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## Field theory of self-organization

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### Abstract

The subject of this study is the self-organizing system whose behavior is governed by the field of an order parameter, a fluctuation amplitude of conjugate field, and a couple of Grassmann conjugated fields that define the entropy as a control parameter. Within the framework of self-consistent approach the macro- and microscopic susceptibilities, as well as memory and nonergodicity parameters, are determined as functions of the intensities of thermal and quenched disorders. The phase diagram is calculated that defines the domains of ordered, disordered, ergodic, and nonergodic phases.

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### 1. Introduction

Nowadays, an original situation has arisen in the theory of self-organizing systems. The synergetic concept has been successfully developing for more than 20 years. It has allowed to explain the self-organization (the ordering) of an open system subjected to the environment disorder [1]. However it has not been used for description of the nonergodicity property manifesting, for e.g., at formation of structural and spin glasses [2,3], traffic jams [4,5], flux steady states (avalanches) [6]. In connection to this the standard synergetic approach requires nontrivial extension—while studying of the ordering self-organizing system, we have to consider not only a symmetry breaking,

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but an ergodicity loosing that induces the clusterization of phase space. This paper is devoted to the solution of this problem.

The suggested approach is based on a synergetic generalization of the thermodynamic theory of phase transitions. The main feature of the latter is that within the closed system (thermostat) a subsystem is picked out to represent a hydrodynamical mode whose amplitude is an order parameter determining the subsystem state [7]. Here it is assumed that the thermostat effects upon the order parameter  $\eta$  both thermally—by varying of a control parameter  $S$ , and immediately—by varying a field  $h$  conjugated to the order parameter. (In the case of a magnet, the values  $\eta, h, S$  mean the magnetization, the magnetic field, and the entropy, respectively.) The special peculiarity of the thermodynamic approach is that a one-sided influence of thermostat on the ordering subsystem is postulated, but not the reverse—the order parameter  $\eta$  variation does not effect upon the thermostat state parameters  $h$  and  $S$ . The synergetic approach considers the whole system corresponding to the thermostat and the picked out subsystem, so that the control parameter  $S$  and conjugate field  $h$  turn out to be dependent on the order parameter  $\eta$ . This dependence is essential for kinetic description of the phase transition. So, the Landau–Khalatnikov dissipative dynamics is realized within the adiabatic relation, when the relaxation time of the order parameter is much longer than the corresponding times for conjugate field and control parameter [8]. At that it is convenient to use the Lorenz system that is the simplest synergetic scheme and the classical model of chaos and strange attractors [9]. It was primarily suggested in order to describe the chaotic fluctuations at convective flow [10] and later on at generation in one-mode laser [11].

This paper is organized as follows. Sections 2, 3, and Appendix A show that the Lorenz system corresponds to the simplest Lagrangian of the supersymmetric field theory, where components give the order parameter  $\eta$ , the conjugate field  $h$ , and the entropy  $S$ . It is rather important that a combination of the Grassmann components of the superfield plays the role of control parameter  $S$ —unlike the usual field theory of a stochastic system [12], where such components are auxiliary variables, which have no physical meaning. The fact that the variables  $\eta, h, S$  form a vector within the supersymmetric space is a reflection of the self-consistent behavior of the synergetic system (in contrast to the statistical field scheme [12], where the superfield is only a convenient technical method). According to the study of the superfield correlator carried out in Section 4, components of such a correlator are not independent—the supersymmetry causes a fluctuation–dissipation relation that connects the usual correlator and response functions being the components of the above supercorrelator [12]. In the case of the quenched disorder an ergodicity loosing occurs that breaks the supersymmetry. This leads to appearance of singular additions to the correlator and response function that define memory and nonergodicity parameters  $q$  and  $\Delta$ . The main purpose of this work is to define quantities  $q$  and  $\Delta$  as functions of thermal and quenched disorder intensities. This allows us to find the conditions of self-organization. Final Section 5 is devoted to the discussion of the results obtained. It is shown that a parameter, determining the transition to time-irreversible regime, is given by squared ratio of a time of the quantum fluctuation to the macroscopic time of order parameter variation. A critical value of effective interaction is found to bound the domain of ordered state.

## 2. Lorenz system

To introduce a microscopic scheme of the self-organization description, let us study firstly the system of Boson and Fermion gases whose interaction is characterized by a potential  $v$ . Within the framework of secondary quantization, Bosons are described by creation and annihilation operators  $b_l^+$ ,  $b_l$ , satisfying the usual commutation relation:  $[b_l, b_m^+] = \delta_{lm}$ , where  $l, m$  are the site numbers. The two-level Fermion subsystem is represented by operators  $a_{l\alpha}^+$ ,  $a_{l\alpha}$ ,  $\alpha = 1, 2$ , which fulfill the anti-commutation relation  $\{a_{l\alpha}, a_{m\beta}^+\} = \delta_{lm}\delta_{\alpha\beta}$ . The occupation numbers  $b_{\mathbf{k}}^+ b_{\mathbf{k}}$  determine the Bosons distribution within  $\mathbf{k}$ -representation that corresponds to the Fourier transform over lattice sites  $l$ . To represent the Fermi subsystem, we introduce the operator  $d_l \equiv a_{l1}^+ a_{l2}$  determining the polarization with respect to the saturation over levels  $\alpha = 1, 2$ , as well as the occupation numbers  $n_{l\alpha} \equiv a_{l\alpha}^+ a_{l\alpha}$ . As a result, the behavior of the considered system is defined by the Dicke Hamiltonian ( $\hbar = 1$ )

$$H = \sum_{\mathbf{k}} \left\{ (E_1 n_{\mathbf{k}1} + E_2 n_{\mathbf{k}2}) + \omega_{\mathbf{k}} b_{\mathbf{k}}^+ b_{\mathbf{k}} + \frac{i}{2} v (b_{\mathbf{k}}^+ d_{\mathbf{k}} - d_{\mathbf{k}}^+ b_{\mathbf{k}}) \right\}, \quad (2.1)$$

where the  $\mathbf{k}$ -representation is used,  $E_{1,2}$  are Fermi energies,  $\omega_{\mathbf{k}}$  is the Boson dispersion law, and the imaginary unit before the interaction  $v$  reflects the Hermitian property.

The Heisenberg equations of motion corresponding to Hamiltonian (2.1) have the form

$$\dot{b}_{\mathbf{k}} = -i\omega_{\mathbf{k}} b_{\mathbf{k}} + (v/2)d_{\mathbf{k}}, \quad (2.2)$$

$$\dot{d}_{\mathbf{k}} = -i\Delta d_{\mathbf{k}} + (v/2)b_{\mathbf{k}}(n_{\mathbf{k}2} - n_{\mathbf{k}1}), \quad (2.3)$$

$$\dot{n}_{\mathbf{k}1} = (v/2)(b_{\mathbf{k}}^+ d_{\mathbf{k}} + d_{\mathbf{k}}^+ b_{\mathbf{k}}), \quad (2.4)$$

$$\dot{n}_{\mathbf{k}2} = -(v/2)(b_{\mathbf{k}}^+ d_{\mathbf{k}} + d_{\mathbf{k}}^+ b_{\mathbf{k}}), \quad (2.5)$$

where the dot stands for a derivative with respect to time and the quantity  $\Delta \equiv E_2 - E_1$  is introduced. In resonance the first terms on the right-hand sides of Eqs. (2.2), (2.3) containing frequencies  $\omega_{\mathbf{k}}$  and  $\Delta$  may be suppressed by introducing the multipliers  $\exp(-i\omega_{\mathbf{k}}t)$  and  $\exp(-i\Delta t)$  for the time dependencies  $b_{\mathbf{k}}(t)$  and  $d_{\mathbf{k}}(t)$ , respectively. On the other hand, to take into account the dissipation these frequencies acquire imaginary terms  $-i/\tau_{\eta}$ ,  $-i/\tau_h$  characterized by relaxation times  $\tau_{\eta}$ ,  $\tau_h$  (here the conditions  $\text{Im } \omega_{\mathbf{k}} < 0$ ,  $\text{Im } \Delta < 0$  reflect the causality principle). As a result, Eqs. (2.2), (2.3) get the dissipative terms  $-b_{\mathbf{k}}/\tau_{\eta}$ ,  $-d_{\mathbf{k}}/\tau_h$ , where  $\tau_{\eta}$  is the relaxation time of Boson distribution and  $\tau_h$  is the Fermion polarization time. We can suppose that the dissipation also influences on the Fermi-level occupancies  $n_{\mathbf{k}\alpha}(t)$ . However, since the stationary values  $n_{\mathbf{k}\alpha}^0 \neq 0$  (in case of an external drive  $n_{\mathbf{k}2}^0 > n_{\mathbf{k}1}^0$ ), the dissipative terms in Eqs. (2.4), (2.5) have a more complicated form  $-(n_{\mathbf{k}\alpha} - n_{\mathbf{k}\alpha}^0)/\tau_S$ , where  $\tau_S$  is the relaxation time of the Fermion distribution over levels  $\alpha = 1, 2$ .

