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Modeling a Viscoelastic Support Considering Its Mass-Inertial Characteristics During Non-Stationary Vibrations of the Beam

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Abstract. Non-stationary loading of a mechanical system consisting of a hinged beam and additional support installed in the beam span was studied using a model of the beam deformation based on the Timoshenko hypothesis with considering rotatory inertia and shear. The system of partial differential equations describing the beam deformation was solved by expanding the unknown functions in the Fourier series with subsequent application of the integral Laplace transform. The additional support was assumed to be realistic rather than rigid. Thus it has linearly elastic, viscous, and inertial components. This means that the effect of a part of the support vibrating with the beam was considered such that their displacements coincide. The beam and additional support reaction were replaced by an unknown concentrated external force applied to the beam. This unknown reaction was assumed to be time-dependent. The time law was determined by solving the first kind of Volterra integral equation. The methodology of deriving the integral equation for the unknown reaction was explained. Analytic formulae and results of computations for specific numerical parameters were given. The impact of the mass value on the additional viscoelastic support reaction and the beam deflection at arbitrary points were determined. The research results of this paper can be helpful for engineers in designing multi-span bridges.

Keywords: Timoshenko multi-span beam, additional viscoelastic support, non-stationary vibration, concentrated mass, Volterra integral equation.

1 Introduction

In engineering and construction, there are many complex mechanical systems consisting of a large number of elements. The approach based on the choice of the central element of the study and taking into account the influence of objects interacting with it using concentrated or distributed forces can be used to build a model of the system in this case.

It is not always known in advance which system parameters significantly influence the stress-strain state, especially under dynamic loading [1]. Therefore, the maximum number of parameters, which at the same time do not overcomplicate the model, should be taken into account.

Thus, in this paper, the Timoshenko-type beam model, taking into account the viscoelastic and mass parameters of intermediate supports, is proposed to study the deformed state of widespread multi-span beams.

2 Literature Review

Beam-like structural elements are widely applied for engineering purposes of mankind. From experience, the Timoshenko hypothesis-based models considering rotatory inertia and shear provide good results for beams [2, 3]. Nevertheless, the classical models, i.e., Euler-Bernoulli beams, are still used for simplification when solving problems involving modeling nonlinear properties of the beam material [4] or in complex identification problems [5].

The paper [4] presents vibration analysis of a simply supported beam with a fractional order viscoelastic material model. The studies show that the selection of appropriate damping coefficients and fractional derivative order of the damping model enables us to fit more accurately the dynamic characteristics of the beam in comparison with using the integer order derivative damping model.

A new method to identify the viscoelastic boundary conditions of Euler–Bernoulli beams under forced response is presented in [5]. The capability of identifying complex boundary conditions under high levels of noise might open the door for the proposed method to be considered in real-life applications of structural health monitoring and model updating with boundary conditions of beam-like structures such as bridges.

For mechanical systems, including structural elements in the form of beams under non-stationary loads, auxiliary restrictions are sometimes imposed on the displacements of specific beam parts (e.g., displacement magnitude) in addition to the robustness requirements. In such cases (i.e., bridges), using additional supports for beams is advisable.

Vibrations of multi-span beams with additional elastic supports under pulse and traveling loads are considered in [6].

Non-stationary direct and inverse problems for multi-span beams with additional elastic supports are solved in [7]. Similar problems in viscoelastic settings are studied in [8].

Implementation of the time-weighted residual method for simulation of flexural waves in multi-span Timoshenko beams subjected to various external loads: from stationary loads to accelerating moving masses considered in [9]. Experimental studies of the dynamic response of multi-span beams under the action of moving masses, among which is [10], is also known.

The paper [11] study extends a frequency domain modified spectral element method (SEM) from single-span beams to multi-span beams subjected to moving point forces. The Timoshenko beam model represents each span. The time history of the moving point force is transformed to the frequency domain as a series of quasi-static or stationary point forces acting on the beam simultaneously. The dynamic responses are obtained by superposing the individual dynamic responses excited by each quasi-static point force. Note that in this work, a similar problem has already been solved at a high level, but each additional support is, in fact, absolutely rigid.

The assumption about the absolute rigidity of the supports is also used by authors in [12] in the analysis of the dynamic behavior of a Rayleigh multi-span uniform continuous beam system traversed by a constant moving force or uniformly distributed loads.

