JOURNAL OF NANO- AND ELECTRONIC PHYSICS
Vol. 14 No 4, 04014(5pp) (2022)
ЖУРНАЛ НАНО- ТА ЕЛЕКТРОННОЇ ФІЗИКИ
Том 14 № 4, 04014(5cc) (2022)

Treatment of Nonlinear Electrical Circuits by the Caputo-Fabrizio Derivative

Z. Korichi[1,*], A. Souigat[1], Y. Benkrima[1], M.T. Meftah[2]

[1] *Ecole Normale Supérieure de Ouargla, 30000 Ouargla, Algeria*
[2] *Laboratoire LRPPS, Faculté de Mathématiques et Sciences de la Matière, Université Kasdi-Merbah, Ouargla 30000, Algeria*

(Received 15 May 2022; revised manuscript received 10 August 2022; published online25 August 2022)

Caputo and Fabrizio have recently proposed a new fractional order derivative without singular kernel, which is suitable for the Laplace transform and has many interesting properties that motivated its use to solve and model many phenomena in various branches of science. In our work, we have treated the fractional differential equations of nonlinear electrical circuits by using the definition of Caputo-Fabrizio derivative. Indeed, we have transformed the fractional differential equations, describing *RC*, *RL* and *LC* circuits, into an ordinary order differential equation. Then we have determined explicit solutions to these differential equations. For the *RC* circuit, we studied changes in charge with time and with derivative order and we found that for all values, the curve retains its general shape. In contrast to the time constant, which increases as alpha increases, we established that q_0 has no relationship to. For the *RL* circuit, time variations of electric current are investigated for various alpha values, and we found that the maximum current I_0 does not change with derivative order. Also, for the *LC* circuit, we studied charge changes for various alpha values, and we showed that the shape of the *LC* vibration is related to the derivative order. For the *LC* circuit, the vibration is a sine wave, and the *RC* circuit vibration is a damped vibration. To validate our obtained results, we found the familiar results.

Keywords: Fractional differential equations, Fractional derivative, Caputo-Fabrizio fractional derivative, Fractional electrical circuits.

DOI: 10.21272/jnep.14(4).04014 PACS numbers: 02.90. + p, 45.10.Hj

1. INTRODUCTION

The fractional derivative is a branch of mathematical analysis that studies the generalization of notions of derivation to arbitrary orders. The theory of this concept is not new, but it is considered an old and its origins dates back to the end of the 17th century [1].

The fractional calculus has several definitions, which do not generally coincide. The Grunwald-Letnikov, Riesz, Riemann-Liouville, and Caputo definitions are the most often used [2, 3].

In the last few years, the concept of the fractional derivative has been getting a lot of attention. It has been used in many different fields of physics, such as classical and quantum physics, plasma physics, thermodynamics, statistical mechanics, and so forth [3-14].

In general, the fractional calculus is used to describe processes that are spatially and temporally nonlocal, especially in electrical circuits, and many authors studied fractional electrical circuits, such as Rousanet et al. [15] proposed using a fractional differential equation to investigate *LC* and *RC* circuits. Ertiket et al. [16] showed how different capacitors are charged and discharged within the framework of fractional calculus, and [17] proposed a number of electrical circuits using the Caputo-Fabrizio fractional operator.

The aim of our work is to study *RC*, *RL* and *LC* circuits by using Caputo-Fabrizio fractional derivative and find solutions to the differential equations of these electrical circuits.

This paper is organized as follows. In section 2, we present the Caputo-Fabrizio derivative and propose a solution to the fractional differential equation using the Laplace transform. In Section 3, *RC* fractional electrical circuit is studied. In Section 4, we solve a fractional differential equation for the *RL* circuit. In Section 5, we study an oscillating system consisting of a coil and a capacitor. Conclusion is given in Section 6.

2. CAPUTO-FABRIZIO DERIVATIVE

Recently Caputo and Fabrizio [18] redefined the Caputo derivative and proposed the following new fractional derivative for the function $g(x)$ belonging to the Sobolev space:

$$D_x^{\alpha} g(x) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^x g'(p) \exp\left(-\alpha \frac{x-p}{1-\alpha}\right) dp \;,\quad (2.1)$$

where α represents the derivation degree $0 < \alpha \le 1$, and $M(\alpha)$ is a constant of normalization that varies according to α.

