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Exactly solvable nonlinear model with two multiplicative Gaussian colored noises

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Abstract

An overdamped system with a linear restoring force and two multiplicative colored noises is considered. Noise amplitudes depend on the system state x as x and $|x|^\alpha$. An exactly soluble model of a system is constructed due to consideration of a specific relation between noises. Exact expressions for the time-dependent univariate probability distribution function and the fractional moments are derived. Their long-time asymptotic behavior is investigated analytically. It is shown that anomalous diffusion and stochastic localization of particles, not subjected to a restoring force, can occur.

Key words: colored noises, Gaussian processes, statistical properties, anomalous diffusion

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1 Introduction

A time evolution of a stochastic system can be described by a Langevin equation [1]. In such approach, the influence of a fluctuating environment is accounted for by means of an external noise term. It is necessary to determine exact statistical characteristics of a system in terms of given statistical characteristics of a noise. Assumption of Gaussian white (delta-correlated) noise considerably simplifies the problem [2]. The time evolution of the system is Markovian process [3], and its univariate probability distribution function (PDF) and transition probability density satisfy the Fokker-Planck equation [2,4,5]. In specific cases one can determine an exact time-dependent solution of such an equation, and a stationary PDF can be obtained in general.

However, Gaussian white noise has some unphysical properties [6], therefore sometimes its application is not justified. Gaussian colored noise with an arbitrary correlation function is more realistic model of a fluctuating environment. Although the time evolution of the stochastic system is non-Markovian process, in specific cases its univariate PDF satisfies the time dependent Fokker-Planck equation [7–9,12]. Exact statistical characteristics of undamped [10] and damped free particles [11] driven by additive Gaussian colored noise, of overdamped particles in a quadratic potential driven by multiplicative Gaussian colored noise [12] are found. In particular, it is shown that anomalous diffusion [10–12] and stochastic localization [11,12] of free particles can occur.

In more general case, a fluctuating environment of a system is modeled by two noise sources. Most commonly, cross-correlated Gaussian white noises are considered. The appropriate Fokker-Planck equation is derived and the two-noise Langevin equation can be reduced to a stochastically equivalent one-noise Langevin equation [13–15]. However, models which demonstrate, for example, nonequilibrium fluctuation-induced transport [16–18], double stochastic resonance [19], require to consider a colored noise source and a white noise

source. The Ornstein-Uhlenbeck process with exponential correlation function is widely used as Gaussian colored noise. But the corresponding Fokker-Planck equation is derived using approximate methods, for instance, conventional small- τ theory [20], the decoupling theory [21], the unified colored noise approximation [22], etc. Within this approximations, systems driven by two Ornstein-Uhlenbeck processes are studied [23,24], the noise self-correlations and the cross-correlation have the same correlation time. There are no general methods for obtaining exact statistical characteristics of systems with two colored noises. Therefore, specific exactly soluble models are needed.

In this paper, we generalize a nonlinear system [12] with colored noise to the case of two colored noises and study the possibility of an exactly solvable model constructing. The system state parameter $x(t)$ evolves according to the following Langevin equation

$$\dot{x}(t) + [\kappa + f_1(t)]x(t) = |x(t)|^\alpha f_2(t), \quad (1)$$

where $x(0) = x_0 (> 0)$, $\kappa (\geq 0)$ and $\alpha (< 1)$ are real-valued parameters, $f_1(t)$ and $f_2(t)$ are colored noises with zero mean and known statistical characteristics. The assumption of positivity of $x(0)$ does not restrict the generality of the model (1).