Now, let us introduce the macroscopic quantities

$$\begin{aligned}\eta_{\mathbf{k}} &\equiv \langle b_{\mathbf{k}}^+ \rangle = \langle b_{\mathbf{k}} \rangle, & h_{\mathbf{k}} &\equiv \langle d_{\mathbf{k}} \rangle = \langle d_{\mathbf{k}}^+ \rangle, \\ S_{\mathbf{k}} &\equiv \langle n_{\mathbf{k}2} - n_{\mathbf{k}1} \rangle, & S_{\mathbf{k}}^0 &\equiv \langle n_{\mathbf{k}2}^0 - n_{\mathbf{k}1}^0 \rangle,\end{aligned}\quad (2.6)$$

where the angular brackets denote thermodynamic averaging. Then, neglecting the correlation in distribution of particles over quantum states, Heisenberg equations (2.2)–(2.5), being complemented by dissipative terms, result in the Lorenz system

$$\tau_{\eta} \dot{\eta} = -\eta + A_{\eta} h, \quad (2.7)$$

$$\tau_h \dot{h} = -h + A_h \eta S, \quad (2.8)$$

$$\tau_S \dot{S} = (S^0 - S) - A_S \eta h. \quad (2.9)$$

Here the dependence on the wave vector  $\mathbf{k}$  is omitted in accordance with one-mode approximation, and the constants defined by relationships  $2A_{\eta} \equiv v\tau_{\eta}$ ,  $2A_h \equiv v\tau_h$ ,  $A_S \equiv 2v\tau_S$  are introduced. Eqs. (2.7)–(2.9) contain the following seven constants: pumping parameter  $S^0$ , three relaxation times  $\tau$ , and three coupling constants  $A$ . But due to the above relations the values  $A$  are fixed by an interaction parameter  $v > 0$ , and only five of these constants are independent. Since four of them fix the scales for quantities  $\eta$ ,  $h$ ,  $S$ ,  $t$ , only the parameter of thermal disorder  $S^0$  plays a substantial role and its value determines system behavior [1,8].

To analyze Eqs. (2.7)–(2.9) we introduce the scales  $\eta_m$ ,  $h_m$ ,  $S_c$  defining the variation for the order parameter  $\eta$ , the conjugate field  $h$ , and the control parameter  $S$ :

$$\begin{aligned}\eta_m^{-2} &\equiv A_h A_S = \tau_h \tau_S v^2, & h_m^{-1} &\equiv A_{\eta} / \eta_m = (\tau_{\eta} / 2) (\tau_h \tau_S)^{1/2} v^2, \\ S_c^{-1} &\equiv A_{\eta} A_h = 2^{-2} \tau_{\eta} \tau_h v^2.\end{aligned}\quad (2.10)$$

Then, using magnitudes  $\eta, h, S$  normalized by  $\eta_m, h_m, S_c$ , we reduce Eqs. (2.7)–(2.9) to the form

$$\tau_{\eta} \dot{\eta} = -\eta + h, \quad (2.11)$$

$$\tau_h \dot{h} = -h + \eta S, \quad (2.12)$$

$$\tau_S \dot{S} = (S^0 - S) - \eta h. \quad (2.13)$$

Within the adiabatic approximation  $\tau_h, \tau_S \ll \tau_{\eta}$  the left-hand sides of Eqs. (2.12), (2.13) may be set equal to zero. Thus, we derive the result

$$h = S^0 \eta / (1 + \eta^2), \quad S = S^0 / (1 + \eta^2). \quad (2.14)$$

With the order parameter growth in physical domain  $\eta \in [0, 1]$  the conjugate field increases and the control parameter decreases monotonically. For  $\eta > 1$  the  $h(\eta)$  dependence becomes decreasing, that corresponds to the negative susceptibility  $\chi = d\eta/dh$  and has no physical meaning.

Inserting (2.14) into (2.11) we obtain Landau–Khalatnikov equation

$$\tau_\eta \dot{\eta} = -\frac{\partial V}{\partial \eta}, \quad V \equiv \frac{1}{2} [\eta^2 - S^0 \ln(1 + \eta^2)]. \quad (2.15)$$

Its form is governed by the synergetic potential  $V(\eta)$  with a minimum at point  $\eta_0 = (S^0 - 1)^{1/2}$ . Thus, a stationary value  $\eta_0 \neq 0$  of the order parameter is realized if the thermal disorder  $S^0 > 1$  ( $S^0 > S_c$  in usual units). Respectively, the magnitude  $S_c$  defined by the last equality (2.10) is the critical value of the control parameter. According to Eq. (2.14) at the stationary state one has  $h_0 = (S^0 - 1)^{1/2}$ ,  $S_0 = 1$ . The last equality means that despite supercritical value  $S^0 > 1$  of thermal disorder the system relaxes, so that the stationary value of the control parameter reduces to the critical one  $S_0 = 1$ .

The mentioned relaxation is caused by the negative feedback of order parameter  $\eta$  and conjugate field  $h$  on control parameter  $S$  that is described by the last term in Eq. (2.13). This feedback, displaying the Le Chatelier principle for the self-organizing system, compensates the  $S(\eta)$ ,  $h(\eta)$  thermostat's state parameters increase which occurs, when this feedback is absent. On the other hand, the positive feedback of quantities  $\eta$  and  $S$  on  $h$  in (2.12) is the reason for the self-organization. It is obvious that the stationary state  $\eta_0$ ,  $h_0$ ,  $S_0$  can be reached only at the condition of inverse effect of the order parameter  $\eta$  on the thermostat parameters  $h, S$ . It is worth to note that inverting the signs of nonlinear terms in Eqs. (2.11)–(2.13) causes the minus appearance on the right-hand side of the first equality (2.14), so that the susceptibility  $\chi = d\eta/dh$  becomes negative and the system is unstable.

Thus, the self-organization process takes place only if both the negative feedback of order parameter  $\eta$  and field  $h$  on control parameter  $S$ , as well as the positive feedback of  $\eta$  and  $S$  on  $h$ , exist. According to (2.12), (2.13) such a choice of signs is determined by the fact that the negative feedback causes the decrease of the control parameter  $S$  in the course of time, whereas the positive one ensures the field  $h$  growth. Further, we shall show that the value  $S$  reduces to the entropy and its decrease reflects the non-conservation of the self-organizing system for which the second law of thermodynamics is broken. The crucial role of the increasing character of field  $h$  is stipulated by the fact that the linear equation (2.11) for the order parameter  $\eta$  contains the field  $h$  itself. As a result, the influence of the field  $h$  on the rate of  $\eta(t)$  dependence increase, as well as on the self-organization process is direct, whereas the influence of control parameter is indirect.

The described scheme of self-organization corresponds to the continuous phase transition type of the second-order one. To describe the first-order transition we should set the relaxation time  $\tau_\eta$  of order parameter to be the function of its value  $\eta$  [1,8]. Such scheme of the self-organization is represented in Ref. [13].

### 3. Lagrange formalism

In the previous section, within the adiabatic approximation, we omitted the fluctuations of conjugate field  $h$  and control parameter  $S$  that made possible the reduction of Lorenz system (2.11)–(2.13) to Landau–Khalatnikov equation (2.15). To form the Lagrange formalism one should make the reverse passage supposing a fluctuation source

$\zeta$  appearance in (2.15). If a nonhomogeneity is taken into account, the basic expression reduces to the Langevin equation (see Ref. [14] for example)

$$\dot{\eta}(\mathbf{r}, t) - \nabla^2 \eta(\mathbf{r}, t) = f(\mathbf{r}, t) + \zeta(\mathbf{r}, t), \quad (3.1)$$

where  $\mathbf{r}$  is the coordinate measured in units of the correlation lengths  $\xi$  and  $t$  is the time related to the scale  $\tau_\eta$ , the force  $f = -V'_0(\eta) \equiv -\partial V_0/\partial \eta$  is defined by dependence  $V_0(\eta)$  for a bare potential related to fluctuations intensity  $T$ . The term  $-\nabla^2 \eta$  on the left-hand side of Eq. (3.1) takes into account the spatial nonhomogeneity within the framework of Ginzburg–Landau model. Expression (3.1) is valid for the nonconserved order parameter, otherwise terms  $-\nabla^2 \eta$  and  $f$  obtain the additional operator  $-\nabla^2$  [15,16]. The fluctuational term is normalized by the white-noise conditions

$$\langle \zeta(\mathbf{r}, t) \rangle = 0, \quad \langle \zeta(\mathbf{r}, t) \zeta(\mathbf{r}', t') \rangle = T \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') \quad (3.2)$$

which correspond to averaging over the Gaussian distribution with variance  $T$ .

To construct the Lagrangian corresponding to Langevin equation (3.1), let us use the standard field scheme [12] based on the generating functional

$$Z\{\eta(\mathbf{r}, t)\} = \left\langle \prod_{(\mathbf{r}, t)} \delta(\dot{\eta} - \nabla^2 \eta - f - \zeta) \det \left| \frac{\delta \zeta}{\delta \eta} \right| \right\rangle \quad (3.3)$$

that is the generalization of the partition function. Herein the continual product of  $\delta$ -functions takes into account that condition (3.1) is satisfied for arbitrary values  $\mathbf{r}$ ,  $t$  and the determinant represents the Jacobian from  $\zeta$  to  $\eta$  transition. Moreover, we apply the Fourier transform for  $\delta$ -function, that results in appearance of the field  $\varphi(\mathbf{r}, t)$ , and introduce the Grassmann conjugate fields  $\psi(\mathbf{r}, t)$ ,  $\bar{\psi}(\mathbf{r}, t)$  for integrated representation of the determinant [12]. Then, equality (3.3) accepts the standard form

$$Z\{\eta\} = \int P\{\eta, \varphi, \bar{\psi}, \psi\} D\varphi D\bar{\psi} D\psi \quad P \propto e^{-S},$$

$$S \equiv \int \mathcal{L}(\eta, \varphi, \bar{\psi}, \psi) d\mathbf{r} dt, \quad (3.4)$$

where the Lagrangian

$$\mathcal{L} = \varphi(\dot{\eta} - \nabla^2 \eta) - \bar{\psi}(\dot{\psi} - \nabla^2 \psi) - \varphi^2/2 + \varphi V'_0(\eta) - V''_0(\eta) \bar{\psi} \psi \quad (3.5)$$

is measured in units of the noise intensity  $T$ . The form of the corresponding Euler equations is following:

$$\dot{\eta} - \nabla^2 \eta = -V'_0(\eta) + \varphi, \quad (3.6)$$

$$\dot{\varphi} + \nabla^2 \varphi = V''_0(\eta) \varphi - V'''_0(\eta) \bar{\psi} \psi \quad (3.7)$$

$$\dot{\psi} - \nabla^2 \psi = -V''_0(\eta) \psi \quad (3.8)$$

$$-\dot{\bar{\psi}} - \nabla^2 \bar{\psi} = -V''_0(\eta) \bar{\psi}. \quad (3.9)$$