Manipulating the dynamic response of a multi-span bridge due to organizational arrangements is considered in the article [13], namely by managing the entry time of the crossing trains (in other words, moving loads).

The analytical solution uses the Euler–Bernoulli beam model. For comparisons, a two-dimensional numerical finite element modeling based on the Timoshenko beam element that includes the effect of the shear force is also presented there.

This solution is more complex because of the number of pavement bridge layers. However, it does not consider the properties of additional supports, is the model considered in [14].

In paper [15], the multi-span Timoshenko beams are investigated, and the interpolation modifies the mode shapes of the beams functions to model the vibration modes of the multi-span beams. Hamilton’s principle is applied to establish the equation of motion of the structure, and the natural circular frequencies and the free vibration responses of the multi-span beams are obtained. Note that in this work, only free vibrations are considered.

The uniform formulation of dynamic vibration analysis of multi-span beams is presented by using an efficient domain decomposition method in the paper [16]. The domain decomposition method divides the structure into several equal sections. Next, the artificial spring is used to simulate the multi-span beam’s complex boundaries and continuity conditions. Finally, the admissible displacement functions are expanded through Jacobi orthogonal polynomials, and the free and forced vibration characteristics of multi-span beam structures can be obtained using Rayleigh–Ritz method.

The paper [17] investigates multi-step Timoshenko beams coupled with rigid bodies on springs. Additional purely elastic supports are considered when In works [16, 17].

Note that in the case of modeling complex multilayer objects, along with beams, more complex objects are used, called strips or strip plates [18]. However, using an object as a strip significantly complicates the model and, in our case, will not give visible refinements of the solution.

The results of the present paper are based on those for non-stationary deforming of mechanical systems consisting of beams and plates having concentrated viscoelastic supports additional to the main support along its edge. Solutions to direct problems for beams and plates with additional supports are found in [19, 20].

After analyzing the existing publications, we conclude that the problem of studying non-stationary oscillations of multi-span beams is quite relevant. At the same time, absolutely rigid supports are most often considered in well-known publications. The proposed model makes it possible to describe the behavior of real mechanical objects more accurately. In addition, introducing additional supports, considering their characteristics can be used to reduce unwanted vibrations, such as in [21, 22].

3 Research Methodology

3.1 Problem settings

A mechanical system consists of a hinged elastic isotropic beam and an additional concentrated viscoelastic support contacting the beam at some point (Figure 1).

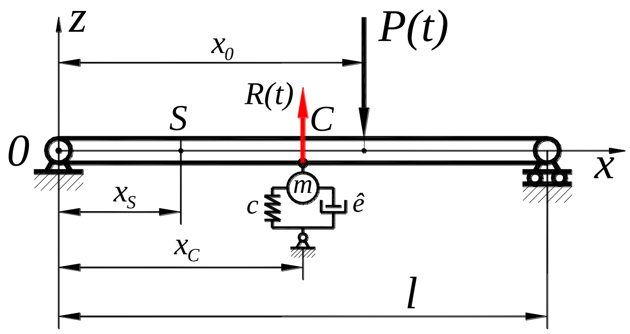

Figure 1 – Beam with additional support loading scheme

The additional support is assumed to be installed transversally to the beam, and its lower end is hinged. The stiffness and damping ratios are considered constant. Transverse pulse load $P(t)$ is applied to the beam at some point, which excites non-stationary vibrations of the beam and additional support.

The impact of the additional viscoelastic support on the beam is modeled by a non-stationary external unknown force (the beam and additional support reaction taken with the opposite sign) applied to the beam at the point of its contact with the support. When solving the problem, we assume the point of the load application as well as the point of contact with the additional support to be arbitrary, i.e., they can be any points of the beam except for its ends.

The displacements of the points of the beam under the simultaneous action of the system of two forces (i.e., the given function $P(t)$ and the unknown function $R(t)$, which is the time-dependent reaction of the beam and additional support considering its mass-inertia characteristics) are to be determined.