We note that for $0 \leq \alpha \leq 1$ the solution of Eq. (1) is not unique at $x = 0$. In this paper, the point $x = 0$ is considered to be a regular point and the solution of Eq. (1) coincides with the solution of the equation

$$\{\dot{x}(t) + [\kappa + f_1(t)]x(t)\}|x(t)|^{-\alpha} = f_2(t) \quad [x(0) = x_0 > 0] \quad (2)$$

for all times $t \geq 0$. Although one can obtain an exact solution of Eq. (2) for the case of arbitrary independent noises $f_1(t)$ and $f_2(t)$, it is not possible to determine exact statistical characteristics of $x(t)$. Therefore, we consider a specific relation between noises $f_1(t)$ and $f_2(t)$, which allows to obtain exactly

the time-dependent univariate PDF and the fractional moments of $x(t)$. We find analytically their long-time asymptotics and show that in the special case $\kappa = 0$ anomalous diffusion and stochastic localization of particles can occur.

The paper is organized as follows. In Sec. II, we solve exactly Eq. (2) with arbitrary independent noises $f_1(t)$, $f_2(t)$ and consider a specific relation between noises. In the same section we obtain the time-dependent univariate PDF and the fractional moments of $x(t)$. Their long-time asymptotic behavior is studied analytically in Sec. III. Our results are summarized in Sec. IV.

2 General analysis

To construct an exactly soluble model of a system describing by the Langevin equation (2), we obtain its exact solution for the case of arbitrary independent noises $f_1(t)$ and $f_2(t)$. By introducing the new variable $y(t) = x(t)|x(t)|^{-\alpha}$, the nonlinear stochastic differential equation (2) is transformed to the linear one

$$\dot{y}(t) + [\omega + (1 - \alpha)f_1(t)]y(t) = (1 - \alpha)f_2(t), \quad (3)$$

where $\omega = (1 - \alpha)\kappa$ and $y(0) = x_0^{1-\alpha}$. The solution of Eq. (3) is given by (see, for example, Ref. [25])

$$y(t) = e^{-\omega t - (1-\alpha)F_1(t)} \left[x_0^{1-\alpha} + (1 - \alpha) \int_0^t d\tau f_2(\tau) e^{\omega\tau + (1-\alpha)F_1(\tau)} \right], \quad (4)$$

where

$$F_1(t) = \int_0^t d\tau f_1(\tau). \quad (5)$$

Substituting Eq. (4) into expression $x(t) = y(t)|y(t)|^{\alpha/(1-\alpha)}$, we can write the solution of Eq. (2) as

$$x(t) = \frac{F(t)|F(t)|^{\alpha/(1-\alpha)}}{e^{F_1(t)}}, \quad (6)$$

where

$$F(t) = e^{-\omega t} \left[x_0^{1-\alpha} + (1-\alpha) \int_0^t d\tau f_2(\tau) e^{\omega\tau + (1-\alpha)F_1(\tau)} \right]. \quad (7)$$

To obtain the time-dependent univariate PDF of $x(t)$, we should determine the univariate PDFs of $F_1(t)$ and $F(t)$. The univariate PDF of $F_1(t)$ is easily found if $f_1(t)$ is Gaussian noise with zero mean and the arbitrary correlation function

$$\langle f_1(t)f_1(t') \rangle = r_1(|t-t'|). \quad (8)$$

Indeed, by virtue of the linear dependence $F_1(t)$ on $f_1(t)$, $F_1(t)$ is Gaussian process with zero mean and the dispersion

$$\sigma_1^2(t) = 2 \int_0^t du r_1(u)(t-u). \quad (9)$$

The simplest way to obtain the univariate PDF of $F(t)$ is to consider the specific relation between noises $f_1(t)$ and $f_2(t)$. We assume that

$$f_2(t) = f(t)e^{(\alpha-1)F_1(t)}, \quad (10)$$

where $f(t)$ is Gaussian noise with zero mean and the arbitrary correlation function

$$\langle f(t)f(t') \rangle = r(|t-t'|). \quad (11)$$

Then $F(t)$ is Gaussian noise with the mean $m(t)$ and the dispersion $\sigma^2(t)$

$$m(t) = x_0^{1-\alpha} e^{-\omega t}, \quad (12)$$

$$\sigma^2(t) = 2(1-\alpha) \frac{e^{-\omega t}}{\kappa} \int_0^t du r(u) \sinh[\omega(t-u)]. \quad (13)$$

