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CONDITIONS FOR THE TIME DISSIPATIVE STRUCTURE FORMATION AT NON-EQUILIBRIUM TRANSITIONS

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We discuss processes of the dissipative structure formation when the non-equilibrium phase transition takes place. The model is considered under the assumptions that dispersion of the relaxation time of the order parameter and influence of the external force are present. It was found that self-organization occurs through the Hopf bifurcation and results in the dissipative structure formation. Analysis was performed according to the Lyapunov and Floquet exponent investigation. It was found that the complex picture of ordering with two reentrant Hopf bifurcations occurs.

Keywords: *NON-EQUILIBRIUM PHASE TRANSITION, DISSIPATIVE STRUCTURE, HOPF BIFURCATION, LIMIT CYCLE, ORDER PARAMETER.*

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1. INTRODUCTION

Modern presentation of non-equilibrium processes is one of the most interesting areas in theoretical physics. Evaluation of the influence of deterministic external force on the structure formation processes and self-organization in whole [1-7] is the central question. Qualitative change in the collective behavior of non-linear dynamic systems that results in the formation of stable space-time (dissipative) structures, determined by the collective hydrodynamic mode [1, 2, 8, 9], is understood under self-organization. Investigation of the systems, which organize themselves, and the processes there is connected with the study of regularities of the space-time ordering in the systems of different nature. Phase transitions are the simplest physical example of self-organization. Their description is based on the thermodynamic scheme: thermostat can influence the specified subsystem, but the latter due to its own smallness does not change thermostat. However, the principle difference in the self-organization processes is in the following: extraction of a small subsystem from thermostat is impossible in connection with the fact that all degrees of freedom, which are reduced to the order parameter, conjugate field, and control parameter, become equivalent. Optically bistable system – laser – is the typical example of such system. There the stated modes determine the induced field amplitude, electrical polarization of environment, and inverse electron level population, respectively. Such system is open and described by self-consistent behavior of the mentioned degrees of freedom. As a result, self-organization process is reduced to the pulse radiation of the specified amplitude and/or frequency (the time dissipative structure formation).

The majority of physical systems, realized in the nature, is under the external action and deviate from the equilibrium position. Moreover, at certain intensities of such action, dependence of the system parameters on the external conditions becomes stepwise. It is significant that at such stepwise change the coherent behavior of separate system elements (self-organization), which was accompanied by the formation of the time, space, or space-time structures, is observed. Such an abrupt transition to the new state resembles the phase transition in the case of thermodynamic equilibrium, and the corresponding stepwise change in the state of non-equilibrium system is called the kinetic or non-equilibrium phase transition [1-5]. In the mathematical sense, qualitative change in the physical system behavior is observed in the bifurcation points. Due to such bifurcations, transition from one stable state to another, stable oscillation processes, change in the vibration period, chaotic behavior mode, etc can occur in the system. Investigation of the formation processes of stable structures in non-equilibrium systems is the subject of theoretical physics starting from the seventies of the last century and is still actual.

The aim of the present work is to investigate the influence of two additional nonlinearities, which appear in synergetic system due to different physical processes, on the processes of time ordering resulted in the formation of the time dissipative structure. Based on the generalized Lorentz-Haken model, which self-consistently describes non-equilibrium transitions in optically bistable systems [1], systems of defects in a solid body [10, 11], polymer and socioeconomic systems [12], we will analyze the possibility of the formation of the time dissipative structures during non-equilibrium phase transitions. Also, we will analyze the combined effect of additional perturbations imposed on the change in the order parameter and nonlinear dependence of the order parameter relaxation time on the behavior of non-equilibrium transitions and time ordering processes of such systems. Using theoretical approach based on the Floquet index, we will define the conditions of coherent behavior of slow system modes and determine the dominant contribution, which induces the time ordering.

This paper is organized as follows. In Section 2 we present the model of the studied system taking into account the dispersion of the order parameter relaxation time and model of the force, which represents the external action. Theoretical investigation of the conditions of dissipative structure initiation is given in Section 3. Different regimes of the system behavior are analyzed and the parameter regions of coherent behavior of the fundamental modes are determined in Section 4. Section 5 contains basic conclusions.