The first one reduces to Langevin equation (3.1) if the stochastic component  $\zeta$  is compared with the field  $\varphi$ . According to the least-action principle from that the Euler

equations follows, this field corresponds to the maximum of a probability distribution  $P$  in (3.4). Thus,  $\varphi$  represents the amplitude of the most probable fluctuation of the field conjugated to the order parameter  $\eta$  (the average value of this field reduces to the force  $f$ ). Obviously, conditions  $\langle \zeta \rangle = 0$ ,  $\varphi \neq 0$  mean that the bare Gaussian distribution transforms from a unimodal into a bimodal form with its maxima at points  $\pm\varphi$  in the course of self-organization. The signs choice of the gradient terms in Eqs. (3.6)–(3.9) is quite remarkable: albeit the standard combination is realized for components  $\eta$ ,  $\psi$  inherent in the relaxation processes like diffusion, but for the fields  $\varphi$ ,  $\bar{\psi}$  inhomogeneities terms contain the opposite signs that means their autocatalytic increase. As is shown in Appendix A, field equations (3.6)–(3.9) reduce to Lorenz system (2.11)–(2.13).

The developed field scheme is represented by the simplest manner if the components  $\eta$ ,  $\psi$ ,  $\bar{\psi}$  and a generalized force  $f \equiv -\delta V_0\{\eta\}/\delta\eta$  are incorporated into a supersymmetrical field

$$\Phi = \eta + \bar{\psi}\chi + \bar{\chi}\psi - \bar{\chi}\chi f, \tag{3.10}$$

where Grassmann coordinates  $\chi$ ,  $\bar{\chi}$  have the same anti-commutation properties as the  $\psi$ ,  $\bar{\psi}$  fields. To represent Lagrangian (3.5) in a supersymmetrical form, first of all one should replace the bare potential  $V_0(\eta)$  with the renormalized one  $\tilde{V}(\eta)$  in equation of motion (A.7),<sup>1</sup> and get rid of gradient terms, making use the variational derivatives  $\tilde{V}'\{\eta\} \equiv \delta\tilde{V}\{\eta\}/\delta\eta = \partial\tilde{V}(\eta)/\partial\eta - \nabla^2\eta$ ,  $\tilde{V}\{\eta\} \equiv \int \tilde{V}(\eta) \mathbf{d}\mathbf{r}$ . Then, expressing fluctuation amplitude  $\varphi$  in terms of generalized force  $f$  according to equality (A.9), we obtain the following form of Lagrangian (3.5):

$$\mathcal{L} = (\dot{\eta}^2/2 - \bar{\psi}\dot{\psi} - f^2/2) + (-\tilde{V}'\{\eta\}f - \bar{\psi}\tilde{V}''\{\eta\}\psi) + \tilde{V}'\{\eta\}\dot{\eta}. \tag{3.11}$$

The last term can be omitted as the total time derivative of  $\tilde{V}\{\eta\}$ , and within superfield representation (3.10), Lagrangian (3.11) assumes the canonical form

$$\mathcal{L} = \int A(\Phi) \mathbf{d}\bar{\chi} \mathbf{d}\chi, \quad A \equiv -(1/2)\Phi\bar{D}D\Phi + \tilde{V}(\Phi). \tag{3.12}$$

Here the kinetic superenergy of the kernel  $A$  corresponds to the first parenthesis of expression (3.11) and the potential superenergy  $\tilde{V}(\Phi)$ —to the second one. Within the framework of the  $\Phi^4$ -model, equalities (A.3), (A.7), and (A.8) give

$$\tilde{V} = \frac{\sigma}{2} + \frac{1-\sigma}{2} \Phi^2 + w \frac{1+6\sigma}{12} \Phi^4, \tag{3.13}$$

where the anharmonicity parameter  $w > 0$  appears because, in contrast with the order parameter  $\eta$  (see (2.10)), superfield (3.10) cannot be scaled by the only magnitude  $\eta_m$ . The supersymmetry group generators have the form

$$D = \frac{\partial}{\partial\bar{\chi}} - \chi \frac{\partial}{\partial t}, \quad \bar{D} = \frac{\partial}{\partial\chi} - \bar{\chi} \frac{\partial}{\partial t}. \tag{3.14}$$

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<sup>1</sup> This renormalization is caused by self-consistency of the superfield components (3.10).

The superequations of motion following from extremum condition of the superaction  $S\{\Phi(z)\} = \int A(\Phi(z)) dz$ ,  $z \equiv \{\mathbf{r}, t, \bar{\chi}, \chi\}$  reads

$$-(1/2)[\bar{D}, D]\Phi + \tilde{V}'\{\Phi\} = 0, \quad (1/2)[\bar{D}, D]\Phi \equiv f + \dot{\bar{\psi}}\chi + \bar{\chi}\dot{\psi} + \bar{\chi}\chi\ddot{\eta}. \quad (3.15)$$

Projecting Eq. (3.15) onto basis vectors 1,  $\bar{\chi}$ ,  $\chi$ ,  $\bar{\chi}\chi$  of the superspace, we arrive at equations

$$\ddot{\eta} = -\tilde{V}''\{\eta\}f - \tilde{V}'''\{\eta\}\bar{\psi}\psi \quad (3.16)$$

$$f = -\tilde{V}'\{\eta\}, \quad (3.17)$$

$$\dot{\psi} = -\tilde{V}''\{\eta\}\psi \quad (3.18)$$

$$\dot{\bar{\psi}} = \tilde{V}''\{\eta\}\bar{\psi}. \quad (3.19)$$

The last of this set can be obtained from (3.8), (3.9) at replacing  $V_0$  by  $\tilde{V}$ , whereas relation (3.17) gives the definition of the generalized force  $f$ . Eq. (3.16) is obtained by time differentiation of Eq. (3.6) and substituting the derivatives  $\dot{\eta}$ ,  $\dot{\phi}$  from (A.9), (3.7) into final expression. Thus, systems (3.6)–(3.9) and (3.16)–(3.19) turn out to be equivalent with accuracy to the bare potential  $V_0(\eta)$  renormalization (A.7). However, though the equations of the first set are symmetrical with respect to the time derivative order, in (3.16)–(3.19) this symmetry is broken at the transition from the fluctuation amplitude  $\phi$  to the force  $f$ .

It is easy to show [17] that, if the gauge condition  $D\Phi=0$  is satisfied, the Grassmann fields  $\bar{\psi}$ ,  $\psi$  fall out of consideration and the single combination  $\bar{\chi}\chi$  is used instead of conjugate coordinates  $\bar{\chi}$ ,  $\chi$ . However, as is shown in Appendix A, the product of Grassmann fields represents the entropy and, consequently, their behavior is essential at the description of the self-organization effects.

#### 4. Correlation technique

Let us introduce the supersymmetrical correlator

$$C(z, z') \equiv \langle \Phi(z)\Phi(z') \rangle, \quad z \equiv \{\mathbf{r}, t, \bar{\chi}, \chi\}. \quad (4.1)$$

According to the equation of motion (3.15) its bare component  $C^{(0)}(z, z')$  meeting the potential  $\tilde{V}^{(0)} = (1 - \sigma)\Phi^2/2$  satisfies the equality

$$L_{\mathbf{k}\omega}(\chi)C_{\mathbf{k}\omega}^{(0)}(\chi, \chi') = \delta(\chi, \chi'), \quad L \equiv (1 - \sigma) - (1/2)[\bar{D}, D], \quad (4.2)$$

where the transition to the time–spatial Fourier transforms is made and the supersymmetrical  $\delta$ -function  $\delta(\chi, \chi') = (\bar{\chi} - \bar{\chi}')(\chi - \chi')$  is introduced. Taking into consideration definition (3.14) and expression  $(\frac{1}{4})[\bar{D}, D]^2 = -\omega^2$ , we obtain

$$C^{(0)}(\chi, \chi') = \frac{1 + (\bar{\chi}\chi + \bar{\chi}'\chi') - [(1 - \sigma) + i\omega]\bar{\chi}\chi' - [(1 - \sigma) - i\omega]\bar{\chi}'\chi + \omega^2\bar{\chi}\chi'\bar{\chi}'\chi'}{(1 - \sigma)^2 + \omega^2}, \quad (4.3)$$

where index  $\omega$  is suppressed and spatial dispersion is not taken into account for brevity. Obviously, Eq. (4.3) represents the expansion in basis components

$$\begin{aligned} \mathbf{B}_0 &= \bar{\chi}\chi, & \mathbf{B}_1 &= \bar{\chi}'\chi', \\ \mathbf{T} &= 1, & \mathbf{T}_1 &= \bar{\chi}\chi\bar{\chi}'\chi', \\ \mathbf{F}_0 &= -\bar{\chi}'\chi, & \mathbf{F}_1 &= -\bar{\chi}\chi'. \end{aligned} \tag{4.4}$$