Thus, the Langevin equation (2) with arbitrary noises $f_1(t)$ and $f_2(t)$ is reduced to the following Langevin equation

$$\{\dot{x}(t) + [\kappa + f_1(t)]x(t)\}|x(t)|^{-\alpha} = f(t)e^{(\alpha-1)F_1(t)} \quad [x(0) = x_0 > 0], \quad (14)$$

where $f_1(t)$ and $f(t)$ are statistically independent Gaussian colored noises with known statistical characteristics.

The random process $p(t) = F(t)|F(t)|^{\alpha/(1-\alpha)}$ is power-normal [12]. Let $P_p(p, t)$ denote the univariate PDF that $p(t) = p$,

$$P_p(p, t) = \frac{1-\alpha}{\sqrt{2\pi}\sigma(t)|p|^\alpha} \exp\left\{-\frac{[p|p|^{-\alpha} - m(t)]^2}{2\sigma^2(t)}\right\}. \quad (15)$$

The random process $l(t) = e^{F_1(t)}$ is logarithmic-normal [26]. Let $P_l(l, t)$ denote the univariate PDF that $l(t) = l$,

$$P_l(l, t) = \frac{1}{\sqrt{2\pi}\sigma_1(t)l} \exp\left[-\frac{\ln^2 l}{2\sigma_1^2(t)}\right] \quad (16)$$

for $l \geq 0$ and $P_l(l, t) = 0$ for $l < 0$. According to Eq. (6), the solution of Eq. (14) can be rewritten as $x(t) = p(t)/l(t)$, where $p(t)$ and $l(t)$ are statistically independent power-normal and lognormal processes. It is not difficult to get the conclusion, that by introducing the new variable $p(t) = x(t)l(t)$, Eq. (14) can be obtained from the Langevin equation with one noise

$$[\dot{p}(t) + \kappa p(t)]|p(t)|^{-\alpha} = f(t) \quad [p(0) = p_0 > 0], \quad (17)$$

where $p_0 = x_0$.

Let $P_x(x, t)$ be the univariate PDF that $x(t) = x$. It can be calculated as (see, for example, Ref. [27])

$$P_x(x, t) = \int_0^{\infty} du u P_l(u, t) P_p(ux, t). \quad (18)$$

Substituting Eq. (15) and Eq. (16) into Eq. (18), we finally obtain the time-dependent univariate PDF of $x(t)$

$$P_x(x, t) = \frac{1 - \alpha}{2\pi\sigma_\alpha(t)\sigma(t)|x|^\alpha} \int_0^{\infty} du \exp \left\{ -\frac{\ln^2 u}{2\sigma_\alpha^2(t)} - \frac{[ux|x|^{-\alpha} - m(t)]^2}{2\sigma^2(t)} \right\}, \quad (19)$$

where $\sigma_\alpha(t) = (1 - \alpha)\sigma_1(t)$.

We note if $f_1(t) \equiv 0$ [$\sigma_\alpha^2(t) \rightarrow 0$ for all times t] the small neighborhood of the point $u = 1$ gives basic contribution and Eq. (19) is reduced to the PDF of the power-normal distribution (15). If $f(t) \equiv 0$ [$\sigma^2(t) \rightarrow 0$ for all times t] we take into account

$$\lim_{\sigma^2(t) \rightarrow 0} \left\{ \left\{ \sqrt{2\pi} \left[\frac{\sigma(t)}{|x|^{1-\alpha}} \right] \right\}^{-1} \exp \left\{ -\frac{1}{2} \left[u - \frac{m(t)}{x|x|^{-\alpha}} \right]^2 \right. \right. \\ \left. \left. \times \left[\frac{\sigma(t)}{|x|^{1-\alpha}} \right]^{-2} \right\} \right\} = \delta \left[u - \frac{m(t)}{x|x|^{-\alpha}} \right], \quad (20)$$