2. SYNERGETIC SYSTEM MODEL

From the mathematical point of view, the Lorentz-Haken system is the simplest model for describing the self-organization effects; it can be written as [1]

$$
\begin{aligned}
\frac{d\eta}{dt} &= -\frac{\eta}{\tau_\eta} + g_\eta h, \\
\frac{dh}{dt} &= -\frac{h}{\tau_h} + g_h \eta S, \\
\frac{dS}{dt} &= \frac{(r-S)}{\tau_S} - g_S \eta h.
\end{aligned}
\tag{1}
$$

Here τ_η, τ_S, τ_h are the relaxation times of the order parameter $\eta(t)$, conjugate field $h(t)$, and control parameter $S(t)$, respectively; g_η, g_S, g_h are the positive feedback constants; r is the external pump parameter. First terms in (1) take into account the dissipation effects inherent to synergetic systems. Relation between the order parameter and the conjugate field is linear (the first equation), while evolution of the conjugate field $h(t)$ and the control parameter $S(t)$ is given by nonlinear feedbacks (the second and the third equations, respectively). It is principle that positive feedbacks, which are provided by the constants g_η and g_h, lead to the increase in the conjugate field inducing self-organization. In accordance with the Le Chatelier principle, these positive feed-backs are compensated by the negative one, which leads to the decrease in the control parameter.

Considering the synergetic systems, which have physical realization, it is popularly believed that change of the conjugate field h occurs much faster than the order parameter η and the control parameter S [1]. Such assumption is true when describing the system of solid-state laser [13] and allows to use the principle of adiabatic approximation suggesting $\tau_h << (\tau_S, \tau_\eta)$ that gives the possibility to neglect the left side of the second equation in system (1). As a result, we obtain the relation $h = \tau_h g_h \eta S$, which leads to the system of two differential equations in the form

$$\begin{aligned} \frac{d\eta}{dt} &= -\frac{\eta}{\tau_\eta} + \tau_h g_h g_\eta \eta S, \\ \frac{dS}{dt} &= \frac{(r-S)}{\tau_S} - \tau_h g_h g_S \eta^2 S. \end{aligned} \tag{2}$$

For the further analysis of the system (2) it is convenient to pass to the dimensionless variables. Such transition is achieved due to the measurement of the time t, order parameter η, conjugate field h, and control parameter S in the following units: τ_η, $\eta_e = (g_h g_S)^{-1/2}$, $h_e = (g_\eta^2 g_h g_S)^{-1/2}$, and $S_e = (g_\eta g_h)^{-1}$. In this case, omitting the index system (2) will take the form

$$\begin{aligned} \frac{d\eta}{dt} &= -\eta(1-S), \\ \frac{dS}{dt} &= \varepsilon^{-1}\left[r - S(1+\eta^2)\right], \end{aligned} \tag{3}$$

where $\varepsilon = \tau_S/\tau_\eta$. System (3) is written with the assumption of linear relaxation of the order parameter η. However, a more realistic situation corresponds to the nonlinear relaxation processes. Taking into account the latter, for the order parameter relaxation time τ_η we take the relation [14]

$$\tau_\eta(\eta) = 1 - \frac{\kappa}{1+\left(\eta/\eta_\tau\right)^2 + \kappa}, \tag{4}$$

where κ and η_τ are the positive constants, which play the role of the dissipation parameter and saturation intensity, respectively. Such form of the dependence $\tau_\eta(\eta)$ is based on the independence of the relaxation time on the order parameter sign reversal. Moreover, relation (4) has practical application, namely, it simulates the action of an optical filter inserted into the resonator of an optically bistable system (for example, solid-state laser) that provides the set of the stable periodic irradiation mode [14]. Using dependence (4), the first equation in (3) is complemented with nonlinear component $f_\kappa = -$ $-(\kappa\eta)/[1 + (\eta/\eta_\tau)^2]$.

On the other hand, formation of the time dissipative structure, which on the phase plane is represented by a stable harmonic cycle, as a rule, is the result of the Andronov-Hopf bifurcation. In accordance with the Hopf theorem, action of the external potential can lead to such bifurcation. In this connection, we generalize system (3) by the introduction of additional perturbations, which are simulated by the potential V_e. According to the standard theory of catastrophes, such potential is given by three types of catastrophes [15]. In the general case, we have

$$V_e = A\eta + \frac{B}{2}\eta^2 + \frac{C}{3}\eta^3 + \frac{D}{4}\eta^4 + \frac{E}{5}\eta^5\,, \tag{5}$$

where A, B, C, D, E are the theory parameters. Thus, for catastrophe A_2 we have $B = D = E = 0$, for catastrophe A_3: $C = E = 0$, and for catastrophe A_4: $D = 0$. In this work, for the external potential we take the simplest case of the catastrophe A_2, which provides the introduction of additional nonlinearity $f_e = -\,d_\eta V_e(\eta)$ [16] to the first equation of the system (3). So, using all the above stated assumptions, generalized synergetic system takes the form of

$$\begin{aligned}
\frac{d\eta}{dt} &= -\eta(1-S) + f_\kappa(\eta) + f_e(\eta),\\
\frac{dS}{dt} &= \varepsilon^{-1}\left[r - S(1+\eta^2)\right],
\end{aligned} \tag{6}$$

and variation of parameters in terms $f_\kappa(\eta)$ and $f_e(\eta)$ can induce the change in the attractor topology in the phase space.