We define the functional product of such components  $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$  by equality

$$X(\chi, \chi') = \int Y(\chi, \chi'')Z(\chi'', \chi') d\chi'' d\chi'. \tag{4.5}$$

Then, components (4.4) satisfy the following multiplication rules:  $\mathbf{B}_0^2 = \mathbf{B}_0, \mathbf{B}_1^2 = \mathbf{B}_1, \mathbf{F}_0^2 = \mathbf{F}_0, \mathbf{F}_1^2 = \mathbf{F}_1, \mathbf{B}_0\mathbf{T}_1 = \mathbf{T}_1, \mathbf{B}_1\mathbf{T} = \mathbf{T}, \mathbf{TB}_0 = \mathbf{T}, \mathbf{TT}_1 = \mathbf{B}_1, \mathbf{T}_1\mathbf{B}_1 = \mathbf{T}_1, \mathbf{T}_1\mathbf{T} = \mathbf{B}_0$ , and the other products are equal to zero. It is seen, that  $\mathbf{B}_{0,1}, \mathbf{T}, \mathbf{T}_1, \mathbf{F}_{0,1}$  form the closed basis, and it is convenient to expand supercorrelator (4.1) in these components:

$$\mathbf{C} = g_+\mathbf{B}_0 + g_-\mathbf{B}_1 + S\mathbf{T} + s\mathbf{T}_1 + G_+\mathbf{F}_0 + G_-\mathbf{F}_1. \tag{4.6}$$

Insertion (3.10) into (4.1) leads to the coefficients

$$\begin{aligned} g_+ &= -\langle f\eta \rangle, & g_- &= -\langle \eta f \rangle, \\ S &= \langle \eta^2 \rangle, & s &= \langle f^2 \rangle, \\ G_+ &= \langle \bar{\psi}\psi^* \rangle, & G_- &= \langle \bar{\psi}^*\psi \rangle. \end{aligned} \tag{4.7}$$

Thus, the magnitudes  $g_{\pm}$  reduce to advanced and retarded response functions of the order parameter  $\eta$  on the action of the field  $f$ ;  $S$  and  $s$  are the autocorrelators of order parameter  $\eta$  and field  $f$ , and the functions  $G_{\pm}$  determine the correlation of Grassmann conjugated fields  $\bar{\psi}, \psi$ . The relations

$$\begin{aligned} g_{\pm}^{(0)} &= S^{(0)} = [(1 - \sigma)^2 + \omega^2]^{-1}, & s^{(0)} &= \omega^2[(1 - \sigma)^2 + \omega^2]^{-1}, \\ G_+^{(0)} &= [(1 - \sigma) + i\omega]^{-1}, & G_-^{(0)} &= [(1 - \sigma) - i\omega]^{-1} \end{aligned} \tag{4.8}$$

are valid for bare supercorrelator (4.3). Hence, accounting for (A.9) we find the relation  $\langle \eta\varphi \rangle_0 = G_-^{(0)}$  that is a special case of the Ward identity [12]. It means that the correlator of the Grassmann fields reduces to response function of the order parameter  $\eta$  to the amplitude of fluctuation  $\varphi$ .

For further consideration it is convenient to pass to the supersymmetrical field

$$\Phi = \eta + \bar{\psi}\chi + \bar{\chi}\psi + \bar{\chi}\chi\varphi, \tag{4.9}$$

differing from initial expression (3.10) by replacing of the force  $-f$  by fluctuations amplitude  $\varphi$ . Then, the component  $\mathbf{T}_1$  falls out of consideration and Fermi components are incorporated with Bose ones in pairs:  $\mathbf{A} \equiv \mathbf{B}_0 + \mathbf{F}_0, \mathbf{B} \equiv \mathbf{B}_1 + \mathbf{F}_1$ . As a result, expansion (4.6) takes the compact form

$$\mathbf{C} = G_+\mathbf{A} + G_-\mathbf{B} + S\mathbf{T}, \tag{4.10}$$

where the basis components  $\mathbf{A} \equiv \mathbf{B}_0 + \mathbf{F}_0$ ,  $\mathbf{B} \equiv \mathbf{B}_1 + \mathbf{F}_1$  satisfy the multiplication rules:  $\mathbf{A}^2 = \mathbf{A}$ ,  $\mathbf{B}^2 = \mathbf{B}$ ,  $\mathbf{B}\mathbf{T} = \mathbf{T}$ ,  $\mathbf{T}\mathbf{A} = \mathbf{T}$  (other products are equal to zero). Expansion (4.10) allows us to treat supercorrelator (4.1) as a vector of the direct product of superspaces.

As a result, the behavior of self-organizing system is described by the Lagrangian

$$\mathcal{L} = (\varphi\dot{\eta} - \bar{\psi}\dot{\psi} - \varphi^2/2) + (\tilde{V}'\{\eta\}\varphi - \bar{\psi}\tilde{V}''\{\eta\}\psi). \tag{4.11}$$

Further, we introduce the quenched disorder

$$p^2 = \frac{\overline{(f(\mathbf{r}) - \bar{f})^2} - (\Delta\varphi)^2}{(\Delta\varphi)^2} \tag{4.12}$$

whose magnitude characterizes the field  $f(\mathbf{r})$  random scattering over volume (bar in (4.12) stands for averaging over coordinate  $\mathbf{r}$ ,  $(\Delta\varphi)^2 \equiv |\varphi_{\omega=0}|^2$  is the mean-square fluctuation). If the quenched disorder is included, then component squared in  $\varphi$  fluctuation of action meeting Lagrangian (4.11), takes the form

$$-\frac{1}{2} \int |\varphi_\omega|^2 \frac{d\omega}{2\pi} - \frac{p^2}{2} \int \delta(\omega) |\varphi_\omega|^2 d\omega. \tag{4.13}$$

Here, we have neglected the integration over  $\mathbf{r}$  and passed to Fourier transform over frequency  $\omega$ . At the equilibrium disorder the field  $f(\mathbf{r})$  scatter reduces to the mean-square fluctuation  $(\Delta\varphi)^2$ , so that  $p = 0$  and expression (4.13) has the canonical form  $-(\frac{1}{2}) \int \varphi^2 dt$ . In the case of quenching, we have  $p > 0$  and the second term in (4.13) leads to renormalization of the bare supercorrelator (4.3) whose component  $S^{(0)}$  gets the multiplier  $1 + 2\pi p^2 \delta(\omega)$  in Eq. (4.8). Respectively, we find the operator  $\mathbf{L}$  in the motion equation (4.2)

$$\mathbf{L} = L_+ \mathbf{A} + L_- \mathbf{B} + L\mathbf{T}; \quad L_\pm = (1 - \sigma) \pm i\omega, \quad L = -[1 + 2\pi p^2 \delta(\omega)]. \tag{4.14}$$

To obtain equation that defines supercorrelator (4.1) one should multiply (3.15) by  $\Phi(z')$  and average the result over distribution  $P\{\Phi\}$  in (3.4). As a result, we get the Dyson superequation

$$\mathbf{C}^{-1} = \mathbf{L} - \mathbf{\Sigma}, \tag{4.15}$$

where within the framework of  $\Phi^4$ -model (3.13) the self-energy superoperator  $\mathbf{\Sigma}$  is defined by equality

$$\Sigma(z, z') = (2/3)w^2(1 + 6\sigma)^2(C(z, z'))^3, \quad z \equiv \{\mathbf{r}, t, \bar{\chi}, \chi\}. \tag{4.16}$$

Herein  $w > 0$  is the anharmonicity parameter related to the noise intensity  $T$  and the condition  $\int C(z, z) dz = 0$  is taken into account that follows from Eqs. (4.4) and (4.10).

If the anharmonicity is omitted,  $\mathbf{\Sigma} = 0$ , according to Eq. (4.15) we obtain components (4.8) diverging at the point  $\sigma = 1$  of transition into self-organization state. Thus, the supersymmetrical field approach allows us to reproduce the basic result following from Lorenz system (2.11)–(2.13) by means of the linear approximation only. In a general case, the self-energy superfunction should be expanded in the similar manner

to supercorrelator (4.10):

$$\Sigma = \Sigma_+ \mathbf{A} + \Sigma_- \mathbf{B} + \Sigma \mathbf{T}. \tag{4.17}$$

Then, accounting for Eq. (4.14) superequation (4.15) reduces to components

$$G_{\pm}^{-1} = [(1 - \sigma) \pm i\omega] - \Sigma_{\pm}, \tag{4.18}$$

$$S = [1 + 2\pi p^2 \delta(\omega) + \Sigma] G_+ G_-. \tag{4.19}$$

The explicit form of expansion coefficients (4.17) is given by expression (4.16). In accordance with Ref. [18] the multiplication of supercorrelators should be understood in usual, but not in functional manner:  $(T(\chi, \chi'))^2 = T(\chi, \chi')$ ,  $A(\chi, \chi')T(\chi, \chi') = T(\chi, \chi')A(\chi, \chi') = A(\chi, \chi')$ ,  $B(\chi, \chi')T(\chi, \chi') = T(\chi, \chi')B(\chi, \chi') = B(\chi, \chi')$ , and the other products are equal to zero. As a result, for the spatially homogeneous case we obtain from (4.16)

$$\Sigma_{\pm}(t) = 2w^2(1 + 6\sigma)^2 S^2(t) G_{\pm}(t), \tag{4.20}$$

$$\Sigma(t) = (2/3)w^2(1 + 6\sigma)^2 S^3(t). \tag{4.21}$$

At insertion of these expressions into Dyson equation (4.18), (4.19) their frequency representation, which contains convolutions, will be necessary. To avoid such a difficulty let us use the fluctuation–dissipation theorem [12,18]

$$S(\omega') = (2/\omega') \text{Im } G_{\pm}(\omega'), \quad \Sigma(\omega') = (2/\omega') \text{Im } \Sigma_{\pm}(\omega'), \tag{4.22}$$

where  $\omega'$  is the real frequency. Using the spectral representation of complex frequency  $\omega$  function, we find after integration of equalities (4.22)

$$S(t \rightarrow 0) = G_{\pm}(\omega \rightarrow 0), \quad \Sigma(t \rightarrow 0) = \Sigma_{\pm}(\omega \rightarrow 0). \tag{4.23}$$

Since the response function  $G_{\pm}(\omega \rightarrow 0)$  gives the susceptibility  $\chi$  in the hydrodynamical limit  $\omega \rightarrow 0$ , we have

$$S(t \rightarrow 0) = \chi \equiv G_{\pm}(\omega \rightarrow 0), \tag{4.24}$$

$$\Sigma_{\pm}(\omega \rightarrow 0) = (2/3)w^2(1 + 6\sigma)^2 \chi^3, \tag{4.25}$$

where in (4.25) expression (4.21) is used. In contrast to Ref. [19] here the self-energy components  $\Sigma_{\pm}$  contain only the second order of the anharmonicity  $w$ .