where $\delta(x)$ is the Dirac delta function. Eq. (19) is reduced to the PDF of the lognormal distribution [12]

$$P_x(x, t) = \frac{1}{\sqrt{2\pi}\sigma_1(t)x} \exp \left[-\frac{1}{2\sigma_1^2(t)} \left(\ln \frac{x}{x_0} + \kappa t \right)^2 \right] \quad (21)$$

for $x \geq 0$ and $P_x(x, t) = 0$ for $x < 0$. Taking into account Eqs. (20) and (21), we note that Eq. (19) reduces to the delta function $\delta(x - x_0)$ at time $t = 0$.

We can now determine the time-dependent fractional moments of $x(t)$. They are defined as

$$m_r^v(t) = \int_{-\infty}^{\infty} dx P_x(x, t) |x|^{r-v} x^v, \quad (22)$$

where r is a real number, and $v = 0$ or 1 . We substitute Eq. (18) into Eq. (22). By introducing the new variable $y = ux$, one can write

$$m_r^v(t) = \int_0^\infty \frac{du}{u^r} P_l(u, t) \int_{-\infty}^\infty dy P_p(y, t) |y|^{r-v} y^v. \quad (23)$$

We take into account that the second integral is the well-known fractional moments of power-normal distribution [12]. Calculating the first integral, we finally obtain the fractional moments of $x(t)$

$$m_r^v(t) = \frac{\Gamma(\xi)}{\sqrt{2\pi}} \sigma^{\xi-1}(t) \exp \left[r^2 \sigma_1^2(t)/2 - a^2(t)/4 \right] \\ \times \{ D_{-\xi}[-a(t)] + (-1)^v D_{-\xi}[a(t)] \}. \quad (24)$$

Here $\xi = 1 + r/(1 - \alpha)$, $a(t) = m(t)/\sigma(t)$, and $D_{-\xi}(z)$ is the integral representation of the Weber parabolic cylinder function [28]

$$D_{-\xi}(z) = \frac{e^{-z^2/4}}{\Gamma(\xi)} \int_0^\infty dy y^{\xi-1} e^{-y^2/2 - zy} \quad (\xi > 0), \quad (25)$$

where $\Gamma(\xi)$ is the gamma function. If $\xi \leq 0$ all fractional moments diverge. Since $m_0^0(t) = 1$, $P_x(x, t)$ is properly normalized.

We note, the asymptotic of $m_r^v(t)$ as $\sigma_1^2(t) \rightarrow 0$ for all times t is obtained directly from Eq. (24)

$$m_r^v(t) = \frac{\Gamma(\xi)}{\sqrt{2\pi}} e^{-a^2(t)/4} \sigma^{\xi-1}(t) \{ D_{-\xi}[-a(t)] + (-1)^v D_{-\xi}[a(t)] \}. \quad (26)$$

Eq. (26) is the fractional moments of the power-normal distribution [12]. In order to determine the fractional moments as $\sigma^2(t) \rightarrow 0$ for all times t , we use the asymptotic form

$$\lim_{\sigma^2(t) \rightarrow 0} \sigma^{\xi-1}(t) \exp \left[-\frac{m^2(t)}{4\sigma^2(t)} \right] D_{-\xi} \left[\pm \frac{m(t)}{\sigma(t)} \right]$$

$$= \frac{\sqrt{2\pi}}{\Gamma(\xi)} m^{\xi-1}(t) \int_0^{\infty} ds s^{\xi-1} \delta(s \pm 1). \quad (27)$$

Substituting Eq. (27) into Eq. (24), we obtain

$$m_r^v(t) = x_0^r \exp \left[\frac{1}{2} r^2 \sigma_1^2(t) - r\kappa t \right]. \quad (28)$$

Eq. (28) is the fractional moments of the logarithmic-normal distribution [12].