3. ALGORITHM FOR THE DETERMINATION OF THE MANIFOLD STABILITY

To investigate the conditions of system (6) self-organization with the time dissipative structure formation we will use the algorithm of bifurcation of the limit cycle generation [17]. This algorithm allows to find the initiation conditions of the limit cycle and determine its stability. According to this procedure, we find the stationary states η_0 and S_0 of the system, which define the position of the stationary points on the phase plane (S, η). Assuming that left sides of the equations in system (6) are equal to zero ($d\eta/dt = 0$ and $dS/dt = 0$), we obtain the following equations for stationary states:

$$\eta_0\left(\frac{r}{1+\eta_0^{\,2}} - \frac{\kappa}{1+\left(\eta_0/\eta_\tau\right)^2} - 2C\eta_0 - 1\right) = A,\;\; S_0 = r(1+\eta_0)^2\,. \tag{7}$$

Phase trajectory behavior in the vicinity of the stationary point (S_0, η_0) is described by the linear stability analysis. For this, time dependence of each dynamic variable in the vicinity of such point is represented in the form of $x \sim e^{\Lambda t}$, where $x \equiv \{\eta, S\}$; $\Lambda = \lambda + i\varpi$; λ determines the stability of the stationary point, ϖ – vibration frequency in its vicinity [18]. Real and imaginary parts of the stability index Λ are calculated using the stability matrix $M_{ij} \equiv (\partial_{ij} f^{(i)})_{xj = xj0}$, where $f^{(i)} \equiv \{d\eta/dt, dS/dt\}$; $x_j \equiv \{\eta, S\}$; $i, j = 1, 2$, index 0 belongs to the stationary state [19]. Expressions for λ and ϖ are determined from the equation for eigenvalues and eigenvectors and take the form

$$\lambda = \frac{1}{2}\left[\left(S_0 - M_0\right) - \varepsilon^{-1}\left(1 + \eta_0^2\right)\right],$$
$$\varpi = \frac{1}{2}\sqrt{8\varepsilon^{-1} S_0 \eta_0^2 - \left[\left(S_0 - M_0\right) + \varepsilon^{-1}\left(1 + \eta_0^2\right)\right]^2}, \quad (8)$$

where $M_0 = 1 + 2C\eta_0 + \kappa[1 - (\eta_0/\eta_\tau)^2]/\kappa[1 + (\eta_0/\eta_\tau)^2]^2$.

Initiation conditions of the time dissipative structure, which is of the form of the stable limit cycle on the phase portrait, are the following: positive real part of the stability index $\lambda > 0$ that provides the phase trajectory removal from the stationary point; non-zero value of the imaginary part of the stability index $\varpi \neq 0$ that determines the vibration frequency; and fulfillment of the stability conditions of the limit cycle by the Floquet index [14] that for the studied system (6) is provided by the fulfillment of the inequality

$$2\alpha\left(\psi_\kappa - C\right)^2 + \alpha\beta\varepsilon S_0\left(1 + 2\beta\varepsilon\eta_0\right) + \varpi\,|_{\lambda=0}\left(\varphi_\kappa + \beta\varepsilon\right) \leq$$
$$\leq \left(C - \psi_\kappa\right)\left(\alpha^2 - 1 + 2\beta\varepsilon S_0 + 2\alpha\beta\varepsilon\eta_0\right). \quad (9)$$

Here we have used the following designations:

$$\alpha = \frac{M_0 - S_0}{\varpi\,|_{\lambda=0}}, \quad \beta = \frac{\eta_0}{\varepsilon\varpi\,|_{\lambda=0}}, \quad \psi_\kappa = -2\frac{\kappa\eta_\tau^2\eta_0(\eta_0 - 3\eta_\tau^2)}{(\eta_0^2 + \eta_\tau^2)^2},$$
$$\varphi_\kappa = 6\kappa\eta_\tau^2\left[1 - \frac{8\eta_\tau^2\eta_0^2}{(\eta_0^2 + \eta_\tau^2)^4}\right].$$

4. BEHAVIOR CHANGE OF THE GENERALIZED TWO-COMPONENT SYSTEM

Firstly, we consider the behavior of the order parameter stationary value η_0 depending on the external pump parameter r at the fixed values of other system parameters. For simplicity, we take the saturation intensity value versus the order parameter relaxation time (4) to be equal to $\eta_\tau = 1{,}0$. The corresponding bifurcation diagram is represented in Fig. 1.