3.2 Analytical solution

The system of Timoshenko beam partial differential equations, which considers the appropriately chosen initial and boundary conditions that describes non-stationary deformation processes, is of the form (see [2]):

$$\begin{cases} G'b_b h_b\left(\dfrac{\partial^2 w}{\partial x^2}-\dfrac{\partial\psi}{\partial x}\right)=\rho b_b h_b\,\dfrac{\partial^2 w}{\partial t^2}-P(x,t)+R_C(x,t); \\ EI\,\dfrac{\partial^2\psi}{\partial x^2}+G'b_b h_b\left(\dfrac{\partial w}{\partial x}-\psi\right)=\rho\cdot I\,\dfrac{\partial^2\psi}{\partial t^2}, \end{cases} \qquad (1)$$

where h_b is the beam thickness; b_b is the beam width; l is the beam length; $G'=k'\cdot G$; k' is the shear coefficient; $I=b_b h^3/12$; w describes the beam middle plane deflection; ψ stands for the rotation angle; ρ, E, ν are the elastic constants of the beam material; t is the time. We also remind that $P(x,t)$ and $R_C(x,t)$ are the exciting load and the reaction of the beam and additional support interaction, respectively.

The system is solved by expanding the unknown functions, namely the displacements and rotation angles, into corresponding Fourier series with time-dependent coefficients. Then a system of ordinary differential equations can be derived for the expansion coefficients, which can be solved, for instance, by using the integral Laplace transform [23]. In this case, after applying the inverse transform, the solutions can be written as convolution-type Duhamel’s integrals, which allows for obtaining analytic representations for the kernels of integral equations.

By solving the system of differential equation (1) for zero initial and hinge boundary conditions, we arrive at the following analytic formula for the deflection functions:

$$w(x,t)=\int_0^t P(\tau)K_P^W(x,t-\tau)d\tau-\int_0^t R(\tau)K_R^W(x,t-\tau)d\tau, \qquad (2)$$

where $K_i^W(x,t)$ are the corresponding kernels of Duhamel’s integrals (i.e., convolutions):

$$K_i^W(x,t)=\sum_{k=1}^{\infty}\frac{C_{ik}}{\Delta_k}\cdot\sin\frac{k\pi\cdot x}{l}\sum_{p=1}^{2}\Omega_{pk}\cdot\sin\omega_{pk}t\,.$$

In the equalities above, we adopt the following notations:

$$a=\frac{G'}{\rho},\qquad b=\frac{G'h_b^{\,2}}{\rho\cdot 12},\qquad d=\frac{E}{\rho},\qquad \lambda_k=\pi\frac{k}{l},$$

$$C_{ik}=\frac{2}{\rho\cdot l\cdot b_b\cdot h_b}\cdot\sin\frac{k\pi\cdot x_i}{l},$$

$$\Delta_k=\sqrt{(\lambda_k^2(a+d)+b)^2-4\cdot a\cdot d\cdot\lambda_k^4},$$

$$\Omega_{1k}=\omega_{1k}-\frac{d\cdot\lambda_k^2+b}{\omega_{1k}};\quad \Omega_{2k}=-\omega_{2k}+\frac{d\cdot\lambda_k^2+b}{\omega_{2k}}.$$

The eigenfrequencies are given by the formula:

$$\omega_{1k}=\sqrt{0.5\left[(\lambda_k^2(a+d)+b)+\Delta_k\right]};$$

$$\omega_{2k}=\sqrt{0.5\left[(\lambda_k^2(a+d)+b)-\Delta_k\right]}.$$

Similar relations can be derived for total displacements, normal rotation angles, and deformation.

In the general case, the additional support can be represented as a combination of mass, stiffness, and damping impact (Figure 1). The formula gives the beam and additional viscoelastic support reaction considering inertia phenomena:

$$R_i(t) = m\frac{d^2 w_i(t)}{dt^2} + \kappa\frac{dw_i(t)}{dt} + c \cdot w_i(t)\,, \tag{3}$$

where c is the additional support stiffness ratio, N/m; κ is the damping, N·s/m; m stands for the mass-inertia characteristic of the additional viscoelastic support, kg; $w_i(t)$ describes the beam deflection at the point of the contact with the additional support, m.

