Ministry of Education and Science of Ukraine
Sumy State University
Faculty of Electronics and Information Technologies

# 5231 Methodological instructions

for practical classes
on the topic ***«Linear algebra»***
on the course **«Higher mathematic»**
for foreign students
of the full-time course of studies

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# CONTENT

# I LINEAR ALGEBRA

## 1 Matrices

In this topic the basic concept of linear algebra, such as matrix is observed. Lecture presents theoretical material, which is connected with this concept, problems, examples of the practical solutions of the problems. We consider here types of the matrices (square matrix, diagonal matrix, union matrix), operations on matrices (sum of the matrices, difference of the matrices, multiplication of a matrix by a scalar, multiplication of the matrices), and its properties.

### 1.1 Definition of a matrix. Types of matrices

**Definition.** A ***matrix*** of size (or dimension) $m \times n$ ***is a rectangular table*** with entries $a_{ij}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ arranged in $m$ rows and $n$ columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}.$$

Note that, for each entry $a_{ij}$, the index *i* refers to the *i*-th row and the index *j* to the *j*-th column.

Matrices are briefly denoted by uppercase letters (for instance, *A*, as here), or by the symbol [ $a_{ij}$ ], sometimes with more details: $A=[a_{ij}]$, ( $i=1,2,\ldots,m, \quad \mathrm{j}=1,2,...,n$ ).

**Definition.** The numbers *m* and *n* are called the ***dimensions*** ***(*** $m\times n$ ***) of the matrix.***

**Definition.** A ***matrix*** is said ***to be finite*** if it has finitely many rows and columns; otherwise, the ***matrix*** is said ***to be infinite***.

**Definition.** The ***null or zero matrix*** is a matrix whose entries are all equal to zero: $a_{ij}=0, \quad i=1,2,\ldots,m, \quad \mathrm{j}=1,2,...,n$ .

**Definition.** ***Column vector*** or *column* is a matrix of size $n=1$.

$$A=\begin{pmatrix} a_{11} \\ a_{21} \\ \ldots \\ a_{m1} \end{pmatrix}.$$

**Definition.** A ***row vector*** or *row* is a matrix of size $m=1$.

$$A=\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \end{pmatrix}.$$

**Definition.** A ***square matrix*** is a matrix of size ( $n \times n$), and $n$ is called the dimension of this square matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

The ***main diagonal*** of a square matrix is its diagonal from the top left corner to the bottom right corner.

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

The ***secondary diagonal*** of a square matrix is the diagonal from the bottom left corner to the top right corner.

$$A = \begin{pmatrix} 0 & 0 & \dots & a_{1n} \\ 0 & a_{2n} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & 0 & \dots & 0 \end{pmatrix}.$$

**Definition.** A square matrix, where main diagonal or secondary diagonal whose entries are all not equal to zero is called ***diagonal matrix.***

**Definition.** Diagonal matrix whose entries are all equal to one $a_{ii}=1,\ i=1,2\ldots n$ is called ***union matrix***

$$E=\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$$

**Definition.** ***Two matrices A and B are equal*** if they are of the same size and their respective entries are equal $a_{ij}=b_{ij}$.

## 1.2 Basic operations with matrices

### 1.2.1 Sum of two matrices

**Definition.** ***The sum of two matrices*** $A=[a_{ij}]$ and $B=[b_{ij}]$ of the same size $m\times n$ is the matrix $C=[c_{ij}]$ of size $m\times n$ with the entries $c_{ij}=a_{ij}+b_{ij}$ $(i=1,2,\ldots,m,\quad \mathrm{j}=1,2,...,n)$.

$$\underset{(m\times n)}{A}+\underset{(m\times n)}{B}=\underset{(m\times n)}{C}$$

The sum of two matrices is denoted by $C = A + B$, and the operation is called ***addition of matrices.***

**Example.** Find matrix $C = A + B$, if

$$A = \underset{(2\times 3)}{\begin{pmatrix} 0 & 1 & -1 \\ 2 & 1 & 2 \end{pmatrix}}, \qquad B = \underset{(2\times 3)}{\begin{pmatrix} 1 & -2 & 2 \\ -1 & 0 & -1 \end{pmatrix}}.$$

$$C = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 2 \\ -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

**Properties of addition for matrices:**

1. $A + O = A$ (property of zero);
2. $A + B = B + A$ (commutativity);
3. $(A + B) + C = A + (B + C)$ (associativity),

where matrices $A$, $B$, $C$, and zero matrix $O$ have the same size.

### 1.2.2 Difference of two matrices

**Definition.** ***The difference of two matrices*** $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size $m \times n$ is the matrix $C = [c_{ij}]$ of size $m \times n$ with entries

$$c_{ij} = a_{ij} - b_{ij} \ \ (i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n).$$

The difference of two matrices is denoted by $C = A - B$, and the operation is called ***subtraction of matrices***.

**Example.** Find matrix $C = A - B$, if

$$A = \underset{(2\times 3)}{\begin{pmatrix} 0 & 1 & -1 \\ 2 & 1 & 2 \end{pmatrix}}, \qquad B = \underset{(2\times 3)}{\begin{pmatrix} 1 & -2 & 2 \\ -1 & 0 & -1 \end{pmatrix}}.$$

$$C = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & -2 & 2 \\ -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 3 & -3 \\ 3 & 1 & 3 \end{pmatrix}.$$

### 1.2.3 Multiplication of a matrix by a scalar

**Definition.** ***The product of a matrix*** $A = [a_{ij}]$ of size $m \times n$ by a scalar $\lambda$ is the matrix $C = [c_{ij}]$ of size $m \times n$ with entries

$$c_{ij} = \lambda a_{ij}, \ (i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n).$$

The product of a matrix by a scalar is denoted by $C = \lambda A$, and the operation is called ***multiplication of a matrix by a scalar***.

**Properties of multiplication of a matrix by a scalar:**

1. $0A = O$ (property of zero);
2. $(\lambda\mu)A = \lambda(\mu A)$ (associativity with respect to a scalar factor);

3. $\lambda(A + B) = \lambda A + \lambda B$ (distributivity with respect to addition of matrices);

4. $(\lambda + \mu)A = \lambda A + \mu A$ (distributivity with respect to addition of scalars), where $\lambda$ and $\mu$ are scalars, matrices $A$, $B$, $C$, and zero matrix $O$ have the same size.

**Example.** Find matrix $C = 3A$, if $\underset{(2\times 2)}{A} = \begin{pmatrix} 1 & -2 \\ 2 & 4 \end{pmatrix}$. .

$$\underset{(2\times 2)}{C} = \begin{pmatrix} 3 & -6 \\ 6 & 12 \end{pmatrix}$$

**Definition.** The ***additively inverse (opposite) matrix*** for a matrix $A = [a_{ij}]$ of size $m \times n$ is the matrix $C = [c_{ij}]$ of size $m \times n$ with entries

$$c_{ij} = -a_{ij} \quad (i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n),$$

or, in matrix form,

$$C = (-1)A.$$

**Remark.** The difference $C$ of two matrices $A$ and $B$ can be expressed as $C = A + (-1)B$.

### 1.2.4 Multiplication of matrices

**Definition.** The ***product*** of a matrix $A=[a_{ij}]$ of size $m\times p$ and a matrix $B=[b_{ij}]$ of size $p\times n$ is the matrix $C=[c_{ij}]$ of size $m\times n$ with entries

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+...+a_{in}b_{nj}\ ,\ \ (i=1,2,\ldots,m,\quad \mathrm{j}=1,2,...,n);$$

i.e., the entry $c_{ij}$ in the $i$-th row and $j$-th column of the matrix $C$ is equal to the sum of products of the respective entries in the $i$-th row of $A$ and the $j$-th column of $B$.

**Remark.** Note that the product is defined for matrices of ***compatible size;*** i.e., the number of the columns in the first matrix should be equal to the number of rows in the second matrix.

The product of two matrices $A$ and $B$ is denoted by $C = AB$, and the operation is called ***multiplication of matrices.***

**Example.** Find $A\cdot B=C$, if

$$\underset{(2\times 2)}{A=\begin{pmatrix}1 & 2\\ -1 & 1\end{pmatrix}},\quad \underset{(2\times 3)}{B=\begin{pmatrix}1 & 0 & -1\\ 3 & 2 & 0\end{pmatrix}}.$$

The product of the matrix $A$ and the matrix $B$ is the matrix $\underset{(2\times2)}{A}\cdot\underset{(2\times3)}{B}=\underset{(2\times3)}{C}$.

$$c_{11}=1\cdot1+2\cdot3=7, \qquad c_{21}=-1\cdot1+1\cdot3=2,$$
$$c_{12}=1\cdot0+2\cdot2=4, \qquad c_{22}=-1\cdot0-1\cdot2=2,$$
$$c_{13}=1\cdot(-1)+2\cdot0=-1, \qquad c_{23}=-1\cdot(-1)+1\cdot0=1.$$
$$C=\begin{pmatrix}7 & 4 & -1\\ 2 & 2 & 1\end{pmatrix}.$$

**Definition.** Two square matrices $A$ and $B$ are said to ***commute*** if $AB=BA$, i.e., if their multiplication is subject to the commutative law.

**Properties of multiplication of matrices:**

1. $AO = O$, (property of zero matrix);
2. $(AB)C = A(BC)$ (associativity of the product of three matrices);
3. $AE = A$ (multiplication by unit matrix);
4. $A(B + C) = AB + AC$ (distributivity with respect to a sum of two matrices);
5. $\lambda(AB) = (\lambda A)B = A(\lambda B)$ (associativity of the product of a scalar and two matrices),
6. $SD = DS$ (commutativity for any square and any diagonal matrices), where $\lambda$ is a scalar, matrices $A$, $B$, $C$, square matrix $S$, diagonal matrix $D$, zero matrices $O$ and unit matrix $I$ have the compatible sizes.