3 Asymptotic behavior

In the previous section, we have constructed an exactly soluble model with two multiplicative Gaussian colored noises. We have obtained exact expressions for the time-dependent univariate probability distribution function and the fractional moments. In this section, we determine the long-time behavior of particles, $t \rightarrow \infty$. Cases $\kappa > 0$ and $\kappa = 0$ are considered separately.

3.1 $\kappa > 0$

In this case $m(\infty) = 0$ and $a(\infty) = 0$. The asymptotic form of Eq. (13) at $t = \infty$ is given by

$$\sigma^2(\infty) = \frac{1-\alpha}{\kappa} \int_0^{\infty} du r(u) e^{-\omega u}. \quad (29)$$

Since $r(u) \rightarrow 0$ as $u \rightarrow \infty$, $\sigma^2(\infty) < \infty$. The asymptotic form of Eq. (9) can be written as

$$\sigma_1^2(t) \sim 2tR_1(t) \quad (t \rightarrow \infty), \quad (30)$$

where $R_1(t) = \int_0^t dur_1(u)$. According to Ref. [11], if

$$R_1(t) = o(1/t) \quad (t \rightarrow \infty), \quad (31)$$

then $\sigma_1^2(\infty) < \infty$, and if $0 < R_1(\infty) \leq \infty$ or $R_1(\infty) = 0$ [$R_1(\infty)$ is the noise intensity], but Eq. (31) does not hold, then $\sigma_1^2(\infty) = \infty$. Thus, there are two qualitatively different cases of the long-time asymptotic behavior of Eq. (9), namely, $\sigma_1^2(\infty) < \infty$ and $\sigma_1^2(\infty) = \infty$.

To determine the asymptotic behavior of $m_r^v(t)$ as $t \rightarrow \infty$ we use the formula

$$D_{-\xi}(0) = 2^{\xi/2-1} \frac{\Gamma(\xi/2)}{\Gamma(\xi)}, \quad (32)$$

which follows from Eq. (25). Substituting Eq. (32) into Eq. (24), we obtain

$$m_r^v(\infty) = \frac{\Gamma(\xi/2)}{\sqrt{2\pi}} 2^{\xi/2} \sigma^{\xi-1}(\infty) e^{r^2 \sigma_1^2(\infty)/2} \frac{1 + (-1)^v}{2} \quad (\xi > 0). \quad (33)$$

If Eq. (31) is fulfilled [$\sigma_1^2(\infty) < \infty$], all fractional moments with $r > \alpha - 1$ (33) are finite, and the stationary PDF $P_{st}(x) = P_x(x, \infty)$ is given by

$$P_{st}(x) = \frac{1 - \alpha}{2\pi\sigma_\alpha(\infty)\sigma(\infty)|x|^\alpha} \int_0^\infty du \exp \left[-\frac{\ln^2(u)}{2\sigma_\alpha^2(\infty)} - \frac{u^2|x|^{2(1-\alpha)}}{2\sigma^2(\infty)} \right]. \quad (34)$$

Note that $P_{st}(x)$ is even, as is also reflected by $m_r^1(\infty) = 0$. The value of $P_{st}(x)$ at the point $x = 0$ depends on α , if $0 < \alpha < 1$ then $P_{st}(0) = \infty$, if $\alpha = 0$ then $P_{st}(0) = e^{\sigma_1^2(\infty)/2} / [\sqrt{2\pi}\sigma(\infty)]$, and if $\alpha < 0$ then $P_{st}(0) = 0$.