For the case under consideration, the functions of deflections at a point are to be derived. Applying the direct Laplace integral transform to (3) for the zero initial condition, we get the formula:

$$R_i(s) = m_i \cdot s^2 \cdot w_i(s) + \kappa_i \cdot s \cdot w_i(s) + c_i \cdot w_i(s)\,. \tag{4}$$

From (4), the Laplace transform of the deflection functions can be easily found:

$$w_i(s) = \frac{R_i(s)}{m_i \cdot s^2 + \kappa_i \cdot s + c_i}\,. \tag{5}$$

Applying the inverse Laplace transform by the convolution theorem, we arrive at the following expression for the deflection at the point of application of the viscoelastic support reaction considering its mass:

$$w_i(t) = \int_0^t K_{fi}(t-\tau) R(\tau) d\tau\,, \tag{6}$$

where $K_{fi}(t) = \frac{1}{m_i}\frac{1}{\omega_{CDi}} \cdot e^{-\frac{\kappa_i}{2 \cdot m_i} \cdot t} \cdot \sin(\omega_{CDi} \cdot t)$ is the finite difference kernel of the convolution type integral accounting for viscous, elastic, and mass-inertia characteristics of the additional support at the *i*-th point, and $\omega_{CDi} = \sqrt{c_i/m_i - 0.25 \cdot \kappa_i^2 / m_i^2}$ is the eigenfrequency corresponding to the *i*-th additional viscoelastic support considering its mass.

Below is an example of solving the problem for one additional support.

Note that (6) is similar to (2) in its structure. Substituting the coordinate of the point of contact with the additional support x_C for the variable x in the beam deflection $w(x,t)$, we can equate the deflections obtained by the two formula: $w(x_C, t) = w_i(t)$.

Hence the right-hand parts of relations (2) and (6) are also equal for $x = x_C$. Then after gathering all the known summands in the right-hand part of the equality and keeping all the unknown ones in its left-hand part, we arrive at a first-kind Volterra equation with respect to the unknown reaction $R(t)$:

$$\int_0^t \left[K_R^W(t-\tau) + K_f(t-\tau)\right] R(\tau) d\tau = \int_0^t K_P^W(t-\tau) P_0(\tau) d\tau\,. \tag{7}$$

By discretization, integral equation (7) is transformed into a system of linear algebraic equations (SLAE) [24], which can be written in the form:

$$\mathbf{A}_\mathbf{R}^* \mathbf{R} = \mathbf{w}_\mathbf{P}\,, \tag{8}$$

where the vector $\mathbf{R}$ corresponds to the change of the reaction $R(t)$ in time; the vector $\mathbf{w}_\mathbf{P}$ is the change in deflection time at the point of contact with the additional support excited solely by the external force $P(t)$ $w_P = \int_0^t K_P^W(t-\tau)P(\tau)d\tau$; the matrix $\mathbf{A}_R^*$ corresponds to the sum of the kernels $K_R^W(t-\tau) + K_f(t-\tau)$.

Finally, the beam and additional support reaction force $R(t)$ is found taking into account the support mass-inertia characteristics, which allows determining the components of the deflection of the beam at all points in time considering the impact of two independent loads $P(t)$ and $R(t)$ rather than the presence of additional supports.

4 Results and Discussion

Consider a specific example of modeling non-stationary transverse vibrations of a beam with additional support. Let, for simplicity purposes, the beam be hinged at its ends and have only additional support attached to the downward side of the beam at an arbitrary point between its ends. The additional support is modeled to be realistic, considering elastic, viscous, and various mass-inertia characteristics.

When undeformed, the midline of the beam is assumed to coincide with the Ox axis of the cartesian coordinate system. The computations are carried out for the following parameters: $\rho = 7890$ kg/m^3, $\nu = 0.3$, and $E = 2.07 \cdot 10^{11}$ Pa, which corresponds to the alloy steel beam mechanical constants; the beam length $l = 0.80$ m; the beam thickness $h_b = 0.04$ m; the beam width $b_b = 0.05$ m.

The coordinates of the point of the exciting load application $x_0 = 0.50$ m, of the point at which the additional viscoelastic support is attached to the beam $x_C = 0.40$ m (i.e., the middle of the beam), the point at which the change of the deflection in time is studied $x_{S1} = 0.20$ m.

The additional support stiffness ratio is $c_1 = 1.0 \cdot 10^7$ N/m, and the ratio of the linear viscous damping is assumed to be $\kappa_1 = 1.0 \cdot 10^3$ N·s/m; the number of the terms in the corresponding Fourie series is 100.

The general scheme of the geometry chosen for the computational example is shown in Figure. 1.