## 1.3 Transpose, complex conjugate matrix, adjoint matrix

**Definition.** The ***transpose*** of a matrix $A=[a_{ij}]$ of size $m\times n$ is the matrix $C=[c_{ij}]$ of size $n\times m$ with entries

$$c_{ij}=a_{ji},\ (i=1,2,\dots,n,\ j=1,2,\dots,m).$$

The transpose is denoted by $C=A^T$

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},\quad (m\times n).$$

$$A^T=\begin{pmatrix} a_{11} & a_{21} & a_{31} & \dots & a_{m1} \\ a_{21} & a_{22} & a_{32} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{mn} \end{pmatrix},\quad (n\times m).$$

**Example.** $\underset{(2\times 3)}{A=\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}}$,

then $\underset{(3\times 2)}{A^T=\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 2 \end{pmatrix}}$

## Properties of transposes:

1. $(A+B)^T = B^T + A^T$ ;
2. $(\lambda A)^T = \lambda A^T$ ;
3. $\left(A^T\right)^T = A$ ;
4. $(A \cdot B)^T = B^T \cdot A^T$ ;
5. $(O)^T = O$ ;
6. $(E)^T = E$ ,

where $\lambda$ is a scalar, matrices *A*, *B*, and zero matrix *O* have size $m \times n$.

## SELF-TEST ASSIGNMENT

A student must be ready to do the following assignments:

1. Concepts, definitions, formulations:

- Matrices.
- Linear operations with matrices.
- Multiplication of matrices.

2. Proofs and conclusions:

- Matrix addition properties.
- Matrix multiplication properties.

3. Assignments:

Find the matrix sum, difference, and product (to your attention simulator "Product of the matrices").

## Exercises

1. Determine which of the matrix products AB and BA are defined. If the product is appropriate, find the size of the matrix obtained.

1) A is $3\times5$ matrix and B is a $5\times2$ matrix.
2) A is $3\times2$ matrix and B is a $2\times2$ matrix.
3) A is $4\times2$ matrix and B is a $4\times2$ matrix.
4) A and B are square matrices of the fifth order.

3. Find the matrix D = 2A- 3B if

$$A=\begin{pmatrix}1 & 3 & 0 & -4\\ 9 & -1 & 3 & 2\end{pmatrix},\quad B=\begin{pmatrix}3 & -2 & 0 & 6\\ 8 & -2 & 3 & 1\end{pmatrix}.$$

4. Find the product of matrices

$$\begin{pmatrix}2 & 1 & 8\\ 8 & -2 & 0\\ 5 & 2 & -1\end{pmatrix}\cdot\begin{pmatrix}1 & 2 & -4\\ 1 & -1 & 3\\ 3 & 2 & 4\end{pmatrix}.$$

5. Find the product of matrices $\begin{pmatrix}1 & 2\\ -3 & 2\\ 4 & -6\end{pmatrix}\cdot\begin{pmatrix}1 & -3\\ 3 & 0\end{pmatrix}$.

6. Find transpose matrix $A=\begin{pmatrix}1 & 2 & -4\\ 1 & -1 & 3\\ 3 & 2 & 4\end{pmatrix}$.

# 2 Determinants

In this topic the concept of linear algebra, such as determinants is observed. Lecture presents theoretical material about determinants, of 2nd and 3rd order, general definition of a determinant of n-th order, properties of determinants, minors and cofactors, inverse matrix, rank of a matrix and its property. We considered calculation of determinants and different problems, which is connected with this notion, examples of the practical solutions of the problems.

## 2.1 Notion of determinant

With each square matrix $A=[a_{ij}]$ of size $n \times n$ one can associate a numerical characteristic, called its ***determinant.*** The determinant of such a matrix can be defined by induction with respect to the size $n$.

1. For a matrix of size $1\times1$ $(n=1)$, the ***first-order determinant*** is equal to its only entry, $\Delta = \det A = a_{11}$.

2. For a matrix of size $2\times2$ $(n=2)$, the ***second-order determinant***, is equal to the difference of the product of its entries on the main diagonal and the product of its entries on the secondary diagonal:

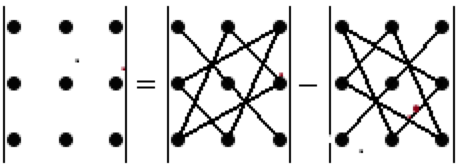
$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

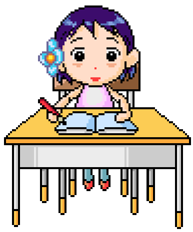
**Example.** $\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 1\cdot 3 - 2\cdot(-1) = 3 + 2 = 5.$

3. For a matrix of size $3\times 3$ ( $n = 3$ ), the ***third-order determinant***,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

This expression is obtained by the ***triangle rule (Sarrus scheme)***.





**Example.** Let us find the determinant $\begin{vmatrix} 2 & 4 & 6 \\ 8 & 0 & 4 \\ -2 & 2 & 8 \end{vmatrix}.$

$$\begin{vmatrix} 2 & 4 & 6 \\ 8 & 0 & 4 \\ -2 & 2 & 8 \end{vmatrix} = 2\cdot 0\cdot 8 + 4\cdot 4\cdot(-2) + 8\cdot 2\cdot 6 - \left(6\cdot 0\cdot(-2) + 4\cdot 8\cdot 8 + 4\cdot 2\cdot 2\right) = 64 - 272 = -208.$$

4. For a matrix of size ($n > 2$), the ***n-th-order determinant*** is defined as follows under the assumption that the ($n-1$)-st-order determinant has already been defined for a matrix of size $(n-1)\times(n-1)$. Consider a matrix $A=[a_{ij}]$ of size $n \times n$.

**Definition.** The ***minor*** $M_{ij}$ corresponding to an entry $a_{ij}$ is defined as the ($n-1$)-st-order determinant of the matrix of size $(n-1)\times(n-1)$ obtained from the original matrix $A$ by removing the $i$-th row and the $j$-th column (i.e., the row and the column whose intersection contains the entry $a_{ij}$ ).

**Example.** For matrix $A=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}$ we have:

$$\Delta(A)=\begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{vmatrix},$$

$$M_{11}=\begin{vmatrix}5 & 6\\ 8 & 9\end{vmatrix}=45-48=-3,$$

$$M_{23}=\begin{vmatrix}1 & 2\\ 7 & 8\end{vmatrix}=8-14=-6.$$

**Definition.** The *cofactor* $A_{ij}$ *of the entry* $a_{ij}$ is defined by $\boldsymbol{A_{ij}=(-1)^{i+j}M_{ij}}$ (i.e., it coincides with the corresponding minor if $i+j$ is even, and is the opposite of the minor if $i+j$ is odd).

**Example.** For matrix $A=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$ we have:

$$\Delta(A)=\begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix},$$

$$A_{11}=(-1)^{1+1}M_{11}=M_{11}=\begin{vmatrix}5&6\\8&9\end{vmatrix}=45-48=-3,$$

$$A_{23}=(-1)^{2+3}M_{23}=-M_{23}=-\begin{vmatrix}1&2\\7&8\end{vmatrix}=-(8-14)=6.$$

.

**Definition.** The ***n th-order determinant*** of the matrix $A$ is defined by

$$\Delta=\boldsymbol{a_{i1}A_{i1}+a_{i2}A_{i2}+\ldots+a_{in}A_{in}}.$$

This formula is also called the ***i-th row expansion of the determinant of A*** and also the ***j-th column expansion of the determinant of A.***

**Example.** Let us find the determinant of the matrix

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} = a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12} =$$

$$= a_{11}(-1)^{2}a_{22} + a_{12}(-1)^{3}a_{21} = a_{11}a_{22} - a_{12}a_{22}.$$

**Example.** Find the determinant by 2-nd column expansion of the determinant.

$$\begin{vmatrix} 2 & 4 & 6 \\ 8 & 0 & 4 \\ -2 & 2 & 8 \end{vmatrix} = 4\cdot(-1)^{1+2}\begin{vmatrix} 8 & 4 \\ -2 & 8 \end{vmatrix} + 0\cdot(-1)^{2+2}\begin{vmatrix} 2 & 6 \\ -2 & 8 \end{vmatrix} + 2\cdot(-1)^{3+2}\begin{vmatrix} 2 & 6 \\ 8 & 4 \end{vmatrix} =$$

$$= -4(64+8) - 2(8-48) = -208.$$

**Example.**

$$\Delta=\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}=a_{21}A_{21}+a_{22}A_{22}+a_{23}A_{23}+a_{24}A_{24}=$$

$$=a_{21}(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}+a_{22}(-1)^{2+2}\begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}+$$

$$+a_{23}(-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}+a_{24}(-1)^{2+4}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}=$$

$$=-a_{21}\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}++a_{22}\begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}-a_{23}\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}+$$

$$+a_{24}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}.$$

## 2.2 Properties of determinants

### Basic properties:

1. **Invariance** with respect to transposition of matrices: $\det A = \det A^T$.

2. **Antisymmetric** with respect to the permutation of two rows (or columns): if two rows (columns) of a matrix are interchanged, its determinant preserves its absolute value, but changes its sign.

3. **Linearity** with respect to a row (or column) of the corresponding matrix: suppose that the *i*-th row of a matrix $A=[a_{ij}]$ is a linear combination of two row vectors, $(a_{i1},...,a_{i3})=\lambda(b_1,...,b_n)+\mu(c_1,...,c_n)$; then $\det A=\lambda\det A_b+\mu\det A_c$, where $A_b$ and $A_c$ are the matrices obtained from *A* by replacing its *i*-th row with $(b_1,...,b_n)$ and $(c_1,...,c_n)$.

This fact, together with the first property, implies that a similar linearity relation holds if a column of the matrix *A* is a linear combination of two column vectors.

### Some useful corollaries from the basic properties:

1. The determinant of a matrix with two equal rows (columns) is equal to zero.

2. If all entries of a row are multiplied by $\lambda$, the determinant of the resulting matrix is multiplied by $\lambda$.

3. If a matrix contains a row (columns) consisting of zeroes, then

its determinant is equal to zero.

