Synergetic theory for a jamming transition in traffic flow

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The theory of a jamming transition is proposed for the homogeneous car-following model within the framework of the Lorenz scheme. We represent a jamming transition as a result of the spontaneous deviations of headway and velocity that is caused by the acceleration/braking rate to be higher than the critical value. The stationary values of headway and velocity deviations and time of acceleration/braking are derived as functions of control parameter (time needed for car to take the characteristic velocity).

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I. INTRODUCTION

In recent years, considerable study has been given to the traffic problems [1]. It is shown, in particular, that the jamming transition is similar to the conventional gas-liquid phase transition, where the freely moving traffic and the jammed traffic correspond to the gas and liquid phases, respectively. The transition between them is caused by the growth of car density above a critical value. The congested traffic flow with an unstable uniform part leads to the formation of traffic jams where the freely moving traffic and jammed traffic coexist. Within the framework of Ref. [2], the jamming transition is represented as a first-order phase transition, whose behavior is defined by headway (car density) that acts as the volume (density) and by the inverted delay time (sensitivity parameter) that reduces to temperature.

Our approach is to take into consideration the complete set of freedom degrees as equivalent variables. We obtain the self-consistent analytical description of the jamming transition as a result of the self-organization caused by the positive feedback of the headway deviation and acceleration/braking time on the one hand, as well as the negative feedback of the deviations of headway and velocity on the other hand.

The paper is organized as follows. In Sec. II the self-consistent Lorenz system of the governed equations for the headway and velocity deviations as well as for the acceleration/braking time is obtained. The jamming transition is shown to be supercritical in character (has the second order) if the relaxation time for the first of the pointed out quantities does not depend on its value; it transforms to the subcritical regime with this dependence appearance. Section III deals with the determination of steady-state values for the headway deviation and the acceleration/braking time within the adiabatic approximation. Out of the latter limit, the time dependences for the headway and velocity deviations are studied on the basis of the phase-portrait method. Section IV contains a short discussion of the used assumptions.

II. BASIC EQUATIONS

Within the framework of the simplest car-following model, the acceleration $\dot{V}$ of a given vehicle as a function of its distance Δx to the front vehicle is defined by equality $\dot{V} = [v_{\rm opt}(\Delta x) - V]/\tau$, where $v_{\rm opt}(\Delta x) = \Delta x/t_0$ is the optimal velocity function (t_0 being a characteristic time interval), $h = Vt_0$ is the optimal headway, and τ is the time of acceleration/braking needed for a car to reach the optimal velocity. It is convenient to introduce deviations $\eta \equiv \Delta x - h$ and $v \equiv \Delta\dot{x} - h/t_0 + V$ of headway Δx and its velocity $\Delta\dot{x}$ from the corresponding optimal values h and $h/t_0 - V$. Then, the flow of cars can be described in terms of the pointed-out quantities η, v, and τ. The key point of our approach is that the above degrees of freedom are assumed to be of dissipative type, so that, when they are not coupled, their relaxation to the steady state is governed by the Debye-type equations with corresponding relaxation times t_η, t_v, t_τ. Within the simplest approach, equations for the time dependences $\eta(t)$, $v(t)$, and $\tau(t)$ are supposed to coincide formally with the Lorenz system that describes the self-organization process [3].

The first of the stated equations has the form

$$\dot{\eta} = -\eta/t_\eta + v, \tag{1}$$

where the dot stands for a derivative with respect to time t. The first term on the right-hand side describes the Debye relaxation during time t_η; the second one is the usual addition. In a stationary state, when $\dot{\eta} = 0$, the solution of Eq. (1) defines the conventional linear relationship $\eta = t_\eta v$, so that the headway deviation is proportional to the velocity deviation.

The equation for the rate of quantity v variation is supposed to have the nonlinear form

$$\dot{v} = -v/t_v + g_v \eta\tau, \tag{2}$$

where t_v, g_v are positive constants. As in Eq. (1), the first term on the right-hand side of Eq. (2) describes the relaxation process of velocity deviation v to the stationary value $v = 0$ determined by a time t_v. The second term describes the positive feedback of the headway deviation η and the time τ of acceleration/braking on the velocity deviation v that results in the increase of value v and, thus, causes the self-organization process.