The computations are carried out for three cases, namely:

1) the minimal possible value of the mass-inertia characteristic, since in the case of the mass less than the value $m_{\min} = \kappa_1^2 / (4 \cdot c_1)$, the natural frequency corresponding to the additional viscoelastic support considering its mass $\omega_{CDi} = \sqrt{c_i / m_i - 0.25 \cdot \kappa_i^2 / m_i^2}$ becomes complex. For the values of the parameters (c_1, κ_1) adopted above we have $m_{\min} = 0.025$ kg. We choose $m_1 = 0.0251$ kg for our computations;

2) the mass m_2 = 1.0 kg;

3) the mass m_3 = 2.5 kg.

Figure 2 shows the external force $P(t)$ exiting the deformations (i.e., non-stationary vibrations) of the beam with additional support. We point out that in Figure 2, the force (in N) is plotted along the vertical axis, and time in seconds is plotted along the horizontal axis.

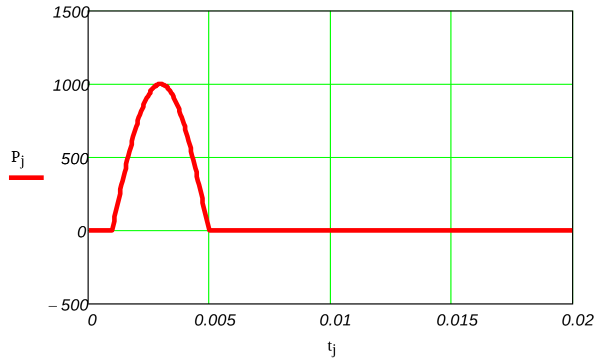

Figure 2 – Exiting load $P(t)$

In Figure 3, the beam deflections $w(x_{S1},t)$, $w(x_{S2},t)$ and $w(x_{S3},t)$ are caused by the influence of only an external perturbing force (illustrate the case – a beam without additional support).

We point out that in Figure 3 the deflection in meters is plotted along the vertical axis, and time in seconds is plotted along the horizontal axis.

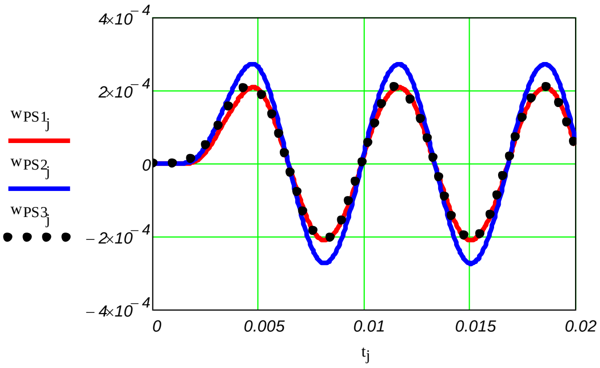

Figure 3 – Deflection of beam point are caused by the influence only of $P(t)$

In Figure 4, the external force $P(t)$ and the reaction of the additional support $R(t)$ determined by solving matrix equation (8) are shown for the masses $m_1 = 0.0251$ kg, $m_2 = 1$ kg, and $m_3 = 2.5$ kg. We point out that in Figure 4, the force (in N) is plotted along the vertical axis, and time in seconds is plotted along the horizontal axis.

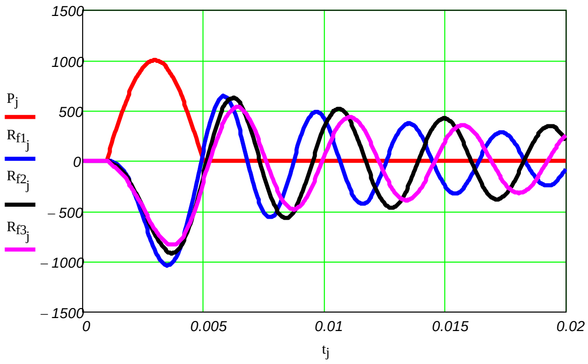

Figure 4 – Exiting load $P(t)$ and reaction determined $R(t)$

In Figure 5, the beam deflections $w(x_{S1},t)$, $w(x_{S2},t)$ and $w(x_{S3},t)$ are shown for the four cases, namely, for the case when the impact of the additional support is not considered and for the three cases of mass-inertia characteristics of the additional support listed above.