4. If a matrix has two proportional rows (columns), its determinant is equal to zero.

5. If a matrix has a row (column) that is a linear combination of its other rows (columns), its determinant is equal to zero.

6. The determinant of a matrix does not change if a linear combination of some of its rows is added to another row of that matrix.

**Theorem.** **(Necessary and sufficient condition for a matrix to be degenerate).** The determinant of a square matrix is equal to zero if and only if its rows (columns) are linearly dependent.

## 2.3 Inverse matrices

Let $A$ be a square matrix of size $n \times n$, and let E be the unit matrix of the same size.

**Definition.** The matrix $A^{-1}$ is called inverse matrix if satisfying the condition $A^{-1}A = AA^{-1} = E$ for a given nondegenerate matrix $A$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}. \qquad A^{-1} = \frac{1}{\Delta} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix},$$ where $A_{ij}$

- cofactor of $a_{ij}$.

**Example.** Find $A^{-1}$ for $A$, if $A=\begin{pmatrix}-1 & 0 & 2\\ 2 & 3 & 2\\ 3 & 7 & 1\end{pmatrix}$.

$$\Delta=\begin{vmatrix}-1 & 0 & 2\\ 2 & 3 & 2\\ 3 & 7 & 1\end{vmatrix}=-3+28+0-(18+0-14)=21.$$

$$A_{11}=(-1)^{1+1}M_{11}=\begin{vmatrix}3 & 2\\ 7 & 1\end{vmatrix}=3-14=-11;$$

$$A_{12}=(-1)^{1+2}M_{12}=-\begin{vmatrix}2 & 2\\ 3 & 1\end{vmatrix}=-(2-6)=4;$$

$$A_{13}=(-1)^{1+3}M_{13}=\begin{vmatrix}2 & 3\\ 3 & 7\end{vmatrix}=14-9=5;$$

$$A_{22}=(-1)^{2+2}M_{22}=\begin{vmatrix}-1 & 2\\ 3 & 1\end{vmatrix}=-1-6=-7;$$

$$A_{23}=(-1)^{2+3}M_{23}=-\begin{vmatrix}-1 & 0\\ 3 & 7\end{vmatrix}=-(-7-0)=7;$$

$$A_{31}=(-1)^{3+1}M_{31}=\begin{vmatrix}0 & 2\\ 3 & 2\end{vmatrix}=0-6=-6;$$

$$A_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} -1 & 2 \\ 2 & 2 \end{vmatrix} = -(-2-4) = 6\,;$$

$$A_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} = -3-0 = -3\,.$$

$$A^{-1} = \frac{1}{21}\begin{pmatrix} -11 & 14 & -6 \\ 4 & -7 & 6 \\ 5 & 7 & -3 \end{pmatrix}.$$

$$A^{-1} \cdot A = \frac{1}{21}\begin{pmatrix} -11 & 14 & -6 \\ 4 & -7 & 6 \\ 5 & 7 & -3 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 2 \\ 2 & 3 & 2 \\ 3 & 7 & 1 \end{pmatrix} =$$

$$= \frac{1}{21}\begin{pmatrix} 11+28-18 & 0+42-42 & -22+28-6 \\ -4-14+18 & 0-21-49 & 8-14+6 \\ -5+14-9 & 0+21-21 & 10+14-3 \end{pmatrix} =$$

$$= \frac{1}{21}\begin{pmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E.$$

## 2.4 Rank of a matrix

**Definition.** Let $A=[a_{ij}]$ be a matrix of size $m\times n$ with at least one nonzero entry. Then there is a positive integer $r \le n$ for which the following conditions hold:

1) the matrix $A$ has an $r$-th-order nonzero minor,

2) any minor of $A$ of order $(r + 1)$ and higher (of it exists) is equal to zero.

The integer $r$ satisfying these two conditions is called the ***rank*** of the matrix $A$ and is denoted by $r = \text{rank}(A)$.

Any nonzero $r$-th-order minor of the matrix $A$ is called its ***basic minor.***

The rows and the columns whose intersection yields its basic minor are called ***basic rows*** and ***basic columns*** of the matrix.

The rank of a matrix is equal to the maximal number of its linearly independent rows (columns). This implies that for any matrix, the number of its linearly independent rows is equal to the number of its linearly independent columns.

When calculating the rank of a matrix $A$, one should pass from submatrices of a smaller size to those of a larger size. If, at some step, one finds a submatrix $A_k$ of size $k\times k$, such that it has a nonzero $k$-th-order determinant and the $(k + 1)$-st-order determinants of all submatrices of size $(k+1)\times(k+1)$ containing $A_k$ are equal to zero, then it can be concluded that $k$ is the rank of the matrix $A$.

### Properties of the rank of a matrix:

Rank matrix does not change if:

1) interchanged two rows (columns);

2) Multiply each element rows (column) on the same, nonzero factor.

3) add to the elements of row (column) the relevant elements of the other row (column), multiplied by one and the same number.

**Example.** Find rank of the matrix $A=\begin{pmatrix}1 & 2 & 3\\ 3 & 6 & 9\end{pmatrix}$.

For matrix we can form 3 determinants of the 2-nd order and 6 determinants of the 1-st order:

$$\begin{vmatrix}1 & 2\\ 3 & 6\end{vmatrix}=0,\quad \begin{vmatrix}1 & 3\\ 3 & 9\end{vmatrix}=0,\quad \begin{vmatrix}2 & 3\\ 6 & 9\end{vmatrix}=0,$$

$$|1|=1,\quad |2|=2,\quad |3|=3,\quad |3|=3,\quad |6|=6,\quad |9|=9.$$

All determinants of the 2-nd-order equal to zero but all determinants of the 1-st-order not equal to zero. Then $rank(A)=1$.

One of the methods finding on rank is method of equivalent transformation.

**Example.** Find rank of the matrix

$$A=\begin{pmatrix}1&1&2&3&-1\\2&-1&0&-4&-5\\-1&-1&0&-3&-2\\6&3&4&8&-3\end{pmatrix}\sim\begin{pmatrix}1&1&1&3&-1\\2&-1&0&-4&-5\\-1&-1&0&-3&-2\\6&3&2&8&-3\end{pmatrix}\sim$$

$$\sim\begin{pmatrix}1&1&1&3&-1\\2&-1&0&-4&-5\\-1&-1&0&-3&-2\\4&1&0&2&-1\end{pmatrix}\sim\begin{pmatrix}0&0&1&0&0\\2&-1&0&-4&-5\\-1&-1&0&-3&-2\\4&1&0&2&-1\end{pmatrix}\sim$$

$$\sim\begin{pmatrix}0&0&1&0&0\\6&0&0&-2&-6\\3&0&0&-1&-3\\4&1&0&2&-1\end{pmatrix}\sim\begin{pmatrix}0&0&1&0&0\\6&0&0&-2&-6\\3&0&0&-1&-3\\0&1&0&0&0\end{pmatrix}\sim$$

$$\sim\begin{pmatrix}0&0&1&0&0\\0&0&0&0&0\\3&0&0&-1&-3\\0&1&0&0&0\end{pmatrix}\sim\begin{pmatrix}0&0&1&0&0\\0&0&0&0&0\\0&0&0&-1&0\\0&1&0&0&0\end{pmatrix}\Rightarrow\quad r(A)=3.$$

## SELF-TEST ASSIGNMENT

A student must be ready to do the following assignments:

1. Concepts, definitions, formulations:

- Determinants of the 2nd, the 3rd and the n-th orders.
- Inverse matrix.
- Minor. Cofactor.
- Rank of matrix.

2. Proofs and conclusions:

- Properties of determinants (2nd and 3rd orders).
- Existence of an inverse matrix.

3. Assignments:

- Calculate the determinants of order 2, 3 and n, to be able to lay out a determinant by the elements of any row or column, to reduce determinant to the triangle form (to your attention Test).
- Find the matrix rank.
- Find an inverse matrix.

## Exercises

1. Calculate determinant of the matrices

$$A=\begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix},\qquad C=\begin{pmatrix} 2 & 1 & -2 \\ 3 & 1 & -3 \\ 4 & 5 & 1 \end{pmatrix},\qquad B=\begin{pmatrix} 4 & 2 & 2 & 5 \\ 3 & 3 & 1 & 3 \\ 4 & 5 & 1 & 5 \\ 2 & -1 & 0 & 1 \end{pmatrix}.$$

2. Let the matrix $A=\begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 3 \\ 4 & 3 & -5 & 0 \\ 3 & 2 & 0 & -5 \end{pmatrix}$ be given. Find minor $M_{13}$ and cofactor $A_{22}$ of the matrix $A$.

3. Find the rank of the matrices $A=\begin{pmatrix} 1 & 3 & 0 & -4 \\ 9 & -1 & 3 & 2 \end{pmatrix}$,

$$B=\begin{pmatrix} 4 & 2 & 2 & 5 \\ 3 & 3 & 1 & 3 \\ 4 & 5 & 1 & 5 \\ 2 & -1 & 0 & 1 \end{pmatrix}.$$

4. Find inverse matrix, if $A=\begin{pmatrix} 2 & 1 & -2 \\ 3 & 1 & -3 \\ 4 & 5 & 1 \end{pmatrix}$.

## Practice report

Solve the problems according to your variant.

You can see the typical solution of the problems below.

**Problem 1** Let the matrix $A = \begin{pmatrix} -3 & 2 & 1 & 0 \\ 2 & -2 & 1 & 4 \\ 4 & 0 & -1 & 2 \\ 3 & 1 & -1 & 4 \end{pmatrix}$ be given.

Calculate *detA*:

*a*) by 1-st row expansion of the determinant,

*b*) by 2-nd column expansion of the determinant using the properties of determinates.