Eqs. (4.18)–(4.21), (4.24) and (4.25) describe the behavior of self-organizing system completely. In particular, they represent not only the ordering phenomena but the effects of ergodicity breaking and memory appearance as well. These effects manifest themselves in elongation of correlators<sup>2</sup>

$$G_-(\omega) = -\Delta + G_{-0}(\omega), \quad S(t) = q + S_0(t) \tag{4.26}$$

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<sup>2</sup> Let us point out the reverse sign of the irreversible response  $\Delta$  in comparison with the definition given for thermodynamical systems, where the ordering corresponds to low values of the noise intensity (temperature).

owing to the Edwards–Anderson memory parameter  $q = \langle \eta(\infty)\eta(0) \rangle$  and irreversible response  $\Delta = \chi - \chi_0$ , which is equal to the difference of the microscopic (isothermal) susceptibility  $\chi \equiv G_{-0}(\omega \rightarrow 0)$  and the macroscopic (adiabatic) Kubo one  $\chi_0 \equiv G_{-}(\omega = 0)$ .<sup>3</sup>

Let us insert the elongated correlators (4.26) into expressions (4.20), (4.21). Then, the renormalized components of self-energy superfunction read

$$\begin{aligned}\Sigma_{\pm}(t) &= 2w^2(1 + 6\sigma)^2 q^2(-\Delta + G_{\pm 0}(t)) + \Sigma_{\pm 0}(t), \\ \Sigma_{\pm 0}(t) &\equiv 2w^2(1 + 6\sigma)^2 S_0(t)G_{\pm 0}(t)(2q + S_0(t)),\end{aligned}\quad (4.27)$$

$$\begin{aligned}\Sigma(t) &= (2/3)w^2(1 + 6\sigma)^2 q^2(q + 3S_0(t)) + \Sigma_0(t), \\ \Sigma_0(t) &\equiv (2/3)w^2(1 + 6\sigma)^2 S_0^2(t)(3q + S_0(t)).\end{aligned}\quad (4.28)$$

Here the nonlinear terms with respect to the correlators  $G_{\pm 0}$  and  $S_0$  are gathered in terms  $\Sigma_{\pm 0}$  and  $\Sigma_0$ , in the second equality of (4.27) relationship (4.24) is taken into consideration, and in Eq. (4.28) the terms containing  $S_0\Delta \simeq 0$  are omitted. Let us note that if the memory is absent, the first terms of  $\Sigma_{\pm}(t)$ ,  $\Sigma(t)$  vanish. Inserting the Fourier transforms of Eqs. (4.26), (4.28) into Dyson equation (4.19), we obtain the following relations within  $\omega$ -representation:

$$q_0[1 - (2/3)w^2(1 + 6\sigma)^2 \chi_0^2 q_0^2] = p^2 \chi_0^2, \quad (4.29)$$

$$S_0 = \frac{(1 + \Sigma_0)G_+ G_-}{1 - 2w^2(1 + 6\sigma)^2 q^2 G_+ G_-}. \quad (4.30)$$

The first equality is caused by the memory appearance that leads to the  $\delta$ -like term at  $\omega = 0$  and the second one meets the frequencies  $\omega \neq 0$ . When  $\omega \rightarrow 0$  the characteristic combination  $G_+ G_- = \chi_0^2$  and the pole of the structure factor (4.30)

$$2w^2(1 + 6\sigma^c)^2 q_0^2 = \chi_0^{-2} \quad (4.31)$$

determines the point  $\sigma^c$  of ergodicity breaking, where  $\chi_0 = \chi$ ,  $q_0 = q$ .

Now, let us insert the Fourier transform of expression (4.27) into Eq. (4.18). Then, using for  $\Sigma_{\pm 0}(\omega)$  equality (4.25) we obtain equation for the retarded Green function  $G_-$  in the hydrodynamical limit  $\omega \rightarrow 0$

$$G_-^{-1} + 2w^2(1 + 6\sigma)^2 q^2 G_- = [(1 - \sigma) - i\omega] - 2w^2(1 + 6\sigma)^2 \chi^2 (q + \chi/3), \quad (4.32)$$

where relation (4.24) is taken into account. As a consequence, the equation for the microscopical susceptibility  $\chi \equiv G_{-}(\omega \rightarrow 0)$  reads

$$1 - (1 - \sigma)\chi + (2/3)w^2(1 + 6\sigma)^2 \chi[(\chi + q)^3 - q^3] = 0. \quad (4.33)$$

<sup>3</sup> The function  $G_{-}(\omega)$  can be used only to define susceptibilities  $\chi_0$ ,  $\chi$  if one takes into account that the magnitudes  $\chi_0 \equiv G_{-}(\omega = 0)$  and  $\chi \equiv G_{-}(\omega \rightarrow 0)$  correspond to equilibrium and nonequilibrium values, respectively. In doing so we should subscript to all correlators in Eqs. (4.23)–(4.25) the index 0 and set  $\omega = 0$ .

The behavior of self-organizing system with quenched disorder is determined by Eqs. (4.29), (4.31) and (4.33) completely. At that one should distinguish the macroscopical magnitudes  $q_0$ ,  $\chi_0$  and microscopical ones  $q$ ,  $\chi$  (the former correspond to frequency  $\omega = 0$  and the latter to limit  $\omega \rightarrow 0$ ). The distinctive feature of this hierarchy is that macroscopical values  $q_0$ ,  $\chi_0$  depend exclusively on quenched-disorder intensity  $p$  while the microscopical ones  $q$ ,  $\chi$ —on thermal disorder  $\sigma$ . Therefore, to determine values  $q_0$ ,  $\chi_0$  we should consider the magnitude  $\sigma = \sigma^c(p)$  on the line of ergodicity breaking, and the quenched-disorder intensity is supposed to take the critical value  $p^c(\sigma)$  for the  $q$ ,  $\chi$  defining. As a result, Eqs. (4.29), (4.31) determine the macroscopical values  $q_0$ ,  $\chi_0$  and Eq. (4.33) defines microscopical ones  $q$ ,  $\chi$ . At that the external addition  $f_{ext}$  to the self-consistent field  $f$  (hereinafter  $f_{ext} = 0$ ), the dispersion  $p$  of field  $f$  fixing the intensity of quenched disorder (4.12), and the parameter of the thermal disorder  $\sigma \equiv S^0/S_c$  act as the state parameters.

Combining equalities (4.29), (4.31), we get the expression for the macroscopical memory parameter

$$q_0 = (3/4)^{1/3}(1 + 6\sigma^c)^{-2/3}(p/w)^{2/3} \quad (4.34)$$

which increases monotonically with  $p$  growth. Insertion of Eq. (4.34) into Eq. (4.31) gives the macroscopical susceptibility

$$\chi_0 = 2^{1/6}3^{-1/3}w^{-1/3}(1 + 6\sigma^c)^{-1/3}p^{-2/3} \quad (4.35)$$

that decreases with  $p$  growth.

Fixing the memory parameter in Eq. (4.33) by expression (cf. with (4.34))

$$q = (3/4)^{1/3}(1 + 6\sigma)^{-2/3}(p^c/w)^{2/3}, \quad (4.36)$$

we find dependencies for the microscopical values  $q$ ,  $\chi$  on  $\sigma$  (see Figs. 1,2). The functions  $q(\sigma)$  and  $\chi(\sigma)$  are two-valued, when the thermal disorder is small. The values  $q$  and  $\chi$ , shown in Figs. 1, and 2 by the dotted lines, correspond to a nonstable state. There is an abruptness with further growth of  $\sigma$  at the ordering point  $\sigma_c$ , where  $d\chi/d\sigma = \infty$ . The growth of anharmonicity parameter  $w$  leads to the decrease of both susceptibilities  $\chi$  and  $\chi_0$ .