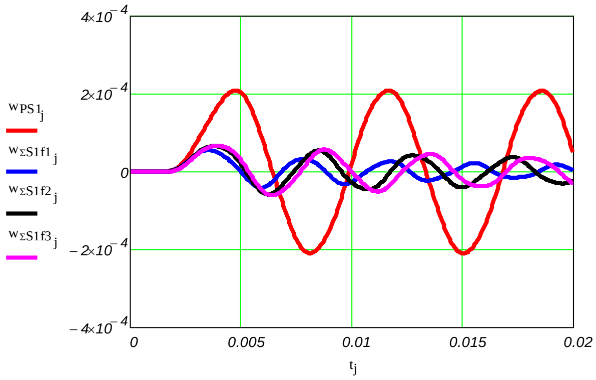

Figure 5 – Deflection of beam point are caused by the influence of $P(t)$ and $R(t)$

We point out that in Figure 5, the deflection in meters is plotted along the vertical axis, and time in seconds is plotted along the horizontal axis. Also, we can see that the impact of the mass on the deflection amplitudes is insignificant (there is a strong dependence of the amplitudes on the elastic and viscous component parameters). Nevertheless, changing the mass results in an essential change in the curve phase characteristics. Increasing the mass leads to increasing the vibration period, which is logical since the system inertia grows with its mass.

5 Conclusions

The article proposes an original approach to model the impact of additional viscoelastic support considering its mass-inertia characteristics by an independent non-stationary external force: the beam and additional support reaction.

Solving the first-kind integral Volterra equation determines the unknown non-stationary load. Modeling non-stationary vibrations of beams with additional supports based on the approach developed in the paper allows obtaining stable analytic and numerical solutions to the problems of mechanics of deformable solids without using iteration schemes.

The impact of the mass on the deflection amplitudes is insignificant (if the mass is not very large). The phase characteristics of the vibrations are changed when the additional support's mass is considered. Increasing the mass leads to increasing the vibration period.

References

1. Smetankina, N.V., Shupikov, A.N., Sotrikhin, S.Yu., Yareschenko, V.G. (2008). A noncanonically shape laminated plate subjected to impact loading: Theory and experiment. *Journal of Applied Mechanics*, Transactions ASME, Vol. 75, No 5. Pp. 051004-1–051004-9. https://doi.org/10.1115/1.2936925.
2. Timoshenko, S. P. (1937). *Vibration problems in engineering*. D. Van Nostrand Company INC. 497 p.
3. Filippov, A.P., Kokhmanyuk, S. S., Yanyutin, Ye. G. (1978). *Deformation of structural elements under impact and impulse loads*. Kyiv, Naukova dumka. 184 p.
4. Freundlich, J. (2013). Vibrations of a Simply Supported Beam with a Fractional Viscoelastic Material Model – Supports Movement Excitation. *Shock and Vibration*, 20, Article ID 126735, https://doi.org/10.3233/SAV-130825.
5. Qiao, G, Rahmatalla, S. (2019). Identification of the viscoelastic boundary conditions of Euler-Bernoulli beams using transmissibility. *Engineering Reports*, 1:e12074, https://doi.org/10.1002/eng2.12074.
6. Kokhmanyuk, S. S., Yanyutin, Ye. G., Romanenko, L. G. (1980). *Vibrations of deformable systems under pulse and moving loads*. Naukova Dumka, Kyiv.
7. Yanyutin, Ye. G., Gnatenko, G. O., Grishakin, V. T. (2007). Solving non-stationary direct and inverse problems for beams with additional elastic supports. *Mashynoznavstvo*, 8, pp. 18–23.
8. Yanyutyn, Ye. G., Grishakin, V. T. (2008). Identification of mobile load for viscoelastic beams. *Bulleting of the National Technical University "Kharkiv Polytechnic Institute".* 47, pp. 178–184.
9. Borji, A., Movahedian, B., Boroomand, B. Implementation of time-weighted residual method for simulation of flexural waves in multi-span Timoshenko beams subjected to various types of external loads: from stationary loads to accelerating moving masses. (2022). *Archive of Applied Mechanics*, 92 (4), pp. https://doi.org/10.1007/s00419-021-02103-z.
10. Dan Stancioiu, Huajiang Ouyang, John E. Mottershead, Simon James, Experimental investigations of a multi-span flexible structure subjected to moving masses, Journal of Sound and Vibration, Volume 330, Issue 9, 2011, Pages 2004-2016, ISSN 0022-460X, https://doi.org/10.1016/j.jsv.2010.11.011.
11. Taehyun, K., Usik, L. (2017). Dynamic analysis of a multi-span beam subjected to a moving force using the frequency domain spectral element method. *Computers & Structures*. 192, pp. 181–195. https://doi.org/10.1016/j.compstruc.2017.07.028.
12. Szyłko-Bigus, O., Śniady, P., Zakęś, F. (2019). Application of Volterra integral equations in the dynamics of a multi-span Rayleigh beam subjected to a moving load. *Mechanical Systems and Signal Processing*, Vol. 121, pp. 777-790, ISSN 0888-3270, https://doi.org/10.1016/j.ymssp.2018.11.056.
13. Vesali, F., Rezvani, M.A., Shadfar, M. (2023). Attuned dynamic response of double track Multi-span railway bridges under the delayed entry of a second train. *Journal of Vibration Engineering and Technologies*. https://doi.org/10.1007/s42417-023-00884-x.
14. Liu, S., Jiang, L., Zhou, W., Xilin, C., Zhang, Y. (2021). Dynamic response analysis of multi-span bridge-track structure system under moving loads. *Mechanics Based Design of Structures and Machines*.
15. Zhao, Z., Wen, S., Li, F., Zhang, C. (2018). Free vibration analysis of multi-span Timoshenko beams using the assumed mode method. *Archive of Applied Mechanics*, Vol. 88, no. 7, pp. 1213–1228. https://doi.org/10.1007/s00419-018-1368-8.
16. Cong Gao, Fuzhen Pang, Haichao Li, Hongfu Wang, Jie Cui, Jisi Huang. (2021). Free and Forced Vibration Characteristics Analysis of a Multispan Timoshenko Beam Based on the Ritz Method. *Shock and Vibration*, Vol. 2021, Article ID 4440250. https://doi.org/10.1155/2021/4440250.
17. Chen, X., Zeng, X., Liu, X. and Rui, X. (2020). Transfer matrix method for the free and forced vibration analyses of multi-step Timoshenko beams coupled with rigid bodies on springs. *Applied Mathematical Modelling*, 87, pp. 152–170.
18. Kantor, B.Ya., Smetankina, N.V., Shupikov, A. N. (2001) Analysis of non-stationary temperature fields in laminated strips and plates. *International Journal of Solids and Structure*. Vol. 38, No 48/49, pp. 8673-8684. https://doi.org/10.1016/S0020-7683(01)00099-3.