*c*) Calculate minor $M_{12}$ and cofactor $A_{32}$ of the matrix $A$.

$$\Delta = \begin{vmatrix} -3 & 2 & 1 & 0 \\ 2 & -2 & 1 & 4 \\ 4 & 0 & -1 & 2 \\ 3 & 1 & -1 & 4 \end{vmatrix}.$$

$$M_{12} = \begin{vmatrix} 2 & 1 & 4 \\ 4 & -1 & 2 \\ 3 & -1 & 4 \end{vmatrix} = (-8) - 16 + 6 + 12 + 4 - 16 = -18,$$

$$M_{32} = \begin{vmatrix} -3 & 1 & 0 \\ 2 & 1 & 4 \\ 3 & -1 & 4 \end{vmatrix} = (-12) + 12 - 12 - 8 = -20,$$

$$A_{32} = (-1)^{3+2} M_{32} = -(-20) = 20.$$

a) $\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + a_{14}A_{14} =$

$$= -3\begin{vmatrix} -2 & 1 & 4 \\ 0 & -1 & 2 \\ 1 & -1 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 & 4 \\ 4 & -1 & 2 \\ 3 & -1 & 4 \end{vmatrix} + 1\begin{vmatrix} 2 & -2 & 4 \\ 4 & 0 & 2 \\ 3 & 1 & 4 \end{vmatrix} =$$

$$= -3(8 + 2 + 4 - 4) - 2(-8 - 16 + 6 + 12 + 4 - 16) + (16 - 12 - 4 + 32) = 38.$$

b) $\Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} + a_{24}A_{24} =$

$$= -2\begin{vmatrix} 2 & 1 & 4 \\ 4 & -1 & 2 \\ 3 & -1 & 4 \end{vmatrix} - 2\begin{vmatrix} -3 & 1 & 0 \\ 4 & -1 & 2 \\ 3 & -1 & 4 \end{vmatrix} + 1\begin{vmatrix} -3 & 1 & 0 \\ 2 & 1 & 4 \\ 4 & -1 & 2 \end{vmatrix} =$$

$$= -2(-8 + 6 - 16 + 12 + 4 - 16) - 2(12 + 6 - 6 - 16) + (-6 + 16 - 12 - 4) = 38.$$

## Problem 1

1) $\begin{vmatrix} 3 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 0 & 1 & -2 & 3 \\ 4 & 0 & 2 & 2 \end{vmatrix}$ 2) $\begin{vmatrix} 3 & -2 & 0 & 1 \\ 2 & 1 & 3 & 1 \\ 4 & 2 & 3 & 3 \\ 0 & 2 & 2 & 2 \end{vmatrix}$ 3) $\begin{vmatrix} 2 & 2 & 1 & 1 \\ 0 & 3 & 3 & 3 \\ -4 & 2 & 3 & 1 \\ 3 & 3 & 2 & 2 \end{vmatrix}$

4) $\begin{vmatrix} 1 & 0 & 3 & 3 \\ -2 & 1 & 2 & 1 \\ 3 & -2 & 1 & 0 \\ 4 & 1 & -2 & 1 \end{vmatrix}$ 5) $\begin{vmatrix} 1 & -2 & 3 & 3 \\ 2 & 4 & 0 & 1 \\ 3 & -1 & 2 & 2 \\ 5 & 4 & 1 & 1 \end{vmatrix}$ 6) $\begin{vmatrix} 0 & 2 & 3 & 2 \\ 1 & 3 & -2 & 1 \\ 2 & -2 & 3 & 2 \\ 4 & 1 & 4 & 1 \end{vmatrix}$

7) $\begin{vmatrix} 1 & -2 & 1 & -2 \\ 3 & 0 & 5 & 1 \\ 4 & 2 & 3 & 2 \\ 5 & 1 & 1 & 1 \end{vmatrix}$ 8) $\begin{vmatrix} 3 & 1 & -2 & 1 \\ 0 & 3 & -1 & 2 \\ 5 & -1 & 1 & -2 \\ -2 & 2 & 1 & 1 \end{vmatrix}$ 9) $\begin{vmatrix} 1 & 2 & -3 & 4 \\ -2 & 1 & 4 & 3 \\ 3 & 2 & 0 & 1 \\ 2 & -2 & 3 & 4 \end{vmatrix}$

10) $\begin{vmatrix} 3 & 2 & 0 & 1 \\ -4 & 2 & 2 & 4 \\ -2 & 1 & 1 & 1 \\ 0 & 3 & 2 & -2 \end{vmatrix}$ 11) $\begin{vmatrix} 4 & 2 & -3 & 4 \\ -2 & 2 & 4 & 3 \\ 3 & 3 & 0 & 1 \\ 2 & -2 & 3 & 2 \end{vmatrix}$ 12) $\begin{vmatrix} 2 & 1 & 2 & 0 \\ 1 & 2 & 3 & 4 \\ 5 & 3 & 4 & -1 \\ 0 & 2 & 2 & 3 \end{vmatrix}$

13) $\begin{vmatrix} 4 & 2 & -4 & 4 \\ 1 & 2 & 3 & -3 \\ 2 & 0 & 1 & 1 \\ 3 & 3 & -3 & -3 \end{vmatrix}$ 14) $\begin{vmatrix} 3 & 2 & 2 & 0 \\ 2 & 4 & 3 & 1 \\ 2 & 4 & 3 & 3 \\ 0 & 2 & 2 & 3 \end{vmatrix}$ 15) $\begin{vmatrix} 0 & 2 & 3 & 3 \\ 2 & 3 & -2 & 1 \\ 2 & -2 & 5 & 2 \\ 4 & 2 & 4 & 1 \end{vmatrix}$

**Problem 2** Find the inverse matrix of the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 3 & 2 & -2 \end{pmatrix}.$$

Check the equality $A^{-1}A = E$.

Inverse matrix is $A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$.

$$\Delta = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 3 & 2 & -2 \end{vmatrix} = 1\cdot 1\cdot(-2) + (-1)\cdot(-1)\cdot 3 + 2\cdot 2\cdot 2 - (2\cdot 1\cdot 3 + (-1)\cdot 2\cdot(-2) +$$
$$+(-1)\cdot 2\cdot 1) = 1.$$

$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} = -2+2 = 0,$ $\qquad A_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix} = 2,$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} = -(-4+3) = 1,$ $\qquad A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = -8,$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4-3 = 1,$ $\qquad A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = -5,$

$A_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = -1,$ $\qquad A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = 5$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3 .$$

$$A^{-1} = \begin{pmatrix} 0 & 2 & -1 \\ 1 & -8 & 5 \\ 1 & -5 & 3 \end{pmatrix}.$$

$$A^{-1} \cdot A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & -8 & 5 \\ 1 & -5 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 3 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 0+4-3 & 0+2-2 & 0-2+2 \\ 1-16+15 & -1-8+10 & 2+8-10 \\ 1-10+9 & -1-5+6 & 2+5-6 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E.$$

## Problem 2

1)

$$\begin{pmatrix} 5 & 3 & 0 \\ 3 & -2 & 1 \\ 2 & 0 & -1 \end{pmatrix}$$

2)

$$\begin{pmatrix} 4 & 1 & -2 \\ 3 & 0 & 5 \\ -2 & 2 & 4 \end{pmatrix}$$

3)

$$\begin{pmatrix} 1 & 3 & 7 \\ 2 & 0 & 3 \\ -1 & 7 & 1 \end{pmatrix}$$

4)

$$\begin{pmatrix} 4 & 1 & -3 \\ 2 & 2 & 7 \\ 5 & 1 & 4 \end{pmatrix}$$

5)

$$\begin{pmatrix} 1 & -2 & 3 \\ 1 & 0 & 3 \\ -3 & 5 & 3 \end{pmatrix}$$

6)

$$\begin{pmatrix} 3 & 5 & 6 \\ -1 & 1 & 3 \\ 7 & 4 & 5 \end{pmatrix}$$

7)

$$\begin{pmatrix} 4 & 1 & -2 \\ 3 & -3 & 2 \\ 6 & 0 & 5 \end{pmatrix}$$

8)

$$\begin{pmatrix} 3 & -2 & 1 \\ 4 & 3 & 5 \\ -1 & 7 & -2 \end{pmatrix}$$

9)

$$\begin{pmatrix} 1 & 3 & -2 \\ 4 & 1 & 0 \\ 3 & -2 & 7 \end{pmatrix}$$

10)

$$\begin{pmatrix} 2 & 3 & -2 \\ 1 & 0 & 5 \\ -3 & 4 & 7 \end{pmatrix}$$

11)

$$\begin{pmatrix} 8 & 1 & 4 \\ 4 & -3 & 2 \\ 0 & 3 & -1 \end{pmatrix}$$

12)

$$\begin{pmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ -2 & 1 & 0 \end{pmatrix}$$

13)

$$\begin{pmatrix} 6 & 3 & 0 \\ -4 & 7 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

14)

$$\begin{pmatrix} 5 & 4 & 1 \\ -3 & 7 & 1 \\ 2 & 0 & 2 \end{pmatrix}$$

15)

$$\begin{pmatrix} 4 & 3 & 0 \\ -1 & 5 & 1 \\ 2 & 2 & 1 \end{pmatrix}$$

16)

$$\begin{pmatrix} 0 & 2 & -3 \\ 4 & 1 & 5 \\ 3 & 1 & 1 \end{pmatrix}$$

**Problem 3** Find the rank of the matrix

$$\begin{pmatrix} 1 & 3 & 5 & -1 \\ 2 & -1 & -3 & 4 \\ 5 & 1 & -1 & 7 \\ 7 & 7 & 9 & 1 \end{pmatrix} \begin{matrix} \sim \\ e_2 - 2e_1 \\ e_3 - 5e_1 \\ e_4 - 7e_1 \end{matrix} \begin{pmatrix} 1 & 3 & 5 & -1 \\ 0 & -7 & -13 & 6 \\ 0 & -14 & -26 & 12 \\ 0 & -14 & -26 & 8 \end{pmatrix} \begin{matrix} \\ \sim \\ e_3 - 2e_2 \\ e_4 - e_3 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 3 & 5 & -1 \\ 0 & -7 & -13 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5 & -1 \\ 0 & -7 & -13 & 6 \\ 0 & 0 & 0 & -4 \end{pmatrix} \begin{matrix} \sim \\ d_3 \leftrightarrow d_4 \end{matrix}$$

