \, d\nu(t) \right) d\nu(s) = 1 + \int_0^{\infty} \mu(s) \left(\frac{r}{s} \right)^{\nu(s)} d\nu(s). \tag{2.9}$$

Proof. Identity (2.8) is an implication of identity (2.6). In the case if $\rho_n < \rho_{n+1}$, then

$$\prod_{m=1}^{n} \left(\frac{r}{\rho_m} \right) = \exp\left(\sum_{m=1}^{n} \ln\frac{r}{\rho_m} \right) = \exp\left(\int_{[0;\rho_n]} \ln\frac{r}{\rho_m} \, d\nu(t) \right).$$

This relation implies the second identity in (2.9).

We then have

$$\int_{[0,s]} \ln\frac{r}{t} \, d\nu(t) = \nu(s) \ln\frac{r}{s} + \int_0^s \frac{\nu(t)}{t} \, dt = \ln\mu(s) + \nu(s) \ln\frac{r}{s}.$$

This gives:

$$M(r, \tilde{f}) = 1 + \int_0^{\infty} \mu(s) \left(\frac{r}{s} \right)^{\nu(s)} d\nu(s).$$

We also observe that if $\nu(s) = 0$ as $s < 1$, then

$$s^{\frac{\mu(s)}{\nu(s)}} = \exp\left(\int_1^s \frac{\nu(t) - \nu(s)}{t} \, d\nu(t) \right) \leqslant 1.$$

The proof is complete. $\qquad\square$

Remark 2.1. *In the case when not all jumps of the function $\nu(r)$ are equal to 1, the function $M(r, \tilde{f})$ is not represented by an integral with respect to the measure ν.*

3. SECOND EXTREMAL PROBLEM

The central index of each transcendental function $f(z)$, $f(0) = 1$, is an integer-valued increasing right continuous function $\nu(r)$ on the semi-axis $[0; \infty)$ and $\nu(0) = 0$. And vice versa, each such function $\nu(r)$ is a central index of some transcendental function $f(z)$, $f(0) = 1$. As in Section 2, let T_ν be the set of all transcendental functions with a prescribed central index $\nu(r)$. This class is easily described in terms of the coefficients of the power series $\sum\limits_{n=0}^{\infty} a_n z^n$, $a_0 = 1$.

We denote by $\mathcal{N}_\nu$ the set of all values of the function $\nu(r)$ and let $\{\rho_n\}_{n=1}^{\infty}$ be the determining sequence of the jump points of the function $\nu(r)$,

$$\tilde{a}_n = \prod_{s=1}^{n} \frac{1}{\rho_s}.$$

Then $f(z) \in T_\nu$ if and only if $|a_n| \leqslant \tilde{a}_n$, while in the case $n \in \mathcal{N}_\nu$ the inequality should be become the identity.

Theorem 3.1. *Let $r_0 > 0$ and $\nu(r)$ be a given central index. Then in the class T_ν, there exists a function $f_0(z)$ such that*

$$M(r_0, f_0) = \inf_{f \in T_\nu} M(r_0, f).$$

Proof. Let $\{f_m(z)\}_{m=1}^{\infty} \subset T_\nu$, $f_m(z) = \sum\limits_{n=0}^{\infty} a_{m,n} z^n$ be a minimizing sequence of functions such that

$$\lim_{m \to \infty} M(r_0, f_m) = \inf_{f \in T_\nu} M(r_0, f).$$

It follows from inequality $|a_{m,n}| \leqslant \tilde{a}_n$ that the sequence $\{f_m(z)\}_{m=1}^{\infty}$ is a compact one in the topology of uniform convergence on compact sets. This is why, without loss of generality, we can suppose that the sequence $\{f_m(z)\}_{m=1}^{\infty}$ is uniformly converging on an arbitrary compact set in the complex plane. Let

$$f_0(z) = \sum_{n=0}^{\infty} a_n z^n = \lim_{m \to \infty} f_m(z).$$

Then

$$M(r_0, f_0) = \lim_{m \to \infty} M(r_0, f_m), \quad a_n = \lim_{m \to \infty} a_{m,n}.$$

Since $|a_{m,n}| \leqslant \tilde{a}_n$, then $|a_n| \leqslant \tilde{a}_n$. Moreover, since $|a_{m,n}| = \tilde{a}_n$ as $n \in \mathcal{N}_\nu$, then for such n we also have $|a_n| = \tilde{a}_n$. Thus, $f_0(z) \in T_\nu$. The proof is complete. $\qquad\square$

Given a fixed $z_0 \neq 0$, $|z_0| = r_0$, we denote by $T_\nu(r_0)$ the set of functions $f(z) \in T_\nu$ such that $M(|z_0|, f) = |f(z_0)|$; it is clear that the set $T_\nu(r_0)$ is non-empty. We consider the following problems.

Problem 1. *Whether there exists a function $f_0(z) \in T_\nu(r_0)$ such that*

$$M(r_0, f_0) = \inf_{f \in T_\nu(r_0)} M(r_0, f)?$$

Problem 2. *Whether there exists a function $f_0(z) \in T_\nu(r_0)$ such that*

$$M(r_0, f_0) = \sup_{f \in T_\nu(r_0)} M(r_0, f)?$$

The arguing in the proof of Theorem 3.1 show that Problems 1 and 2 are solvable. Namely, the following statements are true.

Lemma 3.1. *Let $T_\nu(r_0)$ be the set of functions $f(z) \in T_\nu$ such that*

$$M(|z_0|, f) = |f(z_0)|.$$

Then there exists a function $f_0(z) \in T_\nu(r_0)$ such that

$$M(r_0, f_0) = \inf_{f \in T_\nu(r_0)} M(r_0, f).$$

Lemma 3.2. *Let $T_\nu(r_0)$ be the set of functions $f(z) \in T_\nu$ such that*

$$M(|z_0|, f) = |f(z_0)|.$$

Then there exists a function $f_0(z) \in T_\nu(r_0)$ such that

$$M(r_0, f_0) = \sup_{f \in T_\nu(r_0)} M(r_0, f).$$

Remark 3.1. *In Lemmata 3.1, 3.2, instead of the class $T_\nu(r_0)$, it is possible to consider the class of functions $T_\nu^*(r_0) = \{f \in T_\nu : M(|z_0|, f) = f(z_0)\}$; this set is non-empty for each $z_0 \neq 0$. In this case, the statements of Lemmata 3.1, 3.2 remain true.*

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