19. Voropay, A.V., Povaliaiev, S.I., Yehorov, P.A. (2022) Simulation of intermediate viscoelastic support under non-stationary vibrations of Timoshenko beams. *Bulletin of the National Technical University "KhPI". Series: Mathematical modeling in engineering and technology*, №1, pp. 36-44 (2022). https://doi.org/10.20998/2222-0631.2022.01.05.
20. Voropay, A. V., Yegorov, P. A. (2020). The influence of mass and inertial characteristics of an additional viscoelastic support in the non-stationary deforming of a rectangular plate. *Bulletin of the National Technical University "Kharkiv Polytechnic Institute"*, 1, pp. 15–23. https://doi.org/10.20998/2222-0631.2020.01.02.
21. Yang, F., Sedaghati, R., Esmailzadeh, E. (2021). Vibration suppression of structures using tuned mass damper technology: A state-of-the-art review. *Journal of Vibration and Control*, 28 (7-8), pp. 812-836. https://doi.org/10.1177/1077546320984305.
22. Zhang, L., Chen, Q., Zhang, R., Lei, T. (2023). Vibration control of beams under moving loads using tuned mass inerter systems. *Engineering Structures*, 275, art. no. 115265. https://doi.org/10.1016/j.engstruct.2022.115265.
23. Beerends, R. J., Ter Morsche, H. G., Van Den Berg, J. C., Van De Vrie, E. M. (2003). *Fourier and Laplace Transforms*. Cambridge University Press. 458 p. ISBN: 0521534410, 9780521806893.
24. Voropay, A., Gnatenko, G., Yehorov, P., Povaliaiev, S., Naboka, O. (2022). Identification of the pulse axisymmetric load acting on a composite cylindrical shell, inhomogeneous in length, made of different materials. *Eastern-European Journal of Enterprise Technologies*, 5(7 (119), pp. 21–34. https://doi.org/10.15587/1729-4061.2022.265356.