□ $\begin{pmatrix} 1 & 3 & -1 & 5 \\ 0 & -7 & 6 & -13 \\ 0 & 0 & -4 & 0 \end{pmatrix} \Rightarrow r = 3.$

## Problem 3

1) $\left(\begin{array}{cccc|c} 2 & 1 & 4 & 3 & 9 \\ 3 & -1 & 0 & 2 & 4 \\ -2 & 1 & 4 & 1 & 5 \\ 3 & 1 & 2 & -5 & 6 \end{array}\right)$ 2) $\left(\begin{array}{cccc|c} 4 & 2 & 3 & 1 & 1 \\ 5 & 1 & 2 & 2 & 3 \\ 6 & 2 & 4 & 1 & 0 \\ 3 & -1 & 2 & 2 & 1 \end{array}\right)$ 3) $\left(\begin{array}{cccc|c} 2 & 5 & 3 & 1 & 6 \\ 1 & 4 & 2 & -1 & 5 \\ 3 & 0 & 4 & 1 & 9 \\ 2 & -1 & 3 & 5 & 2 \end{array}\right)$

4) $\left(\begin{array}{cccc|c} 5 & 2 & 6 & 1 & 0 \\ 4 & 3 & -4 & 0 & 2 \\ 6 & 1 & -3 & 2 & -2 \\ -1 & 2 & 4 & 4 & 9 \end{array}\right)$ 5) $\left(\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 7 \\ -2 & 4 & 3 & -4 & 12 \\ 3 & 2 & -1 & 6 & -3 \\ 4 & -3 & 2 & 1 & -3 \end{array}\right)$ 6) $\left(\begin{array}{cccc|c} -2 & 2 & 4 & 3 & 4 \\ 1 & 5 & 2 & 6 & 0 \\ 4 & 3 & 2 & 2 & 5 \\ 5 & 5 & 0 & 1 & 0 \end{array}\right)$

7) $\left(\begin{array}{cccc|c} 3 & 2 & -1 & 4 & 11 \\ 4 & 0 & 1 & -2 & -5 \\ 2 & 4 & 4 & 1 & 2 \\ 3 & 2 & -1 & 3 & 9 \end{array}\right)$ 8) $\left(\begin{array}{cccc|c} 3 & 1 & 4 & 4 & 6 \\ 2 & 4 & 0 & 1 & 3 \\ -1 & 3 & 2 & 2 & -2 \\ 1 & 4 & 1 & 3 & 0 \end{array}\right)$ 9) $\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 7 \\ 2 & 1 & 4 & 0 & 0 \\ 3 & 1 & 3 & 2 & 1 \\ -1 & 2 & 2 & 1 & 6 \end{array}\right)$

10) $\left(\begin{array}{cccc|c} 3 & 1 & 2 & 4 & 2 \\ -1 & 2 & 3 & 5 & 9 \\ 2 & 4 & -2 & 3 & -2 \\ 2 & 3 & 1 & 3 & 3 \end{array}\right)$ 11) $\left(\begin{array}{cccc|c} 2 & 2 & 3 & 6 & 6 \\ -1 & 4 & 0 & 3 & -5 \\ 2 & -2 & 1 & 5 & 6 \\ 3 & 1 & 2 & 3 & 6 \end{array}\right)$ 12) $\left(\begin{array}{cccc|c} 4 & 1 & 2 & 2 & 3 \\ 5 & 3 & 2 & 2 & 5 \\ 1 & 2 & 3 & 2 & 3 \\ 4 & 2 & 3 & 1 & 1 \end{array}\right)$

13) $\left(\begin{array}{cccc|c} 3 & 4 & 2 & 2 & 6 \\ 1 & 5 & 4 & 3 & 3 \\ 3 & 6 & 4 & 5 & 5 \\ 1 & 4 & 2 & 1 & 3 \end{array}\right)$ 14) $\left(\begin{array}{cccc|c} 3 & 1 & 4 & 1 & 0 \\ 5 & 2 & 6 & 4 & 3 \\ 4 & -1 & 2 & 6 & 0 \\ 3 & 2 & 3 & 0 & 1 \end{array}\right)$ 15) $\left(\begin{array}{cccc|c} 4 & 2 & 2 & 5 & 2 \\ 3 & 3 & 1 & 3 & 2 \\ 4 & 5 & 1 & 5 & 3 \\ 2 & -1 & 0 & 1 & -3 \end{array}\right)$

## 3 Systems of linear algebraic equations

In this topic the concept of linear algebra, such as system of linear equations is observed. Lecture presents theoretical material about systems of linear algebraic equations, its consistency, investigation of the system consistency using matrix rank. Undetermined systems and their solutions, homogeneous systems of linear algebraic equations. We considered solving the system by the Cramer's formulas, matrix method, Gauss' method, examples of the practical solutions of the problems.

### 3.1 Notion of a system of linear algebraic equations

**Definition.** A system of m linear equations with *n* unknown quantities has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n = b_3, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m, \end{cases} \quad (1)$$

where $a_{11}, a_{12}, a_{13}, \ldots, a_{mn}$, are the ***coefficients of the system***; $b_1, b_2, b_3, \ldots, b_m$ are its ***free terms***; and $x_1, x_2, x_3, \ldots, x_n$ are the ***unknown quantities***.

**Definition.** System (1) is said to be ***homogeneous*** (2) if all its free terms are equal to zero. Otherwise (i.e., if there is at least one nonzero free term) the system is called ***nonhomogeneous***.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = 0 \end{cases} \qquad (2)$$

If the number of equations is equal to that of the unknown quantities ($m = n$), system (1) is called a ***square system***.

**Definition.** A ***solution of system*** (1) is a set of $n$ numbers $x_1, x_2, x_3, \ldots, x_n$ satisfying the equations of the system.

**Definition.** A system is said to be ***consistent*** if it admits at least one solution. If a system has no solutions, it is said to be ***inconsistent***.

**Definition.** A consistent system of the form (.1) is called a ***determined system*** − it has a unique solution.

**Definition.** A consistent system with more than one solution is said to be ***underdetermined***.

It is convenient to use matrix notation for systems of the form (1), $AX = B$, (.2) where $A=[a_{ij}]$ is a matrix of size $m\times n$ called the ***basic matrix*** of the system; $X=[x_i]$ is a ***column vector*** of size $n$; $B=[b_{ij}]$ is a ***column vector*** of size $m$.

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad X=\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}, \quad B=\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix} \qquad (3)$$

## 3.2 Consistency condition for a general linear system

System (1) or (.2) is associated with two matrices: the basic matrix $A$ of size $m\times n$ and the ***augmented matrix*** $\overline{A}$ of size $m\times(n+1)$ formed by the matrix $A$ supplemented with the column of the free terms, i.e.,

$$\overline{A}=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}. \qquad (4)$$

**Kronecker-Capelli Theorem**. A linear system (1) or (2) is consistent if and only if its basic matrix and its augmented matrix (.4) have the same rank, i.e., rank ($\overline{A}$) = rank($A$).

### 3.3 Matrix method of the solution system of linear equations

A square system of linear equations has the form $AX = B$, (2), where $A$ is a square matrix.

If the determinant of system (1) is different from zero, i.e., $\det A \neq 0$, then the system has a unique solution, $X = A^{-1} \cdot B$.

**Example.** Solve system $\begin{cases} x_1 + 3x_2 - 4x_3 = 5, \\ 2x_1 + x_2 + 3x_3 = -1, \\ 3x_1 - 2x_2 + x_3 = 2. \end{cases}$

$$A = \begin{pmatrix} 1 & 3 & -4 \\ 2 & 1 & 3 \\ 3 & -2 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}.$$

$$\Delta = \begin{vmatrix} 1 & 3 & -4 \\ 2 & 1 & 3 \\ 3 & -2 & 1 \end{vmatrix} = 56 \neq 0.$$

Find $A^{-1}$: $A^{-1} = \frac{1}{56}\begin{pmatrix} 7 & 5 & 13 \\ 7 & 13 & -11 \\ -7 & 11 & -5 \end{pmatrix}$. $X = A^{-1} \cdot B$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{56}\begin{pmatrix} 7 & 5 & 13 \\ 7 & 13 & -11 \\ -7 & 11 & -5 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{56}\begin{pmatrix} 35-5+26 \\ 35-13-22 \\ -35-11-10 \end{pmatrix} = .$$

$$= \frac{1}{56}\begin{pmatrix} 56 \\ 0 \\ -56 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Then $(1; 0; -1)$.

## 3.4 Cramer rule

If the determinant of the matrix of system (1) is different from zero, i.e., $\Delta = \det A \neq 0$, then the system admits a unique solution, which is expressed by

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \ \ldots, \quad x_n = \frac{\Delta_n}{\Delta}, \tag{5}$$

where $\Delta_k$ ($k = 1, 2, \ldots, n$) is the determinant of the matrix obtained from $A$ by replacing its $k$-th column with the column of free terms:

$$\begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} A_{11} & A_{21} & \ldots & A_{n1} \\ A_{12} & A_{22} & \ldots & A_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ A_{1n} & A_{2n} & \ldots & A_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} A_{11}b_1 + A_{21}b_2 + \ldots + A_{n1}b_n \\ A_{12}b_1 + A_{22}b_2 + \ldots + A_{n2}b_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ A_{1n}b_1 + A_{2n}b_2 + \ldots + A_{nn}b_n \end{pmatrix}.$$

Hence

$$x_1 = \frac{1}{\Delta}\left(A_{11}b_1 + A_{21}b_2 + \ldots + A_{n1}b_n\right) = \frac{1}{\Delta}\begin{vmatrix} b_1 & a_{12} & \ldots & a_{1n} \\ b_2 & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_n & a_{n2} & \ldots & a_{nn} \end{vmatrix} == \frac{\Delta_1}{\Delta}.$$

**Example.** Solve system $\begin{cases} x_1 - x_2 + 2x_3 = 5 \\ 2x_1 + x_2 - x_3 = 1 \\ 3x_1 + 2x_2 - 2x_3 = 1 \end{cases}$.

$$\Delta = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 3 & 2 & -2 \end{vmatrix} = 1 \neq 0$$ – system has one solution.