Using the Laplace transform and the definition of Fabrizio and Caputo, which is presented in equation (2.1), Losada and Nieto [19] proposed a solution to the following fractional differential equation:

$$D_x^{\alpha} g(x) = u(x) \;, \quad (2.2)$$

and they deduced that

$$g(x) = \frac{2(1-\alpha)M(\alpha)}{(2-\alpha)}\left[u(x) - u(x)\right] + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^x u(s)ds + g(0). \quad (2.3)$$

*zinebkorichid02@gmail.com

 © 2022 Sumy State University

Taking the normalization condition [19]:

$$\frac{2M(\alpha)}{(2-\alpha)}=1\,. \tag{2.4}$$

So, the solution of equation (2.2) is given by:

$$g(x)=(1-\alpha)\left[u(x)-u(x)\right]+\alpha\int_0^x u(s)ds+g(0)\,. \tag{2.5}$$

3. FRACTIONAL *RC* ELECTRICAL CIRCUIT

Consider an electrical circuit consisting of a capacitor C, a resistor R and a generator.

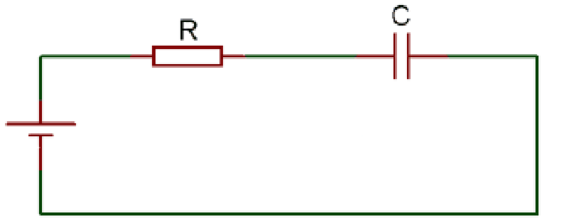

Fig. 1 – *RC* electrical circuit

Using Kirchhoff's laws, we find that the differential equation for $Q(T)$ takes the form:

$$\frac{dq(t)}{dt}+\frac{1}{RC}q(t)=\frac{E}{R}\,. \tag{3.1}$$

This first-order differential equation has the solution:

$$q(t)=q_0\left(1-\exp\left(-\frac{t}{RC}\right)\right). \tag{3.2}$$

Gómez et al. [20] proposed a fractional differential equation for this circuit:

$$\frac{R}{\sigma_R^{(1-\alpha)}}\frac{d^\alpha q(t)}{dt^\alpha}+\frac{1}{C}q(t)=E\,, \tag{3.3}$$

where σ_R is the parameter that determines the fractional structures of R.

It is easy to see that equation (3.3) can be rewritten as:

$$\frac{d^\alpha q(t)}{dt^\alpha}=\frac{\sigma_R^{(1-\alpha)}E}{R}-\frac{\sigma_R^{(1-\alpha)}}{RC}q(t)\,. \tag{3.4}$$

We will now try to solve this fractional differential equation using the Caputo-Fabrizio derivative. According to equation (2.5), the solution of equation (3.4) is derived from:

$$q(t)=-(1-\alpha)\frac{\sigma_R^{(1-\alpha)}}{RC}\left[q(t)-q(0)\right]+ \\ \alpha\int_0^t\left[\frac{\sigma_R^{(1-\alpha)}E}{R}-\frac{\sigma_R^{(1-\alpha)}}{RC}q(s)\right]ds+q(0). \tag{3.5}$$

By calculating the derivative of the above equation, we can immediately obtain the solution by solving the following ordinary differential equation:

$$\left(1+(1-\alpha)\frac{\sigma_R^{(1-\alpha)}}{RC}\right)\frac{dq(t)}{dt}+\alpha\frac{\sigma_R^{(1-\alpha)}}{RC}q(t)=\alpha\frac{\sigma_R^{(1-\alpha)}E}{R}\,. \tag{3.6}$$

The last equation has the solution:

$$q(t)=q_0\left(1-\exp\left(-\frac{t}{\tau}\right)\right), \tag{3.7}$$

where

$$\tau=\frac{RC\left(1+(1-\alpha)\frac{\sigma_R^{(1-\alpha)}}{RC}\right)}{\alpha\sigma_R^{(1-\alpha)}} \tag{3.8}$$

is a fractional time constant.

Substituting charge expression (3.8) into equation (3.6), we get

$$q_0=CE\,. \tag{3.9}$$

Fig. 2 shows how the charge in the RC circuit changes over time for $\alpha=\{1, 0.8, 0.6, 0.4\}$.