The ergodicity-breaking point is fixed by equation

$$\begin{aligned} 3A + (A + 1)^3 &= 1 + 2(1 - \sigma^c)p^{-2}, \\ A &\equiv 2^{5/6}3^{-2/3}(1 + 6\sigma^c)^{1/3}w^{1/3}p^{-4/3}, \end{aligned} \quad (4.37)$$

obtained from condition  $\chi = \chi_0$  according to equalities (4.33)–(4.36). As is seen from Fig. 2 the quantity  $\sigma_c$  that meets the ordering point is defined by the maximum value  $\sigma^c$  corresponding to the ergodicity breaking. The dependencies  $\sigma^c(p)$ ,  $\sigma_c(p)$  which represent the phase diagram of self-organizing system are shown in Fig. 3. It is seen, that the boundary  $\sigma_c$  of the disordered domain that corresponds to small values of  $\sigma$  does not depend on  $p$ , and width  $\sigma^c$  of the ergodicity region decreases with growth of quenched disorder. Comparing Fig. 3a with b shows that the behavior of self-organizing system is rather sensitive to the value of anharmonicity parameter  $w$  whose growth

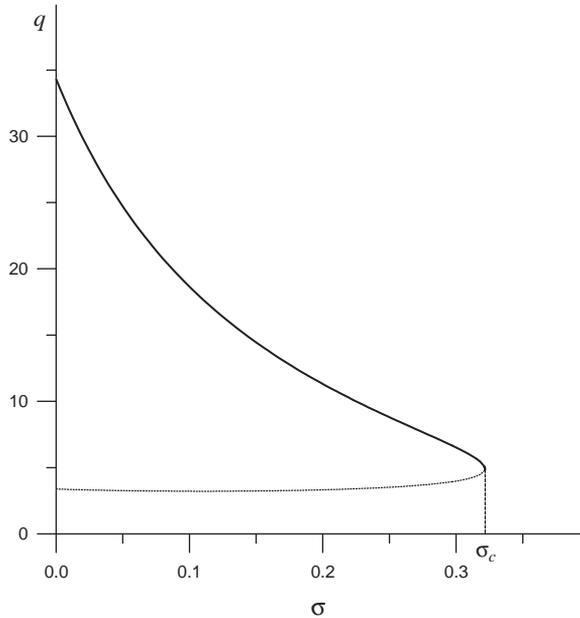


Fig. 1. Dependence of microscopic memory parameter  $q$  on thermal disorder intensity  $\sigma$  at anharmonicity parameter  $w = 0.01$  (dotted line meets the unstable state,  $\sigma_c$  is the ordering point).

decreases the domain of disordered and ergodic phases. The corresponding dependence  $\sigma_c(w)$  for the thermal disorder parameter meeting the ordering is shown in Fig. 4. The  $\sigma_c$  takes its maximum value  $\sigma_c = 1$ , when  $w = 0$ , and the growth of  $w$  causes the  $\sigma_c$  monotonous decrease until  $\sigma_c = 0$  at  $w_c = 0.064$ . The maximum value  $p_m \equiv p^c(\sigma = 0)$  of the quenched disorder at the ergodicity breaking line changes with the value  $w$  according to the dependence shown in Fig. 5. When  $w$  is small, the quantity  $p_m \propto w^{-1/2}$  increases infinitely and the ergodicity region disappears with the growth of anharmonicity parameter above the critical value  $w_c$ .

The nonergodicity parameter is defined by solution of Eqs. (4.33)–(4.36). At the constant value of quenched disorder  $p$  (Fig. 6a) the three regimes are possible. At small  $p$  the macroscopic susceptibility  $\chi_0$  exceeds microscopic quantity  $\chi(\sigma)$  for arbitrary values of  $\sigma$  (see Fig. 2), and the system is always in the ergodic state. When parameter  $p$  reaches the values which exceed threshold  $p_c$  meeting the condition  $\chi_0(p_c) = \chi(\sigma^c)$ , the nonergodicity parameter  $\Delta = \chi - \chi_0$  takes the nonzero values within the region  $\sigma > \sigma^c$ . The microscopic susceptibility  $\chi$  exceeds the macroscopic one  $\chi_0$  for arbitrary values of  $\sigma$  starting from value  $p_m$  that meets the condition  $\chi_0(p_m) = \chi(\sigma = 0)$  and the system is always nonergodic. At the fixed value of thermal disorder  $\sigma$  (Fig. 6b) the dependence  $\Delta(p)$  is defined by infinite increase of macroscopic susceptibility  $\chi_0 \sim p^{-2/3}$  within the domain of weak quenched disorder  $p \rightarrow 0$ . As a result, nonergodicity parameter takes nonzero values starting from critical value  $p^c$  and increases with further growth of quenched disorder monotonically.

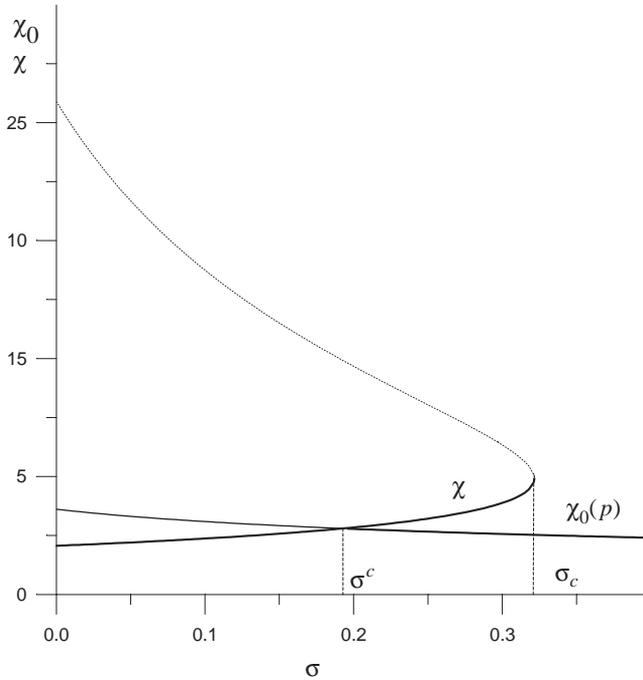


Fig. 2. Dependence of microscopic  $\chi$  and macroscopic  $\chi_0$  susceptibilities on thermal disorder intensity  $\sigma$  at unharmonicity parameter  $w = 0.01$  (dotted and thin solid lines meet the unstable state; thick line—stable state;  $\sigma^c$  is the point of ergodicity breaking,  $\sigma_c$  is the ordering point).

Analytic representation of  $\Delta(\sigma, p)$  dependence is possible only near the line of ergodicity breaking  $\sigma^c(p)$ . Setting in Eq. (4.33)  $0 < \sigma - \sigma^c \ll \sigma^c$ ,  $\chi = \chi_0 + \Delta$ ,  $\Delta \ll \chi$ , accounting for Eq. (4.35) within the first order over small values  $(\sigma - \sigma^c)/\sigma^c$ ,  $\Delta/\chi$ , we find

$$\Delta = B(p)(\sigma^c - \sigma),$$

$$B \equiv \left[ 2 + \frac{2\chi_0}{q_0} + \left( \frac{\chi_0}{q_0} \right)^2 \right] \left[ (1 - \sigma^c) - \frac{2}{\chi_0} - \frac{3}{q_0} - \frac{4\chi_0}{3q_0^2} \right]^{-2}. \tag{4.38}$$

Coefficient  $B(p)$  diverges at the point corresponding to divergence of derivative  $\partial\chi/\partial\sigma$  at  $\sigma = \sigma_c$  (see Fig. 2). At fixed value of thermal disorder it is possible to make the expansion in series over difference  $q_0^c - q_0$  in equality (4.35). Then, taking into account dependence (4.34), we obtain in linear approximation

$$\Delta = q_0^c [(p/p^c)^{2/3} - 1], \tag{4.39}$$

where the critical value of memory parameter  $q_0^c$  meets the ergodicity breaking point  $p^c$ .

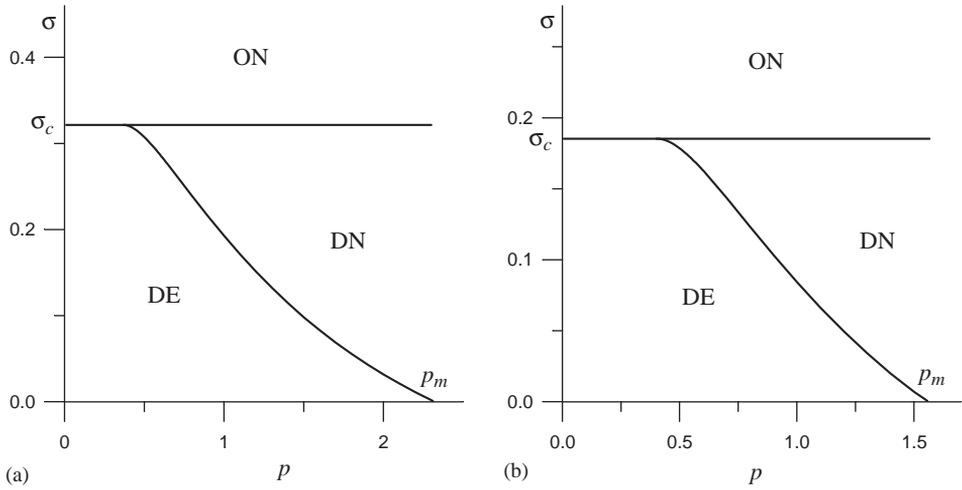


Fig. 3. Phase diagram: (a) at  $w = 0.01$ , (b) at  $w = 0.02$  ( $O$  is the ordered phase,  $D$ —disordered one,  $E$ —ergodic one,  $N$ —nonergodic one).