Let find $\Delta_1$, $\Delta_2$, $\Delta_3$ and solution:

$$\Delta_1 = \begin{vmatrix} 5 & -1 & 2 \\ 1 & 1 & -1 \\ 1 & 2 & -2 \end{vmatrix} = 1, \quad \Delta_2 = \begin{vmatrix} 1 & 5 & 2 \\ 2 & 1 & -1 \\ 3 & 1 & -2 \end{vmatrix} = 2, \quad \Delta_3 = \begin{vmatrix} 1 & -1 & 5 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 3,$$

$$x_1 = \frac{\Delta_1}{\Delta} = 1, \quad x_1 = \frac{\Delta_2}{\Delta} = 2, \quad x_1 = \frac{\Delta_3}{\Delta} = 3.$$

## 3.5 Gaussian elimination of unknown quantities

Two systems are said to be ***equivalent*** if their sets of solutions coincide.

The method of Gaussian elimination consists in the reduction of a given system to an equivalent system with an upper triangular basic matrix. The latter system can be easily solved. This reduction is carried out in finitely many steps. On every step, one performs an ***elementary transformation of the system*** (or the corresponding augmented matrix) and obtains an equivalent system.

**The elementary transformations are of the following three types:**

1. Interchange of two equations (or the corresponding rows of the augmented matrix).

2. Multiplication of both sides of one equation (or the corresponding row of the augmented matrix) by a nonzero constant.

3. Adding to both sides of one equation both sides of another equation multiplied by a nonzero constant (adding to some row of the augmented matrix its another row multiplied by a nonzero constant).

Suppose that $\det A \neq 0$. Then by consecutive elementary transformations, the augmented matrix of the system $\overline{A}$ [see (6)] of size

$n \times (n+1)$ can be reduced to the diagonal form and one obtains an equivalent system with an upper triangular basic matrix,

$$\begin{cases} x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = c_1, \\ \qquad x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = c_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \qquad\qquad\qquad\qquad x_n = c_n. \end{cases} \tag{6}$$

This system is solved by the so-called "backward substitution": inserting $x_n = c_n$ (obtained from the last equation) into the preceding ($n$ – 1) - st equation, one finds $x_{n-1}$. Then inserting the values obtained for $x_n$, $x_{n-1}$ into the ($n$ – 2) - nd equation, one finds $x_{n-2}$. Proceeding in this way, one finally finds $x_1$.

**Suppose that** $\det A = 0$ **and rank (*A*) = *r*, 0 < *r* < *n*.**

In this case, the system is either inconsistent (i.e., has no solutions) or has infinitely many solutions.

By elementary transformations one obtains a system of the form

$$\begin{cases} x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = c_1, \\ \qquad x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = c_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \qquad\qquad\qquad\qquad 0 = c_n, \end{cases}$$

where the matrix $[a_{ij}]$ ($i, j = 1, 2, \ldots, r$) of size $r \times r$ is nondegenerate.

If at least one of the right-hand sides $c_{r+1},\dots,c_n$ is different from zero, then the system is inconsistent.

If $c_{r+1}=\dots=c_n=0$, then the last $n-r$ equations can be dropped, and it remains to find all solutions of the first $r$ equations.

Transposing all terms containing the variables $x_{r+1},\dots,x_n$ to the right-hand sides and regarding these variables as arbitrary free parameters, we obtain a linear system for the unknown quantities $x_1,\dots,x_r$ with the nondegenerate basic matrix $[a_{ij}]$ ($j, j$ = 1, 2, . . , $r$).

**Example.** Let us find a solution of the system $\begin{cases} x_1-x_2+2x_3=5, \\ 2x_1+x_2-x_3=1, \\ 3x_1+2x_2-2x_3=1 \end{cases}$ by the Gaussian elimination method.

By elementary transformations of the augmented matrix, we obtain

$$\bar{A}=\left(\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 2 & 1 & -1 & 1 \\ 3 & 2 & -2 & 1 \end{array}\right)\begin{array}{c} e_2-2e_1 \\ \sim \\ e_3-3e_1 \end{array}\left(\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 3 & -5 & -9 \\ 0 & 5 & -8 & -14 \end{array}\right)\begin{array}{c} e_2/3 \\ \sim \\ \phantom{x} \end{array}\left(\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 1 & -5/3 & -3 \\ 0 & 5 & -8 & -14 \end{array}\right)\begin{array}{c} \phantom{x} \\ \sim \\ e_3-5e_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 1 & -5/3 & -3 \\ 0 & 0 & 1/3 & 1 \end{array}\right) \underset{3e_3}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 1 & -5/3 & -3 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{c} e_1 - 2e_3 \\ \sim \\ e_2 + (5/3)e_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{c} e_1 + e_2 \\ \sim \\ \phantom{x} \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \Rightarrow \begin{cases} x_1 = 1, \\ x_2 = 2, \\ x_3 = 3. \end{cases}$$

**Example.** Let us find a solution of the system $\begin{cases} x_1 + 2x_2 + x_3 = 4, \\ 2x_1 + 4x_2 + 2x_3 = 3 \end{cases}$ by the Gaussian elimination method.

By elementary transformations of the augmented matrix, we obtain

$$\overline{A} = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 4 & 2 & 3 \end{array}\right) \underset{e_2 - 2e_1}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & -5 \end{array}\right).$$

The system is inconsistent.

**Example.** Let us find a solution of the system

$$\begin{cases} x_1 + 2x_2 + 4x_3 - x_4 - 3x_5 = 7, \\ 2x_1 + 9x_3 + x_5 = 4, \\ x_2 + 2x_4 - x_5 = 6 \end{cases}$$ by the Gaussian elimination method.

$$\bar{A} = \left(\begin{array}{ccccc|c} 1 & 2 & 4 & -1 & -3 & 7 \\ 2 & 0 & 9 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 & -1 & 6 \end{array}\right) \begin{array}{c} e_2 - 2e_1 \\ \sim \\ \phantom{x} \end{array} \left(\begin{array}{ccccc|c} 1 & 2 & 4 & -1 & -3 & 7 \\ 0 & -4 & 1 & 2 & 7 & -10 \\ 0 & 1 & 0 & 2 & -1 & 6 \end{array}\right)_{e_2 \leftrightarrow e_3} \sim$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 2 & 4 & -1 & -3 & 7 \\ 0 & 1 & 0 & 2 & -1 & 6 \\ 0 & -4 & 1 & 2 & 7 & -10 \end{array}\right)_{e_3 + 4e_2} \sim \left(\begin{array}{ccccc|c} 1 & 2 & 4 & -1 & -3 & 7 \\ 0 & 1 & 0 & 2 & -1 & 6 \\ 0 & 0 & 1 & 10 & 3 & 14 \end{array}\right) \begin{array}{c} e_1 - 4e_3 \\ \sim \\ \phantom{x} \end{array}$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 2 & 0 & -41 & -15 & -49 \\ 0 & 1 & 0 & 2 & -1 & 6 \\ 0 & 0 & 1 & 10 & 3 & 14 \end{array}\right) \begin{array}{c} e_1 - 2e_2 \\ \sim \\ \phantom{x} \end{array} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -45 & -13 & -61 \\ 0 & 1 & 0 & 2 & -1 & 6 \\ 0 & 0 & 1 & 10 & 3 & 14 \end{array}\right).$$

The transformed system has the form

$$\begin{cases} x_1 + 0 \cdot x_2 + 0 \cdot x_3 - 45x_4 - 13x_5 = -61 \\ 0 \cdot x_1 + x_2 + 0 \cdot x_3 + 2x_4 - x_5 = 6 \\ 0 \cdot x_1 + 0 \cdot x_2 + x_3 + 10x_4 + 3x_5 = 14 \end{cases} \Leftrightarrow \begin{cases} x_1 = -61 + 45x_4 + 13x_5 \\ x_2 = 6 - 2x_4 + x_5 \\ x_3 = 14 - 10x_4 - 3x_5 \end{cases}$$

The system has infinitely many solutions.

$x_1$, $x_2$, $x_3$ – basic quantities, $x_4$, $x_5$ – free parameters.

## 3.6 Gauss-Jordan elimination of unknown quantities

This method consists of applying elementary transformations for reducing a system with a nondegenerate basic matrix to an equivalent system with the identity matrix. On the $k$ – th step ($k$ = 1, 2, . . ., $n$) the rows of the augmented matrix $\bar{A}'$ obtained on the preceding step can be transformed as follows: provided that the diagonal element obtained on each step is not equal to zero. After $n$ steps, the basic matrix is transformed to the identity matrix and the right-hand side turns into the desired solution. The diagonal element obtained on some step of the above elimination procedure may happen to be equal to zero. In this case, the formulas become more complicated and reindexing of the unknown quantities may be required.