The kinetic equation for the acceleration/braking time τ,

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$$\dot{\tau}=(\tau_0-\tau)/t_\tau-g_\tau\eta v, \tag{3}$$

differs from Eqs. (1) and (2) as follows: the relaxation of quantity τ occurs not to the zero but to the finite value τ_0, representing the stationary time needed for a car to reach the characteristic velocity (in other words, τ_0 is the car characteristic); t_τ is a corresponding relaxation time. In Eq. (3) the negative feedback of the quantities η and v on τ is introduced to imply the decrease of acceleration/braking time τ with the growth of the headway and velocity deviations ($g_\tau>0$ is a corresponding constant).

Equations (1), (2), and (3) constitute the basis for self-consistent description of the car-following model driven by the control parameter τ_0. The distinguishing feature of these equations is that nonlinear terms that enter Eqs. (2) and (3) are of opposite signs, while Eq. (1) is linear. Physically, the latter means just that the velocity deviation is the derivative of headway deviation with respect to time. The negative sign of the last term in Eq. (3) can be regarded as a manifestation of the Le Chatelier principle, i.e., since a decrease in the acceleration/braking time promotes the formation of a stable car flow, the headway and velocity deviations η and v tend to impede the growth of the acceleration/braking time and, as a consequence, the jamming. The positive feedback of η and τ on v in Eq. (2) plays an important part in the problem. As we shall see later, it is precisely the reason behind the self-organization that brings about the traffic jam.

To explain the relaxation transition to the stable jamming state, we shall show further that it is quite enough to use the adiabatic approximation: $t_v=0$, $t_\tau=0$. Therefore, we could proceed not from Eqs. (2) and (3) but from much simple expressions,

$$v=a_v\eta\tau,\quad a_v\equiv t_v g_v,\quad \tau=\tau_0-a_\tau\eta v,\quad a_\tau\equiv t_\tau g_\tau, \tag{4}$$

which are related to the stationary case $\dot{v}=0,\dot{\tau}=0$ in Eqs. (2) and (3), respectively. The equalities (4) have an absolutely clear physical meaning: the increase of the headway deviation η or acceleration/braking time τ leads to a growth of the velocity deviation v, whereas the increase of the headway η and velocity v deviations should cause the decrease of acceleration/braking time τ in comparison with characteristic time τ_0 if the car flow is not broken.

After introducing the suitable scales for quantities η,v,τ,

$$\eta_m\equiv(a_v a_\tau)^{-1/2},\ v_m\equiv\eta_m/t_\eta=t_\eta^{-1}(a_v a_\tau)^{-1/2},$$
$$\tau_c\equiv(t_\eta a_v)^{-1}, \tag{5}$$

Eqs. (1), (2), and (3) can be rewritten in the simplest form of the well-known Lorenz system:

$$\dot{\eta}=-\eta+v, \tag{6}$$

$$\epsilon\ \dot{v}=-v+\eta\tau, \tag{7}$$

$$\delta\ \dot{\tau}=(\tau_0-\tau)-\eta v, \tag{8}$$

where the relaxation time ratios $\epsilon\equiv t_v/t_\eta$, $\delta\equiv t_\tau/t_\eta$ are introduced and the dot now stands for the derivative with respect to the dimensionless time t/τ_η. In general, the system (6)–(8) cannot be solved analytically, but in the simplest case $\epsilon\ll 1$ and $\delta\ll 1$, the left-hand sides of Eqs. (7) and (8) can be neglected. Then, the adiabatic approximation can be used to express the velocity deviation v and the acceleration/braking time τ in the form of the equalities (4). As a result, the dependences of τ and v on the headway deviation η are given by

$$\tau=\frac{\tau_0}{1+\eta^2},\quad v=\frac{\tau_0\eta}{1+\eta^2}. \tag{9}$$

Note that, although η is in the physically meaningful range between 0 and 1, the acceleration/braking time is a monotonically decreasing function of η, whereas the velocity deviation v increases with η (at $\eta>1$ we have $dv/d\eta<0$, which has no physical meaning).