If Eq. (31) is not fulfilled [$\sigma_1^2(\infty) = \infty$], then $m_r^0(\infty) = \infty$ and $m_r^1(\infty) = 0$. The stationary PDF $P_{st}(x) = P_x(x, \infty)$ (19) can be written as

$$P_{st}(x) \sim \frac{1}{2\pi\sigma(\infty)|x|^\alpha \sqrt{2tR_1(t)}} \times \int_0^\infty du \exp \left[-\frac{\ln^2(u)}{4(1-\alpha)^2 t R_1(t)} - \frac{u^2|x|^{2(1-\alpha)}}{2\sigma^2(\infty)} \right] \quad (t \rightarrow \infty). \quad (35)$$

For $x \neq 0$, the first term of the exponent in Eq. (35) can be neglected compared to the second one, and we find

$$P_{st}(x)|_{x \neq 0} = \lim_{t \rightarrow \infty} \frac{1}{4|x|\sqrt{\pi t R_1(t)}} = 0. \quad (36)$$

For $x = 0$ Eq. (35) is reduced to

$$P_{st}(0) = \lim_{\substack{t \rightarrow \infty \\ x \rightarrow 0}} \frac{(1 - \alpha) \exp[(1 - \alpha)^2 t R_1(t)]}{\sqrt{2\pi\sigma(\infty)}|x|^\alpha} = \infty. \quad (37)$$

To gain more insight into the behavior of $P_{st}(x)$ in the vicinity of $x = 0$, we define the probability

$$W_\varepsilon(t) = \int_{-\varepsilon}^{\varepsilon} dx P_x(x, t), \quad (38)$$

that $x(t) \in (-\varepsilon, \varepsilon)$. Substituting Eq. (35) into Eq. (38), we obtain

$$W_\varepsilon(t) \sim \frac{1}{\sqrt{2\pi t A(t)}} \int_{-\infty}^{\infty} dz \exp\left[-\frac{z^2}{2tA(t)}\right] \operatorname{erf}\left[\frac{\varepsilon^{1-\alpha} e^z}{\sqrt{2}\sigma(\infty)}\right] \quad (t \rightarrow \infty), \quad (39)$$

where $A(t) = 2(1 - \alpha)^2 R_1(t)$. To evaluate $W_\varepsilon(\infty)$, we introduce the new variable $z = s\sqrt{t}$, and take into account that

$$\lim_{t \rightarrow \infty} \operatorname{erf}\left[\frac{\varepsilon^{1-\alpha} e^{s\sqrt{t}}}{\sqrt{2}\sigma(\infty)}\right] = h(s), \quad (40)$$

where $h(s)$ is the Heaviside function. Substituting Eq. (40) into Eq. (39), we obtain $W_\varepsilon(\infty) = 1/2$. Thus, in the case $\sigma_1^2(\infty) = \infty$ no well-defined stationary PDF exists.

3.2 $\kappa = 0$

In this subsection, we study the statistical behavior of free particles (particles not subjected to a restoring force [12]) under influence of two colored noises.

As mentioned in the Introduction, free particles driven by additive colored noise or multiplicative Gaussian colored noise can display anomalous diffusion and stochastic localization.

In this case ($\kappa = 0$) $m(t) = x_0^{1-\alpha}$ for all times t , and Eq. (13) is reduced to

$$\sigma^2(t) = 2(1 - \alpha)^2 \int_0^t dur(u)(t - u). \quad (41)$$

Eq. (41) is qualitatively similar to Eq. (9), and its long-time asymptotic has the form

$$\sigma^2(t) \sim 2(1 - \alpha)^2 t R(t) \quad (t \rightarrow \infty), \quad (42)$$

where $R(t) = \int_0^t dur(u)$. If

$$R(t) = o(1/t) \quad (t \rightarrow \infty), \quad (43)$$

then $\sigma^2(\infty) < \infty$, and if $0 < R(\infty) \leq \infty$ or $R(\infty) = 0$ [$R(\infty)$ is the noise intensity], but Eq. (43) does not hold, then $\sigma^2(\infty) = \infty$. So, there are two different cases of the long-time asymptotic behavior of Eq. (41), namely, $\sigma^2(\infty) < \infty$ and $\sigma^2(\infty) = \infty$.