**Example.** Find solution of the system

$$\begin{cases} x_1 - x_2 + 2x_3 = 5, \\ 2x_1 + x_2 - x_3 = 1. \\ 3x_1 + 2x_2 - 2x_3 = 1. \end{cases}$$

$$\bar{A} = \left(\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 2 & 1 & -1 & 1 \\ 3 & 2 & -2 & 1 \end{array}\right) \begin{matrix} e_2 - 2e_1 \\ \sim \\ e_3 - 3e_1 \end{matrix} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 3 & -5 & -9 \\ 0 & 5 & -8 & -14 \end{array}\right) \begin{matrix} e_2/3 \\ \sim \\ \phantom{x} \end{matrix} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 1 & -5/3 & -3 \\ 0 & 5 & -8 & -14 \end{array}\right) \sim$$

$$\begin{matrix} e_1+e_2 \\ \sim \\ e_3-5e_2 \end{matrix}\left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 2 \\ 0 & 1 & -5/3 & -3 \\ 0 & 0 & 1/3 & 1 \end{array}\right) \underset{e_3\cdot 3}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 2 \\ 0 & 1 & -5/3 & -3 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{matrix} e_1-\frac{1}{3}e_3 \\ \sim \\ e_2+\frac{5}{3}e_3 \end{matrix} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \Leftrightarrow$$

$$\begin{cases} x_1 = 1, \\ x_2 = 2, \\ x_3 = 3. \end{cases}$$

**Example.** Find solution of the system $\begin{cases} x_1+5x_2-x_3=3, \\ 2x_1+4x_2-3x_3=2, \\ 3x_1-x_2-3x_3=-7. \end{cases}$

$$\overline{A}=\begin{pmatrix} 1 & 5 & -1 & 3 \\ 2 & 4 & -3 & 2 \\ 3 & -1 & -3 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -6 & -1 & -4 \\ 3 & -16 & 0 & -16 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -1 & -4 \\ 0 & -16 & 0 & -16 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -1 & -4 \\ 0 & -16 & 0 & -16 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -1 & -4 \\ 0 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -1 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -1 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \Rightarrow$$

$r(A) = r(\overline{A}) = n = 3 \Rightarrow$ the system has one solution.

$x_1 = 1,\ x_2 = 1,\ x_3 = -1$ .

## 3.7 Existence of nontrivial solutions of a homogeneous system

Consider a homogeneous system $AX = O_m$ (2)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = 0, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = 0, \end{cases}$$

where $A = [a_{ij}]$ is its basic matrix of size $m \times n$, $X = [x_i]$ is a column vector of size *n*, and $O_m = [o]$ is a column vector of size *m*. System (2) is always consistent since it always has the so-called ***trivial solution*** $X = O_n$ ($x_1 = x_2 = \ldots = x_n = 0$).

**Theorem.** A homogeneous system (3) has a nontrivial solution if and only if the rank of the matrix $A$ is less than the number of the unknown quantities $n$.

It follows that a square homogeneous system has a nontrivial solution if and only if the determinant of its matrix of coefficients is equal to zero, det$A$ = 0.

**Example.** Find solution of the system

$$\begin{cases} 2x-3y+z=0, \\ x+y+z=0, \\ 3x-2y+2z=0. \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & -3 & 1 \\ 1` & 1 & 1 \\ 3 & -2 & 2 \end{vmatrix} = 4-2-9-(3-6-4)=0,$$

$$\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = 2+3=5,$$

$$x = t\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix} = t\cdot(-3-1) = -4t,$$

$$y=-t\begin{vmatrix}2 & 1\\ 1 & 1\end{vmatrix}=-t\cdot(2-1)=-t\,,$$

$$z=t\begin{vmatrix}2 & -3\\ 1 & 1\end{vmatrix}=t\cdot(2+3)=5t\,.$$

$$(-4t;\,-t;\,5t).$$

**Example.** Let us find a solution of the system

$$\begin{cases} x_1+2x_2+x_3=0,\\ -2x_1+3x_2+2x_3=0,\\ -x_1+5x_2+3x_3=0\end{cases}$$ by the Gaussian elimination method.

$$\overline{A}=\left(\begin{array}{ccc|c}1 & 2 & 1 & 0\\ -2 & 3 & 2 & 0\\ -1 & 5 & 3 & 0\end{array}\right)\begin{matrix}e_2+2e_1\\ \sim\\ e_3+e_1\end{matrix}\left(\begin{array}{ccc|c}1 & 2 & 1 & 0\\ 0 & 7 & 4 & 0\\ 0 & 7 & 4 & 0\end{array}\right)\begin{matrix}\\ \sim\\ e_3-e_2\end{matrix}\left(\begin{array}{ccc|c}1 & 2 & 1 & 0\\ 0 & 7 & 4 & 0\\ 0 & 0 & 0 & 0\end{array}\right)\begin{matrix}e_2/7\\ \sim\\ \end{matrix}$$

$$\sim\left(\begin{array}{ccc|c}1 & 2 & 1 & 0\\ 0 & 1 & 4/7 & 0\end{array}\right)\begin{matrix}e_1-2e_2\\ \sim\end{matrix}\left(\begin{array}{ccc|c}1 & 0 & -1/7 & 0\\ 0 & 1 & 4/7 & 0\end{array}\right)\Rightarrow\begin{cases}x_1=x_3/7,\\ x_2=-4x_3/7.\end{cases}$$

## SELF-TEST ASSIGNMENT

A student must be ready to do the following assignments

1. Concepts, definitions, formulations:

- Definite, indefinite, consistent, inconsistent systems.
- Cramer`s formulas. Matrix form of the system. Gauss' method.
- Kronecker-Capelli theorem.

2. Proofs and conclusions:

- Matrix method of solution.
- Cramer's rule.
- Kronecker-Capelli theorem.

3. Assignments:

- Solve the square systems by Cramer's method, through inverse matrix.
- Solve the arbitrary systems by Gauss' method (to your attention simulator "Solving of the systems by Gauss-Jordan method").
- Analyse of the systems on the consistence (compatibility) according to Kronecker-Capelli theorem.

## Questions

1. In what case the system of linear equations can be solved:

a) by Cramer`s rule; b) by matrix method?

2. How can be proved a) inconsistency of the system; b) the existence of unique solution using Cramer`s method?

3. Can the system of n equations with n unknowns has:

a) exactly n solutions; b) less than n solutions; c) more than n solutions?

4. In what case nonhomogeneous system of n equations with n unknowns:

a) has a unique solution; b) a set of solutions; c) has no solutions?

5. In what case homogeneous system of n equations with n unknowns:

a) has a unique solution; b) a set of solutions; c) has no solutions?

6. Can the system of three equations with five unknowns:

a) has a unique solution; b) does not have solutions?

7. How using Gaussian elimination for solving of the system of linear equations can be defined:

a) inconsistency of the system, b) the existence of unique solution?

## Exercises

1. Find solution of the systems: a) by Cramer's method; b) by matrix method.

1) $\begin{cases} x_1 + 2x_2 + 3x_3 = 5, \\ x_1 + 3x_2 + 4x_3 = 6, \\ 2x_1 - x_2 - x_3 = 1. \end{cases}$ 2) $\begin{cases} x_1 - 2x_2 + 2x_3 + x_4 = 3, \\ 2x_2 + 3x_3 - x_4 = 4, \\ 2x_1 - x_3 + 5x_4 = 6, \\ x_1 + x_2 - x_3 - x_4 = 0. \end{cases}$

2. Find solution of the systems by Gauss' method.

1) $\begin{cases} x_1 + 2x_2 + x_3 = 4, \\ 3x_1 - 5x_2 + 3x_3 = 1, \\ 2x_1 + 7x_2 - x_3 = 8. \end{cases}$ 2) $\begin{cases} 2x_1 + 7x_2 + 3x_3 + x_4 = 5, \\ x_1 + 3x_2 + 5x_3 - 2_4 = 3, \\ x_1 + 5x_2 - 9x_3 + 8x_4 = 1, \\ 5x_1 + 18x_2 + 4x_3 + 5x_4 = 12. \end{cases}$ 3) $\begin{cases} 2x_1 + 3x_2 + 11x_3 + 5_4 = 2, \\ x_1 + x_2 + 5x_3 - 2x_4 = 1, \\ 2x_1 + x_2 + 3x_3 + 2x_4 = -3, \\ x_1 + x_2 + 3x_3 + 4x_4 = -3. \end{cases}$

3. Investigate systems for consistency and find solution.

1) $\begin{cases} x_1 - x_2 + x_3 - x_4 = -3, \\ 2x_1 - 3x_2 - 4x_3 + x_4 = 1, \\ x_1 + 7x_3 - 4x_4 = -10. \end{cases}$ 2) $\begin{cases} x_1 + 2x_2 + 3x_3 - x_4 = 0, \\ x_1 - x_2 + x_3 + 2x_4 = 4, \\ x_1 + 5x_2 + 5x_3 - 4x_4 = -4, \\ x_1 + 8x_2 + 7x_3 - 4x_4 = -8. \end{cases}$

4. Find solution of the homogeneous systems:

1) $\begin{cases} 2x_1 + x_2 - x_3 = 0, \\ x_1 + 2x_2 + x_3 = 0, \\ 2x_1 - x_2 + 3x_3 = 0. \end{cases}$ 2) $\begin{cases} 3x_1 - x_2 + 4x_3 = 0, \\ -x_1 + 5x_2 - x_4 = 0, \\ 3x_1 + 4x_2 + 3x_4 = 0. \end{cases}$ 3) $\begin{cases} 3x_1 + x_2 - 8x_3 + x_4 = 0, \\ 2x_1 - 2x_2 - 3x_3 + 2x_4 = 0, \\ x_1 + 11x_2 - 12x_3 - 5x_4 = 0, \\ x_1 - 5x_2 + 2x_3 + 3x_4 = 0. \end{cases}$

## Practice report

Solve the problems according to your variant.

You can see the typical solution of the problems below.