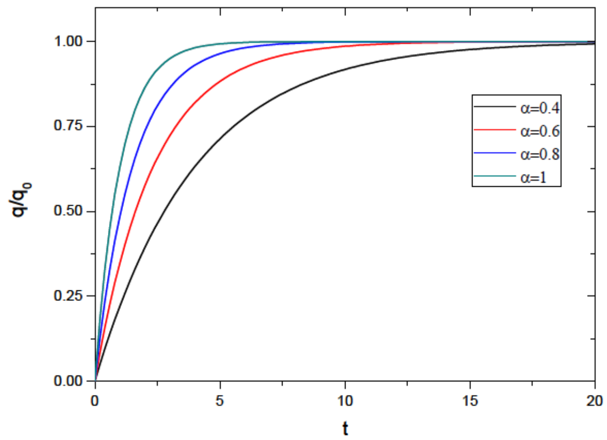

Fig. 2 – Plots of q/q_0 with parameters $\alpha=\{1, 0.8, 0.6, 0.4\}$ and $RC=1$

- For all α values, the curve retains its general shape, and it is identical to the non-fractional case.
- q_0 has no relation to α and it relates only to E and C.
- The degree of derivation α has an effect on the time constant τ.
- There is an inverse relationship between the degree of derivation α and the time constant τ.

4. FRACTIONAL *RL* ELECTRICAL CIRCUIT

Consider a RL circuit (resistor-inductor circuit) consisting of an inductor L and a resistor R, which will be connected in series.

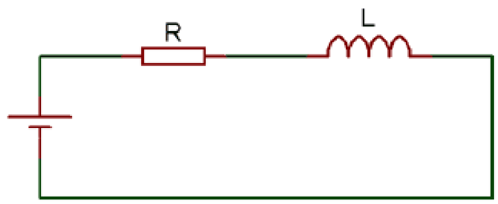

Fig. 3 – *RL* electrical circuit

According to Kirchhoff's voltage law, the sum of the voltages in a circuit must be zero. As a result, the differential equation shown below is generated:

$$L\frac{di(t)}{dt}+Ri(t)=E\,. \tag{4.1}$$

The solution of this differential equation is:

$$i(t)=\frac{E}{R}\left(1-\exp\left(-\frac{R}{L}t\right)\right). \tag{4.2}$$

For this circuit, the fractional differential equation takes the form:

$$\frac{L}{{\sigma_L}^{(1-\alpha)}}\frac{d^{\alpha}i(t)}{dt^{\alpha}}+Ri(t)=E\,. \tag{4.3}$$

In order to solve the above fractional differential equation with the Fabrizio derivative, we use the same method that we used in the previous section to solve the previous differential equation.

So,

$$\begin{gathered}i(t)=-(1-\alpha)\frac{R{\sigma_L}^{(1-\alpha)}}{L}\left[i(t)-i(0)\right]+\\ \alpha\int_0^t\left[\frac{{\sigma_L}^{(1-\alpha)}E}{L}-\frac{R{\sigma_L}^{(1-\alpha)}}{L}i(s)\right]ds+i(0).\end{gathered} \tag{4.4}$$

When we take the first derivative of the last equation with respect to t, we get:

$$\begin{gathered}\left(1+(1-\alpha)\frac{R{\sigma_L}^{(1-\alpha)}}{L}\right)\frac{di(t)}{dt}+\\ \alpha\frac{R{\sigma_L}^{(1-\alpha)}}{L}i(s)=\alpha\frac{{\sigma_L}^{(1-\alpha)}E}{L}.\end{gathered} \tag{4.5}$$

The solution of equation (4.5) is written in the form:

$$i(t)=I_0\left(1-\exp\left(-\frac{t}{\tau}\right)\right), \tag{4.6}$$

where

$$\tau=\alpha\frac{R{\sigma_L}^{(1-\alpha)}}{L\left(1+(1-\alpha)\frac{R{\sigma_L}^{(1-\alpha)}}{L}\right)}. \tag{4.7}$$

In order to find I_0, we can use equations (4.5) and (4.6). We obtain:

$$I_0=\frac{E}{R}. \tag{4.8}$$

The electric current variations in the *RL* circuit are shown in Fig. 4 in terms of time.