Substituting the second equality (9) into Eq. (6) yields the Landau-Khalatnikov relation:

$$\dot{\eta}=-\frac{\partial\Phi}{\partial\eta} \tag{10}$$

with the effective potential given by

$$\Phi=\tfrac{1}{2}\eta^2-\tfrac{1}{2}\tau_0\ln(1+\eta^2). \tag{11}$$

For $\tau_0<1$, the η dependence of Φ is monotonically increasing and the only stationary value of η equals zero, $\eta_e=0$, so that there is no headway deviation in this case. If the parameter τ_0 exceeds the critical value, $\tau_c=1$, the effective potential assumes the minimum with nonzero steady-state headway deviation $\eta_e=\sqrt{\tau_0-1}$ and the acceleration/braking time $\tau_e=1$.

The above scenario represents the supercritical regime of the traffic-jam formation and corresponds to the second-order phase transition. The latter can be easily seen from the expansion of the effective potential (11) in a power series of $\eta^2\ll 1$:

$$\Phi\approx\frac{1-\tau_0}{2}\eta^2+\frac{\tau_0}{4}\eta^4. \tag{12}$$

So the critical exponents are identical to those obtained within the framework of the mean-field theory [4].

The drawback of the outlined approach is that it fails to account for the subcritical regime of the self-organization that is the reason for the appearance of the traffic jam and analogous to the first-order phase transition, rather than the second-order one. So one has to modify the above theory by taking the assumption that the effective relaxation time $t_\eta(\eta)$ increases with headway deviation η from the initial value $t_\eta/(1+m)$ fixed by a parameter $m>0$ to the final one t_η [5]. The simplest two-parameter approximation is as follows:

$$\frac{t_\eta}{t_\eta(\eta)}=1+\frac{m}{1+(\eta/\eta_0)^2}, \tag{13}$$

where $0<\eta_0<1$. The expression for the effective potential (11) then changes by adding the term

$$\Delta\Phi=\frac{m}{2}\eta_0^2 \ln\left(1+\frac{\eta^2}{\eta_0^2}\right) \tag{14}$$

and the stationary values of η are

$$\eta_e^m=\eta_{00}\{1\mp[1+\eta_0^2\eta_{00}^{-4}(\tau_0-\tau_c)]^{1/2}\}^{1/2}, \tag{15}$$

$$2\eta_{00}^2\equiv(\tau_0-1)-\tau_c\eta_0^2, \quad \tau_c\equiv1+m.$$

The upper sign on the right-hand side of Eq. (15) is for the value at the unstable state η^m where the effective potential $\Phi+\Delta\Phi$ has the maximum; the lower one corresponds to the stable state η_e. The corresponding value of the stationary acceleration/braking time

$$\tau^m=\frac{1+\eta_{00}^2+\sqrt{(1+\eta_{00}^2)^2-(1-\eta_0^2)\tau_0}}{1-\eta_0^2} \tag{16}$$

smoothly increases from the value

$$\tau_m=1+\eta_0 \sqrt{\frac{m}{1-\eta_0^2}} \tag{17}$$

at the parameter $\tau_0=\tau_{c0}$ with

$$\tau_{c0}=(1-\eta_0^2)\tau_m^2 \tag{18}$$

to the marginal value $\tau_c=1+m$ at $\tau_0=\tau_c$.

III. RESULTS

The τ_0 dependences of η_e, η^m, and τ_e are depicted in Fig. 1. As is shown in Fig. 1(a), when the adiabatic condition $t_\tau, t_v \ll t_\eta$ is met and the parameter τ_0 slowly increases to below τ_c, no traffic jam can form. At the point $\tau_0=\tau_c$, the stationary headway deviation η_e jumps upward to the value $\sqrt{2}\eta_{00}$ and its further smooth increase is determined by Eq. (15). If the parameter τ_0 then goes downward, the headway deviation η_e continuously decreases up to the point where $\tau_0=\tau_{c0}$ and $\eta_e=\eta_{00}$. At this point, the jumplike headway deviation goes down to zero. Referring to Fig. 1(b), the stationary acceleration/braking time τ_e shows a linear increase from 0 to τ_c with the parameter τ_0 being in the same interval. Then, after the jump down to the value $(1-\eta_0^2)^{-1}$ at $\tau_0=\tau_c$, the stationary time τ_e smoothly decays to 1 at $\tau_0 \gg \tau_c$. When the parameter τ_0 then decreases from above τ_c down to τ_{c0}, the acceleration/braking time τ_e grows. When the point (18) is reached, the traffic becomes freely moving, so that the stationary acceleration/braking time undergoes the jump from the value (17) up to the one defined by Eq. (18). For $\tau_0<\tau_{c0}$, again the parameter τ_e does not differ from τ_0. Note that this subcritical regime is realized provided the parameter m, which enters the dispersion law (13), is greater than the value