**Problem 1.** Solve the system of linear equations using

a) Cramer’s rule,

b) the matrix method,

c) Gauss method.

| | | |
|---|---|---|
| 1 $\left(\begin{array}{ccc\|c} 2 & 1 & 3 & 7 \\ 2 & 3 & 1 & 1 \\ 3 & 2 & 1 & 6 \end{array}\right)$ | 2 $\left(\begin{array}{ccc\|c} 2 & -1 & 2 & 3 \\ 1 & 1 & 2 & -4 \\ 4 & 1 & 4 & -3 \end{array}\right)$ | 3 $\left(\begin{array}{ccc\|c} 3 & -1 & 1 & 12 \\ 1 & 2 & 4 & 6 \\ 5 & 1 & 2 & 3 \end{array}\right)$ |
| 4 $\left(\begin{array}{ccc\|c} 3 & 2 & -4 & 8 \\ 2 & 4 & -5 & 11 \\ 1 & -2 & 1 & 1 \end{array}\right)$ | 5 $\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -5 \\ 2 & 0 & 3 & -2 \end{array}\right)$ | 6 $\left(\begin{array}{ccc\|c} 2 & -1 & 4 & 15 \\ 3 & -1 & 1 & 8 \\ 5 & -2 & 5 & 0 \end{array}\right)$ |
| 7 $\left(\begin{array}{ccc\|c} 2 & -1 & 3 & -4 \\ 1 & 3 & -1 & 11 \\ 1 & -2 & 2 & -7 \end{array}\right)$ | 8 $\left(\begin{array}{ccc\|c} 3 & -2 & 4 & 12 \\ 3 & 4 & -2 & 6 \\ 2 & -1 & -1 & -9 \end{array}\right)$ | 9 $\left(\begin{array}{ccc\|c} 8 & 3 & -6 & -4 \\ 1 & 1 & -1 & 2 \\ 4 & 1 & -3 & -5 \end{array}\right)$ |

| | | |
|---|---|---|
| 10 $\left(\begin{array}{ccc\|c} 3 & -3 & 2 & 2 \\ 4 & -5 & 2 & 1 \\ 1 & -2 & 0 & 5 \end{array}\right)$ | 11 $\left(\begin{array}{ccc\|c} 3 & 2 & -4 & 8 \\ 2 & 4 & -5 & 1 \\ 5 & 6 & -9 & 2 \end{array}\right)$ | 12 $\left(\begin{array}{ccc\|c} 3 & 1 & 2 & -3 \\ 2 & 2 & 5 & 5 \\ 5 & 3 & 7 & 1 \end{array}\right)$ |
| 13 $\left(\begin{array}{ccc\|c} 4 & 1 & -3 & 9 \\ 1 & 1 & -1 & -2 \\ 8 & 3 & -6 & 12 \end{array}\right)$ | 14 $\left(\begin{array}{ccc\|c} 2 & 3 & 4 & 33 \\ 7 & -5 & 0 & 24 \\ 4 & 0 & 11 & 39 \end{array}\right)$ | 15 $\left(\begin{array}{ccc\|c} 2 & 3 & 4 & 12 \\ 7 & -5 & 1 & -33 \\ 4 & 0 & 1 & -7 \end{array}\right)$ |
| 16 $\left(\begin{array}{ccc\|c} 4 & -7 & -2 & 0 \\ 2 & -3 & -4 & 6 \\ 2 & -4 & 2 & 2 \end{array}\right)$ | 17 $\left(\begin{array}{ccc\|c} 5 & -9 & -4 & 6 \\ 1 & -7 & -5 & 1 \\ 4 & -2 & 1 & 2 \end{array}\right)$ | 18 $\left(\begin{array}{ccc\|c} 1 & -5 & 1 & 3 \\ 3 & 2 & -1 & 7 \\ 4 & -3 & 0 & 1 \end{array}\right)$ |
| 19 $\left(\begin{array}{ccc\|c} 1 & 4 & -1 & 6 \\ 0 & 5 & 4 & -20 \\ 3 & -2 & 5 & -22 \end{array}\right)$ | 20 $\left(\begin{array}{ccc\|c} 3 & -2 & 4 & 21 \\ 3 & 4 & -2 & 9 \\ 2 & -1 & -1 & 10 \end{array}\right)$ | 21 $\left(\begin{array}{ccc\|c} 4 & -3 & 1 & 3 \\ 1 & 1 & -1 & 4 \\ 3 & -4 & 2 & 2 \end{array}\right)$ |
| 22 $\left(\begin{array}{ccc\|c} 5 & -5 & -4 & -3 \\ 1 & -1 & 5 & 1 \\ 4 & -4 & -9 & 0 \end{array}\right)$ | 23 $\left(\begin{array}{ccc\|c} 7 & -2 & -1 & 2 \\ 6 & -4 & -5 & 3 \\ 1 & 2 & 4 & 5 \end{array}\right)$ | 24 $\left(\begin{array}{ccc\|c} 3 & -2 & -5 & 5 \\ 2 & 3 & -4 & 12 \\ 1 & -2 & 3 & -1 \end{array}\right)$ |
| 25 $\left(\begin{array}{ccc\|c} 4 & 1 & 4 & 19 \\ 2 & -1 & 2 & 11 \\ 1 & 1 & 2 & 8 \end{array}\right)$ | 26 $\left(\begin{array}{ccc\|c} 2 & -1 & 2 & 0 \\ 4 & 1 & 4 & 6 \\ 1 & 1 & 2 & 4 \end{array}\right)$ | 27 $\left(\begin{array}{ccc\|c} 2 & -1 & 2 & 8 \\ 1 & 1 & 2 & 11 \\ 4 & 1 & 4 & 22 \end{array}\right)$ |
| 28 $\left(\begin{array}{ccc\|c} 3 & 1 & 2 & 1 \\ 2 & 2 & -3 & 9 \\ 1 & -1 & 1 & 2 \end{array}\right)$ | 29 $\left(\begin{array}{ccc\|c} 6 & 3 & -5 & 0 \\ 9 & 4 & -7 & 3 \\ 3 & 1 & -2 & 5 \end{array}\right)$ | 30 $\left(\begin{array}{ccc\|c} 8 & -1 & 3 & 2 \\ 4 & 1 & 6 & 1 \\ 4 & -2 & -3 & 7 \end{array}\right)$ |

**Problem 2.** Solve the homogenous system $\begin{cases} x_1+2x_2+x_3=0, \\ -2x_1+3x_2+2x_3=0, \\ -x_1+5x_2+3x_3=0. \end{cases}$

$$\bar{A}=\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -2 & 3 & 2 & 0 \\ -1 & 5 & 3 & 0 \end{array}\right)\begin{matrix} e_2+2e_1 \\ \sim \\ e_3+e_1 \end{matrix}\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 7 & 4 & 0 \\ 0 & 7 & 4 & 0 \end{array}\right)\begin{matrix} \\ \sim \\ e_3-e_2 \end{matrix}\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 7 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)\begin{matrix} e_2/7 \\ \sim \\ \end{matrix}$$

$$\sim\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 4/7 & 0 \end{array}\right)\begin{matrix} e_1-2e_2 \\ \sim \end{matrix}\left(\begin{array}{ccc|c} 1 & 0 & -1/7 & 0 \\ 0 & 1 & 4/7 & 0 \end{array}\right)\Rightarrow\begin{cases} x_1=x_3/7, \\ x_2=-4x_3/7. \end{cases}$$

**Problem 2.**

1. $\begin{cases} 5x_1-3x_2+4x_3=0, \\ 3x_1+2x_2-x_3=0, \\ 8x_1-x_2+3x_3=0. \end{cases}$

2. $\begin{cases} 5x_1-6x_2+4x_3=0, \\ 3x_1-3x_2+x_3=0, \\ 2x_1-3x_2+3x_3=0. \end{cases}$

3. $\begin{cases} x_1+2x_2-5x_3=0, \\ 2x_1-4x_2+x_3=0, \\ 3x_1-2x_2-4x_3=0. \end{cases}$

4. $\begin{cases} x_1+x_2+x_3=0, \\ 2x_1-3x_2+4x_3=0, \\ 3x_1-2x_2+5x_3=0. \end{cases}$

5. $\begin{cases} x_1+2x_2+4x_3=0, \\ 5x_1+x_2+2x_3=0, \\ 4x_1-x_2-2x_3=0. \end{cases}$

6. $\begin{cases} 3x_1-x_2+x_3=0, \\ 2x_1+3x_2-4x_3=0, \\ 5x_1+2x_2-3x_3=0. \end{cases}$

7. $\begin{cases} x_1 - 2x_2 + x_3 = 0, \\ 3x_1 + 3_2 + 5x_3 = 0, \\ 4x_1 + x_2 + 6x_3 = 0. \end{cases}$

8. $\begin{cases} 2x_1 + x_2 - 3x_3 = 0, \\ x_1 + 2x_2 - 4x_3 = 0, \\ x_1 - x_2 + x_3 = 0. \end{cases}$

9. $\begin{cases} 2x_1 - x_2 + 2x_3 = 0, \\ 4x_1 + x_2 + 5x_3 = 0, \\ 2x_1 + 2x_2 + 3x_3 = 0. \end{cases}$

10. $\begin{cases} 4x_1 + x_2 + 4x_3 = 0, \\ 3x_1 - 2x_2 - x_3 = 0, \\ 7x_1 - x_2 + 3x_3 = 0. \end{cases}$

11. $\begin{cases} 3x_1 - 2x_2 + x_3 = 0, \\ 2x_1 + 3x_2 - 5x_3 = 0, \\ 5x_1 + x_2 - 4x_3 = 0. \end{cases}$

12. $\begin{cases} 5x_1 + x_2 + 2x_3 = 0, \\ 3x_1 + 2x_2 - 3x_3 = 0, \\ 2x_1 - x_2 + x_3 = 0. \end{cases}$

13. $\begin{cases} x_1 + 2x_2 - 5x_3 = 0, \\ x_1 - 2x_2 - 4x_3 = 0, \\ 2x_1 - 9x_3 = 0. \end{cases}$

14. $\begin{cases} x_1 - 3x_2 + 5x_3 = 0, \\ x_1 + 2x_2 - 3x_3 = 0, \\ 2x_1 - x_2 + 2x_3 = 0. \end{cases}$

15. $\begin{cases} 2x_1 - x_2 + 2x_3 = 0, \\ 3x_1 + 2x_2 - 3x_3 = 0, \\ 5x_1 + x_2 - x_3 = 0. \end{cases}$

16. $\begin{cases} 2x_1 - x_2 + 3x_3 = 0, \\ x_1 - 3x_2 + 2x_3 = 0, \\ x_1 + 2x_2 + x_3 = 0. \end{cases}$

Електронне навчальне видання

# Методичні вказівки

до практичних занять
на тему ***«Лінійна алгебра»***
з курсу **«Вища математика»**
для іноземних здобувачів
очної форми здобуття вищої освіти

(Англійською мовою)

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