The graphs above show that the maximum current value i_0 does not change with the degree of derivation α. Whereas, as α increases, the time constant reduces τ.

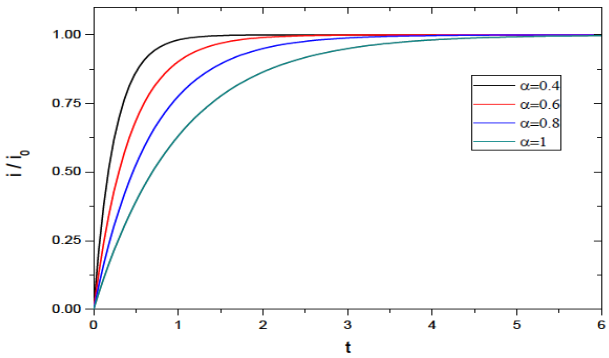

Fig. 4 – Plots of I/I_0 with parameters $\alpha = \{1, 0.8, 0.6, 0.4\}$ and $R/L = 1$

5. FRACTIONAL *LC* CIRCUIT

Consider an electrical circuit containing a capacitor of capacitance C, which stores electric charge q_0, and a coil of inductance L. The circuit is closed at $t = 0$.

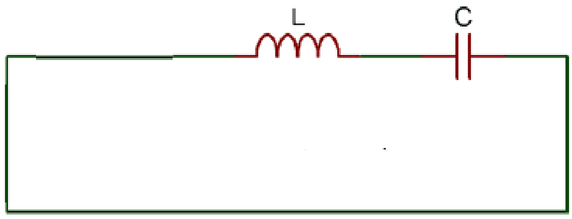

Fig. 5 – *LC* electrical circuit

The differential equation for the charge evolution q is given by:

$$\frac{d^2q(t)}{dt^2}+\frac{1}{LC}q(t)=0\,. \tag{5.1}$$

The solution of equation (5.1) is:

$$q(t)=q_0\cos(\omega_0 t)\,, \tag{5.2}$$

where ${\omega_0}^2=1/LC$.

For a fractal element, the fractional differential equation for the *LC* circuit is [30]:

$$\frac{L}{{\sigma_L}^{2(1-\alpha)}}\frac{d^{2\alpha}q(t)}{dt^{2\alpha}}+\frac{1}{C}q(t)=0\,, \tag{5.3}$$

where σ_R is the parameter that determines the fractional structures of L.

Equation (5.3) can be written as:

$$\frac{d^{\alpha}}{dt^{\alpha}}f(t)=-\frac{{\sigma_L}^{2(1-\alpha)}}{LC}q(t)\,, \tag{5.4}$$

where:

$$f(t)=\frac{d^{\alpha}q(t)}{dt^{\alpha}}\,. \tag{5.5}$$

The solution of the fractional equation (5.4) with the Caputo-Fabrizio fractional derivative is derived from equation (2.5):

$$f(t) = -(1-\alpha)\frac{\sigma_L^{2(1-\alpha)}}{LC}\left[q(t)-q(0)\right] - \alpha\frac{\sigma_L^{2(1-\alpha)}}{LC}\int_0^t q(s)ds + f(0). \quad (5.6)$$

The expression of $q(t)$ can be obtained via solving the following equation:

$$f(t) = \frac{d^\alpha q(t)}{dt^\alpha}. \quad (5.7)$$

Then

$$q(t) = (1-\alpha)\left[f(t)-f(0)\right] + \alpha\int_0^t f(s)ds + q(0). \quad (5.8)$$

The second derivative of the last equation gives:

$$\frac{d^2q(t)}{dt^2} = (1-\alpha)\frac{d^2f(t)}{dt^2} + \alpha\frac{df(t)}{dt}. \quad (5.9)$$

When we replace $\frac{d^2f(t)}{dt^2}$ and $\frac{df(t)}{dt}$ by their expressions

$$\frac{df(t)}{dt} = -(1-\alpha)\frac{\sigma_L^{2(1-\alpha)}}{LC}\frac{dq(t)}{dt} - \alpha\frac{\sigma_L^{2(1-\alpha)}}{LC}q(t), \quad (5.10)$$

$$\frac{d^2f(t)}{dt^2} = -(1-\alpha)\frac{\sigma_L^{2(1-\alpha)}}{LC}\frac{d^2q(t)}{dt^2} - \alpha\frac{\sigma_L^{2(1-\alpha)}}{LC}\frac{dq(t)}{dt}, \quad (5.11)$$

we find

$$\left(LC+(1-\alpha)^2\sigma_L^{2(1-\alpha)}\right)\frac{d^2q(t)}{dt^2} + 2\alpha(1-\alpha)\sigma_L^{2(1-\alpha)}\frac{dq(t)}{dt} + \alpha^2\sigma_L^{2(1-\alpha)}q(t) = 0. \quad (5.12)$$