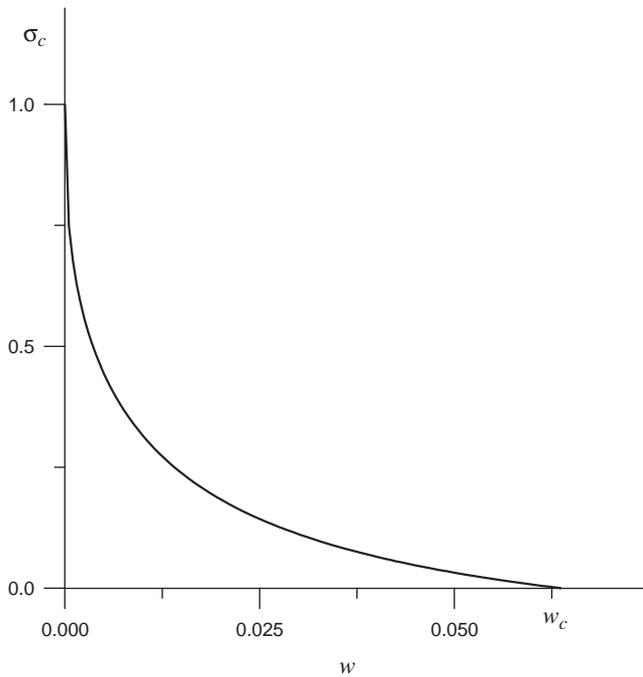


Fig. 4. Dependence of thermal disorder intensity  $\sigma_c$  corresponding to the ordering point on anharmonicity parameter  $w$ .

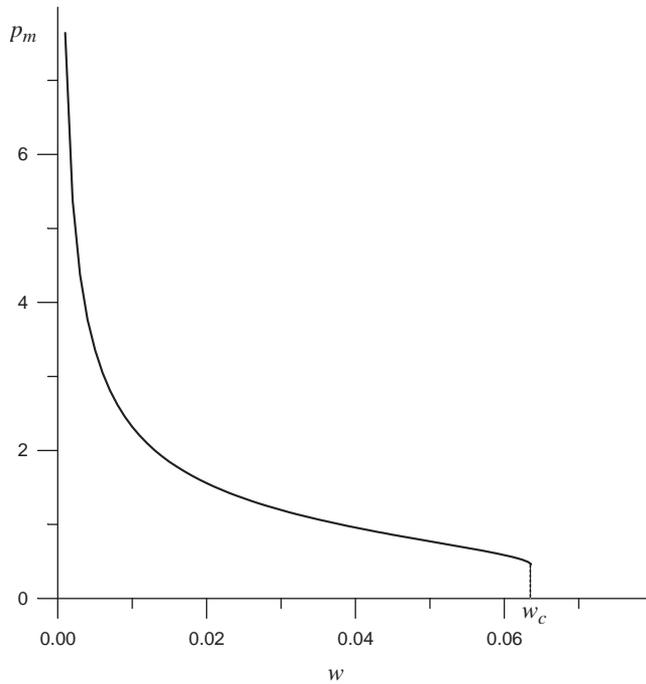


Fig. 5. Dependence of maximal intensity  $p_m$  of quenched disorder at the boundary of ergodic region on anharmonicity parameter  $w$ .

## 5. Conclusion

As is shown above, at fixed values of thermal and quenched-disorder intensities  $\sigma$ ,  $p$  the behavior of self-organizing system is represented by the fields of the order parameter  $\eta$  and the fluctuation amplitude  $\varphi$  of conjugate field—on the one hand, and by the couple of Grassmann conjugated fields  $\psi$ ,  $\bar{\psi}$ —on the other hand. The Boson and Fermion gases, interacting between themselves by means of potential  $v$  in Hamiltonian (2.1), meet these fields within the framework of microscopic representation. As is known with transition to self-consistent scheme a couple of three-tail vertexes, each meeting the  $v$ , forms a four-tail one corresponding to the anharmonicity parameter  $w$  in bare superpotential (3.13). Hence, it is possible to suppose that the relation  $w = v^2$  is fulfilled. Then, expression (A.6) for the critical value of thermal disorder  $S_c$ , in which we should take into consideration the factor  $w^{-1}$  (see after (3.13)), takes the form

$$S_c = \frac{\varepsilon}{2} \frac{\tau_S}{\tau_\eta} \left( \frac{T}{v} \right)^2, \quad (5.1)$$

where we pass to dimension quantities,  $T$  is the noise intensity corresponding to temperature for thermodynamic systems. The obtained  $S_c(v)$  dependence has the same character as in the last formula (2.10). Identifying them, we find the expression for the

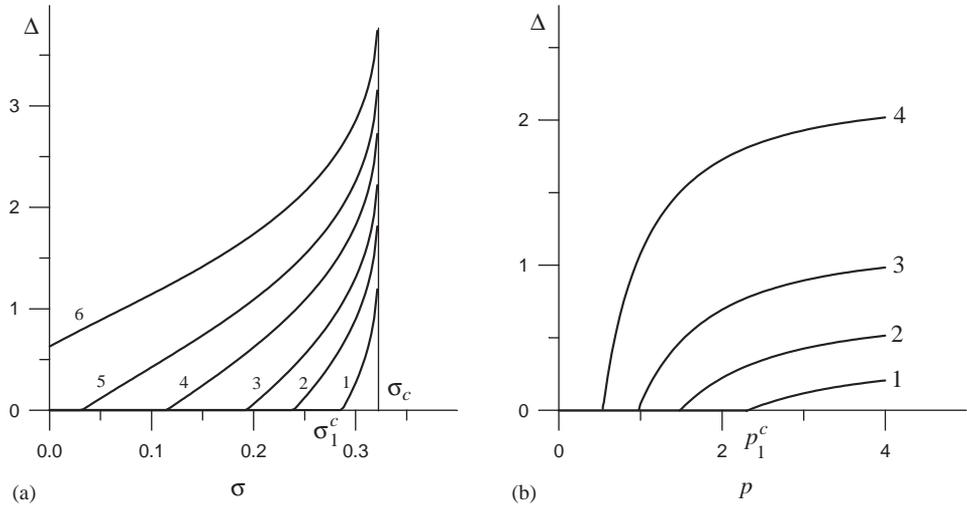


Fig. 6. Dependences of nonergodicity parameter  $\Delta$ : (a) on thermal disorder intensity  $\sigma$  (curves 1–6 meet the  $p = 0.6, 0.8, 1.0, 1.4, 2.0,$  and  $4.0,$  respectively), (b) on quenched disorder intensity  $p$  (curves 1–4 meet the  $\sigma = 0.0, 0.1, 0.2,$  and  $0.3,$  respectively). The anharmonicity parameter  $w = 0.01$ .

detuning parameter  $\varepsilon$  that provides the time irreversibility in entropy balance equation (A.2):

$$\varepsilon = c \left( \frac{\tau_0}{\tau_\eta} \right)^2, \quad c \equiv \frac{2}{\pi^2 \hbar^2} \frac{\tau_\eta^2}{\tau_h \tau_S}, \quad \tau_0 \equiv \frac{2\pi \hbar}{T}. \tag{5.2}$$

Here the dimension units are used,  $\hbar$  is the Planck constant. The coefficient  $c$  is defined by ratio of characteristic times  $\tau_h \tau_S / \tau_\eta^2$ . It is possible to suppose that the factor  $c$  is a constant, and detuning parameter (5.2) is determined by squared ratio of quantum fluctuation time  $\tau_0$  to macroscopic time  $\tau_\eta$  of the order parameter variation. Obviously, the condition  $\varepsilon \ll 1$  is always satisfied.

The peculiarity of self-consistent supersymmetric scheme presented in Sections 3–4 is that it allows us to escape the adiabatic approximation. By the latter conditions,  $v \ll v_c$ , the critical value of thermal disorder is defined by equality (5.1). With the growth of interaction parameter  $v$  the complete suppression of disordered and ergodic states takes place, when value  $v$  exceeds the critical one  $v_c = 0.252$ . In this end, the effects of mutually coordinated influence of superfield components manifest themselves substantially, and the dependencies type of represented in Figs. 1–6 would be used.

We have shown for the simplest example of sand flow on the inclined surface [6] that the above scheme represents the avalanche formation in self-organized criticality (SOC) phenomena [20–30]. The theory of SOC explains spontaneous (avalanche-type) dynamics, unlike the typical phase transitions and self-organization processes that occur only when a control parameter is driven to a critical value. The earthquakes and snow avalanches are the most bright displays of such behavior. Besides, the mode of SOC manifests itself at natural selection in the biological systems, forest fires, percolation

of liquid in porous mediums, etc. The substantial feature of SOC consists in the irregular character of process corresponding to the intermittency mode: during the basic time dissipation impedes the spontaneous energy accumulation and the system is in the subcritical state; uncompensated pumping of energy can arise up spontaneously leading to the self-organization; after this during short-time discharging of the stored energy takes place which is called avalanche. Similar to the spin glass [3] the effective potential forms the complex landscape in the system's configuration space, where the domains of allowed states are separated by barriers separating the subensembles of the total statistical ensemble [25,31]. Therefore, the distribution over set of statistical ensembles of nonergodic system is studied at complete description of SOC. Such set corresponds to the different avalanche sizes, the distribution over which has been studied in Ref. [6].

The above approach pretends to describe only the conditions for single-avalanche formation. As is seen from Fig. 4 one can distinguish two-system behaviors depending on the interaction parameter  $v \equiv \sqrt{w}$ : at subcritical values  $v < v_c$  the system passes into self-organization mode if the thermal disorder intensity is above the critical value  $\sigma_c$ ; in the opposite case  $v > v_c$  the ordering process realizes independently on external conditions. The former of this cases meets the phase transition (the self-organization) that is caused by external influence, the latter—the SOC mode whose intermittent character is realized for nonergodic systems only. In the phase diagram in Fig. 3 it is seen that the system is nonergodic if the quenched-disorder intensity exceeds the maximum value  $p_m$  fixed by  $w$  value. According to the dependence represented in Fig. 5 at the supercritical values  $w > w_c$  the self-organization process is nonergodic even in the absence of quenched disorder. Thus, the microscopic condition of self-organized criticality implies that the anharmonicity parameter in bare superpotential (3.13) is bounded by the minimal value  $w_c = 0.064$ . It ensures both the possibility of system's self-organization and intermittent character of its evolution.