$$m_{\min}=\frac{\eta_0^2}{1-\eta_0^2}. \tag{19}$$

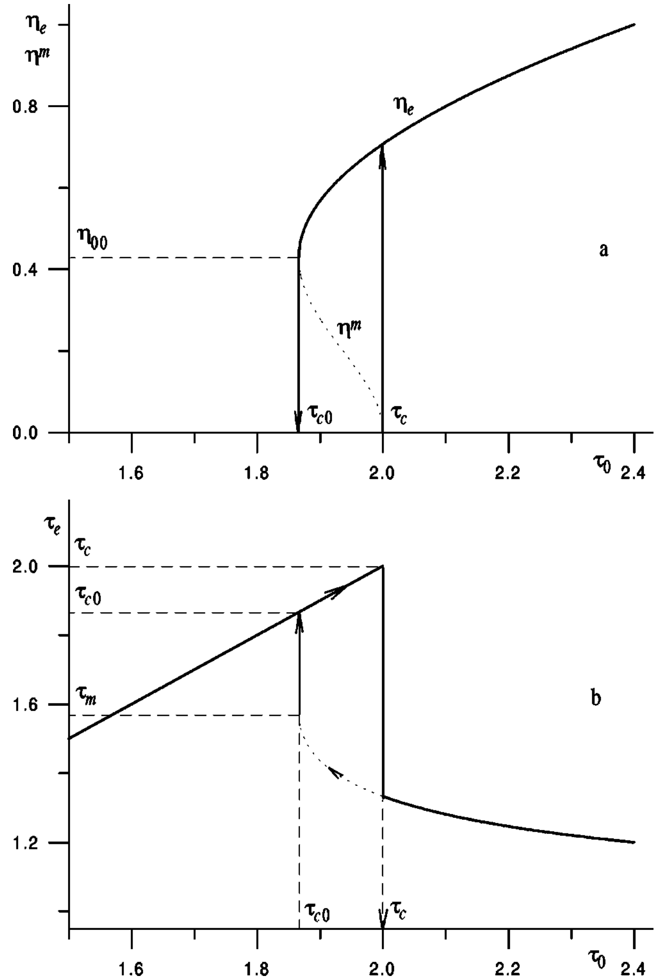

FIG. 1. The τ_0 dependences of the stationary values of (a) headway deviations η_e, η^m; (b) acceleration/braking time τ_e. The arrows indicate the hysteresis loop.

Clearly, according to the picture described, the jamming generation is characterized by the well pronounced hysteresis: the cars initially at motion with optimal headway between them begin to deviate only if the acceleration/braking time τ_0 of cars exceeds its limiting value $\tau_c=1+m$, whereas the acceleration/braking time τ_{c0} needed for uniform car flow is less than τ_c [see Eqs. (17) and (18)]. This is the case in the limit $t_\tau/t_\eta \to 0$ and the hysteresis loop shrinks with the growth of the adiabaticity parameter $\delta\equiv t_\tau/t_\eta$. In addition to the smallness of δ, the adiabatic approximation implies that the ratio $t_v/t_\eta\equiv\epsilon$ is also small. In contrast to the former, the latter does not seem to be realistic for the system under consideration, where, in general, $t_v\approx t_\eta$. So it is of interest to study to what extent the finite value of ϵ could change the results.