The last equation has a specific solution of the following form:

$$q(t) = A\exp(rt). \quad (5.13)$$

So, equation (5.12) becomes

$$\left(LC+(1-\alpha)^2\sigma_L^{2(1-\alpha)}\right)r^2 + 2\alpha(1-\alpha)\sigma_L^{2(1-\alpha)}r + \alpha^2\sigma_L^{2(1-\alpha)} = 0. \quad (5.14)$$

It is easy to see that the last equation has the following solutions:

$$\begin{cases} r_1 = -\lambda - i\omega, \\ r_2 = -\lambda + i\omega, \end{cases} \quad (5.15)$$

where

$$\lambda = \frac{\alpha(1-\alpha)\sigma_L^{2(1-\alpha)}}{\left(LC+(1-\alpha)^2\sigma_L^{2(1-\alpha)}\right)} \quad (5.16)$$

and

$$\omega = \frac{\alpha\sigma_L^{(1-\alpha)}\sqrt{LC}}{\left(LC+(1-\alpha)^2\sigma_L^{2(1-\alpha)}\right)}. \quad (5.17)$$

So,

$$q(t) = A\exp(-\lambda t)\cos(\omega t+\varphi). \quad (5.18)$$

By taking the following initial conditions:

$$q(t) = q_0 \;;\mathrm{t}=0, \quad (5.19)$$

$$\frac{dq(t)}{dt} = 0 \;;\mathrm{t}=0, \quad (5.20)$$

we get

$$A = \frac{q_0}{\cos(\varphi)}, \quad (5.21)$$

$$\varphi = a\tan\left(-\frac{\lambda}{\omega}\right). \quad (5.22)$$

It is worth noting here that when $\alpha = 1$, we find the familiar results.

Fig. 6 shows time changes of charge for $\alpha = 1$, $\alpha = 0.8$, and $\alpha = 0.6$.

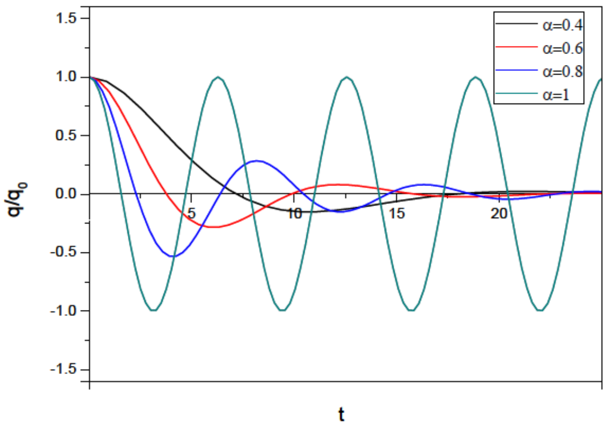

Fig. 6– Plots of q/q_0 with parameters $\alpha = \{1, 0.8, 0.6, 0.4\}$ and $LC = 1$

- For $\alpha = 1$, it can be clearly seen that the LC circuit's vibration is a sinusoidal oscillation.
- For $\alpha < 1$, vibration of the LC circuit is a damped vibration that eventually reaches 0.

6. CONCLUSIONS

In this work, we have applied the Caputo-Fabrizio fractional derivative to *RC*, *RL* and *LC* electrical circuits equations. First, we have transformed the fractional differential equations into a linear integral equation. Then, the solutions of *RC*, *RL* and *LC* fractional equations have been systematically established. When we limit our results to $\alpha \to 1$, we obtain the standard ones.