It is not difficult to verify that if $\sigma_1^2(\infty) = \infty$ and $\sigma^2(\infty) < \infty$ or $\sigma^2(\infty) = \infty$ then $m_r^0(\infty) = \infty$ and $m_r^1(\infty) = 0$, $P_x(0, \infty) = \infty$ and $P_x(x, \infty)|_{x \neq 0} = 0$, $W_\varepsilon(\infty) = 1/2$. So, in this cases no well-defined stationary PDF exists.

If $\sigma^2(\infty) < \infty$ and $\sigma_1^2(\infty) < \infty$, all fractional moments (24) with $r > \alpha - 1$ are finite

$$m_r^v(t) = \frac{\Gamma(\xi)}{\sqrt{2\pi}} \sigma^{\xi-1}(\infty) \exp \left[r^2 \sigma_1^2(\infty)/2 - a^2(\infty)/4 \right] \\ \times \{ D_{-\xi}[-a(\infty)] + (-1)^v D_{-\xi}[a(\infty)] \}, \quad (44)$$

$a(\infty) = x_0^{1-\alpha}/\sigma(\infty)$, and the stationary PDF is given by

$$P_{st}(x) = \frac{1 - \alpha}{2\pi\sigma_1(\infty)\sigma(\infty)|x|^\alpha} \int_0^\infty du \exp \left\{ -\frac{\ln^2(u)}{2\sigma_\alpha^2(\infty)} - \frac{[ux|x|^{-\alpha} - x_0^{1-\alpha}]^2}{2\sigma^2(\infty)} \right\}. \quad (45)$$

In contrast to the case $\kappa > 0$, where the stationary PDF (34) is even, the stationary PDF (45) is not even, as is also reflected by the fractional moments, $m_r^1(\infty) \neq 0$.

Thus, if Eqs. (31) and (43) are fulfilled, then Gaussian colored noises $f_1(t)$ and $f(t)$ lead to stochastic localization of free particles. This phenomenon was first described for free particles driven by additive [11] and multiplicative [12] colored noise.

If $\sigma^2(\infty) = \infty$ and $\sigma_1^2(\infty) < \infty$, then $m_r^0(\infty) = \infty$ and $m_r^1(\infty) = 0$, $P_x(x, \infty) = 0$ and $W_\varepsilon(\infty) = 0$. Consequently, the stationary PDF does not exist, Gaussian colored noises $f(t)$ give rise to diffusive behavior of free particles. This case is qualitatively similar to one considered in Ref. [12]. It is not difficult to verify, if $0 < R(\infty) < \infty$, then $\sigma^2(t) \propto t$ as $t \rightarrow \infty$, $m_r^0(t) \propto t^{(\xi-1)/2}$ and $m_r^1(t) \propto t^{\xi/2-1}$. The dispersion of the particle position, $\sigma_x^2(t) = \langle x^2(t) \rangle - \langle x(t) \rangle^2$, can be rewritten as $\sigma_x^2(t) = m_2^0(t) - [m_1^1(t)]^2$. Hence, $\sigma_x^2(t) \propto t^{1/(1-\alpha)}$ as $t \rightarrow \infty$, and the conditions $\alpha = 0$, $\alpha < 0$, and $0 < \alpha < 1$ correspond to normal diffusion, subdiffusion, and superdiffusion, respectively.

4 Conclusions

We have constructed an exactly solvable nonlinear model of an overdamped system with two multiplicative Gaussian colored noises and studied its statistical properties. Starting from the exact solution of the Langevin equation for the case of arbitrary independent noises, we have considered the specific relation between noises. In such approach, the time evolution of the system state has been represented as the ratio of the power-normal process to the lognor-

mal one, and the two-noise Langevin equation can be reduced to the one-noise Langevin equation. We have obtained exact expressions for the time-dependent univariate probability distribution function of the system state and the fractional moments. Analyzing their analytically found long-time asymptotics, we have shown that anomalous diffusion and stochastic localization of particles, not subjected to a restoring force, can occur.

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