Owing to the condition $\delta\ll1$, Eq. (8) is still algebraic and τ can be expressed in terms of η and v. As a result, we derive the system of two nonlinear differential equations that can be studied by the phase portrait method [5]. The phase portraits for various values of ϵ are displayed in Fig. 2, where the center O represents the stationary state and the saddle point S is related to the maximum of the effective

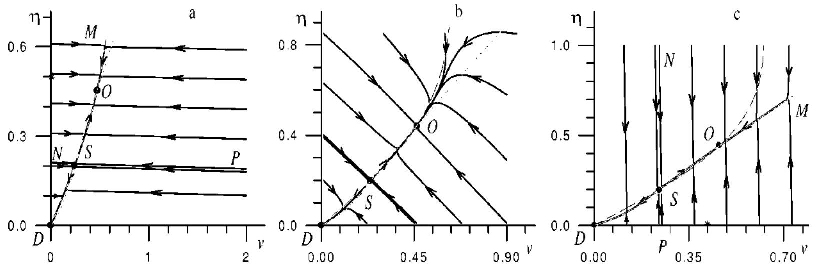

FIG. 2. Phase portraits in the η-v plane at $m=1$, $\eta_0=0.1$, and $\tau_0=1.25\tau_c$ for (a) $\epsilon=10^{-2}$; (b) $\epsilon=1$; (c) $\epsilon=10^2$.

potential. As is seen from the figure, independently of ϵ, there is the universal section, the ''mainstream,'' that attracts all phase trajectories and its structure appears to be almost insensitive to changes in ϵ. Analysis of time dependences $v(t)$ and $\eta(t)$ reveals that the headway and velocity deviations slow down appreciably in this section in comparison to the rest of the trajectories that are almost rectilinear (it is not difficult to see that this effect is caused by the smallness of δ). Since most of the time the system is in the vicinity of the ''mainstream,'' we arrive at the conclusion that finite values of ϵ do not affect qualitatively the above results obtained in the adiabatic approximation.

IV. DISCUSSION

According to the above consideration, the simplest picture of the dissipative dynamic of traffic flow in a homogeneous car-following model can be represented within the framework of the Lorenz model, where the headway η and velocity v deviations play the role of an order parameter and its conjugate field, respectively, and the acceleration/braking time τ is a control parameter. The model is examined to show that a jam is created if the car characteristic τ_0 is larger than the critical magnitude τ_c. The above pointed-out dissipative regime is inherent in the systems with small values of the relaxation time t_τ for acceleration/braking, being apparently a characteristic of a car-driver, and large ones t_η, t_v for the headway and velocity deviations. According to Ref. [5], in the opposite case $t_\tau \geqslant t_\eta, t_v$, the system behaves in autooscillation or stochastic manners.

It is worthwhile to note that the above synergetic scheme allows us to explain the collective phenomena of jamming transitions in the N-body problem with $N\to\infty$. Then the following question arises: why do exactly three variables (the headway and velocity deviations η, v and acceleration/braking time τ) permit us to explain the nontrivial behavior of the N-body problem? The answer to this question gives the theorem by Ruelle and Takens: the nontrivial collective behavior of the many-body system (the type of strange attractor) can be represented only in the case in which the number of variables is not less than three [3]. The interpretation of this fact is the simplest: the first of the freedom degrees can be chosen as the way along the phase trajectory, and the second one corresponds to the negative Lyapunov exponent, ensuring an attraction to this trajectory, the third one acts in the opposite manner to give repulsion. In our case of the self-organization process, the second v and third τ freedom degrees provide the positive and negative feedbacks in Eqs. (2) and (3).

The last question in our approach is why does only the Lorenz scheme allow us to describe the main peculiarities of the jamming transition? The answer is that this is the simplest approach, permitting us to understand the self-organization effects, just as the Landau phenomenological theory of phase transitions describes the great variety of thermodynamical transformations in the simplest way [6]. Let us note in this connection that the effective potential given by the sum of equalities (11) and (14) plays a part in the Landau free energy. But the above-stated synergetic scheme has a principal difference from the Landau-type theory [2] because the former takes into account feedback of the thermostat (the velocity deviation and the acceleration/braking time) with the subsystem under consideration (the headway deviation), whereas the latter does not.

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