REFERENCES

1. K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations* (John Wiley & Sons, New York: 1993).
2. I. Podlubny, *Fractional Differential Equations* (1999).
3. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations* 204 (2006).
4. R. Hilfer, *Applications of Fractional Calculus in Physics* (World scientific: 2000).
5. N. Laskin, *Phys. Lett. A* **268**, 298 (2000).
6. N. Bouzid, M. Merad, *Few-Body. Systems* **58**, 131 (2017).
7. G.M. Zaslavsky, M.A. Edelman, *Physica D* **193**, 128 (2004).
8. V.E. Tarasov, G.M. Zaslavsky, *Chaos* **16**, 023110 (2006).
9. R.R. Nigmatullin, *Physica A* **363**, 282 (2006).
10. H. Sun, Y. Zhang, D. Baleanu, W. Chen, Y. Chen, *Communications in Nonlinear Science and Numerical Simulation* **64**, 213 (2018).
11. A.V. Milovanov, J.J. Rasmussen, *Phys. Rev. B* **66**, 134505 (2002).
12. V.E. Tarasov, *Phys. Plasmas* **12**, 082106 (2005).
13. Z. Korichi, M.T. Meftah, *J. Math. Phys.* **55**, 033302 (2014).
14. Z. Korichi, M.T. Meftah, *Theor. Math. Phys.* **186**, 374 (2016).
15. A.A. Rousan, N.Y. Ayoub, F.Y. Alzoubi, H. Khateeb, M. Al-Qadi, M.K. Hasan, B.A. Albiss, *Fractional Calculus and Applied Analysis* **9**, 33 (2006).
16. H. Ertik, A.E. Calik, H. Şirin, M. Şen, B. Öder, *Revista mexicana de física* **61**, 58 (2015).
17. J.F. Gómez-Aguilar, T. Córdova-Fraga, J.E. Escalante-Martínez, C. Calderón-Ramón, R.F. Escobar-Jiménez, *Revista mexicana de física* **62**, 144 (2016).
18. M. Caputo, M. Fabrizio, *Prog. Fract. Differ. Appl.* **1**, 73 (2015).
19. J. Losada, J.J. Nieto, *Prog. Fract. Differ. Appl.* **1**, 87 (2015).
20. J.F. Gómez Aguilar, J.R. Razo Hernández, J. Rosales García, M. Guía Calderón, *Ingeniería. Investigacióny Tecnología* **15**, 2 (2014).

Обробка нелінійних електричних кіл за допомогою похідної Капуто-Фабріціо

Z. Korichi[1], A. Souigat[1], Y. Benkrima[1], M.T. Meftah[2]

[1] *Ecole Normale Supérieure de Ouargla, 30000 Ouargla, Algeria*
[2] *Laboratoire LRPPS, Faculté de Mathématiques et Sciences de la Matière, Université Kasdi-Merbah, Ouargla 30000, Algeria*

Капуто та Фабріціо нещодавно запропонували нову похідну дробового порядку без сингулярного ядра, яка підходить для перетворення Лапласа та має багато цікавих властивостей, які спонукали до її використання для вирішення та моделювання багатьох явищ у різних галузях науки. У роботі ми обробили дробові диференціальні рівняння нелінійних електричних кіл, використовуючи визначення похідної Капуто-Фабріціо. Дійсно, ми перетворили дробові диференціальні рівняння, що описують *RC*, *RL* та *LC* кола, у диференціальне рівняння звичайного порядку. Потім ми визначили явні розв'язки цих диференціальних рівнянь. Для *RC*-ланцюга ми досліджували зміни заряду з часом і з порядком похідних і виявили, що для всіх значень крива зберігає свою загальну форму. На відміну від постійної часу, яка зростає зі збільшенням альфа, ми встановили, що q_0 не має відношення до неї. Для ланцюга *RL* досліджено зміни електричного струму з часом для різних значень альфа, і ми виявили, що максимальний струм I_0 не змінюється з порядком похідної. Крім того, для *LC*-ланцюга ми досліджували зміни заряду для різних значень альфа та показали, що форма вібрації *LC* пов'язана з порядком похідної. Для *LC*-ланцюга вібрація є синусоїдальною, а вібрація *RC*-ланцюга є затухаючою вібрацією. Щоб підтвердити отримані результати, ми знайшли схожі результати в літературі.

Ключові слова: Дробові диференціальні рівняння, Дробова похідна, Дробова похідна Капуто-Фабріціо, Дробові електричні кола.