## Appendix A

To obtain Eqs. (2.11)–(2.13) of synergetic system from the field equations (3.6)–(3.9) let us multiply Eq. (3.8) by  $\bar{\psi}$  from left-hand side and Eq. (3.9) by  $\psi$  from right-hand side, and add the results. Then, quantities

$$S = \bar{\psi}\psi, \quad \mathbf{j} = (\nabla\bar{\psi})\psi - \bar{\psi}\nabla\psi \quad (\text{A.1})$$

are governed by the continuity equation  $\dot{S} + \nabla\mathbf{j} = 0$  that, obviously, expresses the entropy conservation law for conserved systems. Respectively, the first combination of Grassmann fields in Eq. (A.1) determines the entropy  $S$  and the second one—its current  $\mathbf{j}$ . It is characteristic that Eqs. (3.8), (3.9) for Grassmann conjugated fields  $\psi$ ,  $\bar{\psi}$  differ only by the sign in front of the time derivative, so that dependencies  $\psi(t)$ ,  $\bar{\psi}(t)$  coincide at time inversion. Namely, this circumstance provides the condition of entropy conservation albeit for each of the fields  $\psi$ ,  $\bar{\psi}$  this condition is not fulfilled: according to (3.8), in the homogeneous case the quantity  $\psi(t)$  decreases exponentially with decrement  $t^{-1} \int_0^t V_0''(\eta(t')) dt'$ , whereas the conjugate field  $\bar{\psi}(t)$  increases with the

same increment. For the entropy  $S \equiv \bar{\psi}\psi$  the pointed out processes are compensated and magnitude of  $S$  is conserved. As a result, the obtained continuity equation does not contain the feedback with order parameter  $\eta$ .

To examine such feedback we ought to take into account the detuning of right-hand sides of Eqs. (3.8), (3.9) that is caused by the macroscopic time irreversibility. With this purpose we introduce the coefficient  $1 + \varepsilon$  into the right-hand side of Eq. (3.8) defined by the detuning parameter  $\varepsilon \ll 1$  (its value is determined in Section 5). Then, the term  $-\varepsilon V_0''(\eta)S$  appears on the right-hand side of continuity equation. In addition, we take into consideration that self-organization process realizes only at stationary current  $\mathbf{j}$  which leads to thermostat entropy  $S$  increase with constant velocity  $-\nabla \mathbf{j} \equiv (\tau_\eta/\tau_S)S^0$  (at that, of course, the entropy of self-organizing system  $\Delta S \equiv S^0 - S$  decreases). As a result, entropy balance equation takes the form

$$\dot{S} = (\tau_\eta/\tau_S)S^0 - \varepsilon V_0''(\eta)S. \tag{A.2}$$

From here, the equality  $S = (\tau_\eta/\varepsilon\tau_S)S^0/V_0''(\eta)$  is obtained in the stationary regime  $\dot{S} = 0$ . This equality reduces to the form (2.14) for the bare potential

$$V_0 = \eta^2/2 + \eta^4/12. \tag{A.3}$$

Now, let us examine Eq. (3.7) for the fluctuation field  $\varphi(\mathbf{r}, t)$ . In contrast to the conjugate field  $h$  the fluctuational one  $\varphi$  is the nonhomogeneous even in the stationary state. Let us use the approximation  $\nabla^2 \varphi = (\xi/a)^2 \varphi$ , where  $a$  is the measure scale of stationary fluctuation,  $\xi$  is the correlation length being the scale of the coordinate  $\mathbf{r}$ . Then, from Eq. (3.7) the relation follows for the stationary state  $\dot{\varphi} = 0$

$$\varphi = \frac{V_0'''(\eta)S}{V_0''(\eta) - (\xi/a)^2}, \tag{A.4}$$

showing that the values  $\varphi, S$  are related through an anharmonicity  $V_0'''(\eta) \neq 0$  of bare potential only. Besides, the stability condition  $\varphi > 0$  implies that the scale of nonhomogeneity exceeds the value  $(V_0''(\eta))^{-1/2}\xi$ . At more rigorous requirement  $(a/\xi)^2 V_0''(\eta) \gg 1$  the term  $-(\xi/a)^2$  in the denominator may be omitted. This inequality meets the adiabatic condition that, however, relates not the time, but the spatial scales.

Within the adiabatic approximation equation (3.7) for amplitude  $\varphi(t)$  of the most probable fluctuation takes the form

$$\dot{\varphi} = V_0''(\eta)\varphi - V_0'''(\eta)S. \tag{A.5}$$

Taking into consideration  $S(\eta)$  dependence (see after (A.2)) at the stationary condition  $\dot{\varphi} = 0$ , we obtain  $\varphi = (\sigma/2)V_0'''(\eta)/[V_0''(\eta)]^2$ ,  $\sigma \equiv S^0/S_c$ , where the characteristic value of control parameter is introduced

$$S_c \equiv (\varepsilon/2)(\tau_S/\tau_\eta). \tag{A.6}$$

At last, let us consider Eq. (3.6) for the order parameter field  $\eta(\mathbf{r}, t)$ . Inserting in it the dependence  $\varphi(\eta)$ , we arrive at the Ginzburg–Landau–Khalatnikov equation

$$\dot{\eta} - \nabla^2 \eta = -\partial \tilde{V}/\partial \eta, \quad \tilde{V}(\eta) \equiv V_0(\eta) + \frac{\sigma}{2V_0''(\eta)} \tag{A.7}$$

that differs from (2.15) by gradient term. For dependence (A.3) the synergetic potential assumes the form

$$\tilde{V}(\eta) = V_1(\eta) + \eta^4/12, \quad V_1(\eta) \equiv \frac{1}{2} \left( \eta^2 + \frac{\sigma}{1 + \eta^2} \right), \quad (\text{A.8})$$

differing from dependence (2.15). At supercritical parameter  $\sigma > 1$  dependence  $V_1(\eta)$  takes the minimum at the point  $\eta_0^2 = \sqrt{\sigma} - 1$  related to stationary synergetic potential  $V_1(\eta_0) = \sqrt{\sigma} - \frac{1}{2}$  whose value is lower than initial magnitude  $V_1(0) = \sigma/2$  [32]. The distinction of (2.15) and (A.8) is caused by the circumstance that the potential  $V$  is defined at constant field  $h$ , whereas  $\tilde{V}$ —at fluctuation amplitude  $\varphi$  fixed. In other words, the first potential is the field  $h$  function, whereas the second one depends on fluctuation amplitude  $\varphi$ . The quantities  $h, \varphi$  represent the couple of conjugated steady-state parameters (like the volume and pressure in thermodynamics) and the synergetic potentials  $V(h), \tilde{V}(\varphi)$  are related by Legendre transformation  $V = \tilde{V} - h\varphi$ . The state equation governing the  $h(\varphi)$  dependence follows from the usual condition  $h = -\partial\tilde{V}/\partial\varphi$ . It is simpler to introduce the field  $f = -V'_0(\eta) + \nabla^2\eta \equiv -\delta V_0\{\eta\}/\delta\eta$  that reduces to the generalized force  $f$ . Then, Eq. (3.6) assumes the form

$$\dot{\eta} = f + \varphi. \quad (\text{A.9})$$

Comparing Eqs. (A.9) and (2.11), we find the relation

$$h = \eta + (f + \varphi) \equiv (\eta - V'_0(\eta) + \nabla^2\eta) + \varphi. \quad (\text{A.10})$$

For the stationary state ( $\dot{\eta} = 0$ ) the amplitude of the most probable fluctuation  $\varphi = -f$  coincides with the generalized force with accuracy to sign and field  $h = \eta$  reduces to the order parameter. In the general case, the disagreement  $f + \varphi \neq 0$  results in variation of order parameter in the course of time and difference between the fields  $h, \varphi$  is conditioned by nonlinear component of generalized force  $f$ .

Note that the obtained Eqs. (A.7), (A.5), (A.2) and the Lorenz equations (2.11)–(2.13) coincide in their mathematical structure only. So, the entropy balance equation (A.2) contains the negative feedback that is similar to Eq. (2.13), however it is expressed by the term  $-\varepsilon V''_0(\eta)S$  that is proportional to the entropy, whereas the corresponding term  $-A_S\eta h$  does not contain the entropy. The term  $A_h\eta S$  in (2.12) and  $-V'''_0(\eta)S$  in (A.5) coincide in case of bare potential (A.3) with accuracy to sign. Finally, field equation (A.7) for the order parameter differs from Landau–Khalatnikov equation (2.15) by accounting for spatial nonhomogeneity only. As was noted, the physical reason for the above distinctions is that the Lorenz equations contain the field  $h$  conjugated to order parameter, whereas the amplitude  $\varphi$  of most-probable fluctuation appears in the initial field equations (3.6)–(3.9). Since the  $h$  and  $\varphi$  play a role of conjugated parameters of the system's steady state, then the developed field formalism and Lorenz scheme are mutually supplementary approaches—the first one is used at fixed value of most-probable fluctuation amplitude  $\varphi$ , and the second one—at fixed field  $h$ . Obviously, the second case is realized more naturally (to pass to this we should exploit the state equation (A.10)).

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