Here the solid curve corresponds to the case of the absence of additional perturbations ($A = C = 0$) and shows that at small values of the external pump parameter the system is characterized by a single trivial stationary state: $S_0^{(1)} = r$, $\eta_0^{(1)} = 0$. In the case of overcoming the energy barrier $r_0 = \kappa + 1$, bifurcation of generation of two new solutions $\eta_0^{(2,3)} = \pm\sqrt{r - \kappa - 1}$, $S_0^{(2,3)} = r/(r - \kappa)$

(non-equilibrium second-order transition) occurs in the system. Hatched curves illustrate the influence of the parameter C at $A = 0$, dashed lines – influence of A at $C = 0$. It is seen from the figure that in the case of $A = 0$ and $C = 0{,}1$ (curve 1) behavior of the order parameter stationary value is topologically the same as in the case of $A = C = 0$. At $A = 0$ and $C = -0{,}1$ (curve 1′) we observe the hysteresis behavior of $\eta_0(r)$. In this case, with the increase in r stationary value $\eta_0 \neq 0$ abruptly disappears at $r = r_2$, while with the decrease in r – abruptly appears at $r = r_0$ (non-equilibrium first-order transition). In the case of $C = 0$ and $A = 0{,}1$ two stationary states abruptly appear in the system at overcoming the energy barrier $r = r_1$ (curve 2). In the case of $C = 0$ and $A = -0{,}1$ we have a single stationary value of the order parameter on the whole axis of the external pump parameter (curve 2′).

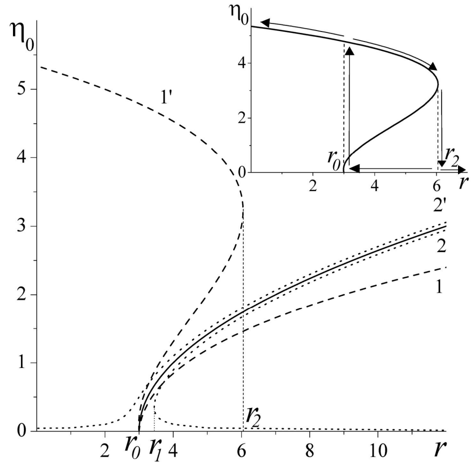


***Fig. 1** – Dependence of the order parameter stationary value η_0 on the external pump parameter r at $C = 0$, $\kappa = 5{,}0$ and different values of the parameter A*

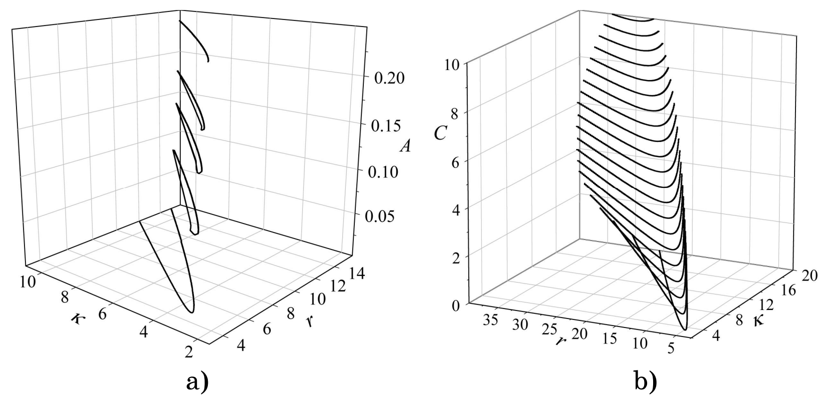


***Fig. 2** – Phase diagram: a) influence of the parameter A at $C = 0$; b) influence of the parameter C at $A = 0$ (dissipative structure is realized inside the limited region)*

Then, satisfying the realization conditions of the stable limit cycle in the system (9), we obtain the phase diagrams, which show the parameter region of the time dissipative structure formation. Solutions of the inequality (9) are shown in Fig. 2, which represents the influence of the external potential parameters A and C on the change in the existence region of the time dissipative structure.

It is seen from Fig. 2a that in the case of $C = A = 0$ a semi-restricted domain of the system parameter values, at which phase space is characterized by the presence of the time dissipative structure, is realized. At $C = 0$ and non-zero values of the parameter A this domain is restricted and substantially narrowed with the increase in A degenerating into unique dependence $\kappa(r)$. Fig. 2b illustrates the influence of the parameter C on the change in the existence region of the time dissipative structure. As seen from this figure, increase in the values of C expands this region and shifts it toward the large values of the dissipation parameter κ and external pump parameter r. Thus, influence of additional linear perturbations on the system destroys the region of the time ordering parameters, while nonlinear perturbations increase this region. Now we consider in detail the phase diagram of the dependence of the dissipation parameter on the external pump parameter shown in Fig. 3.

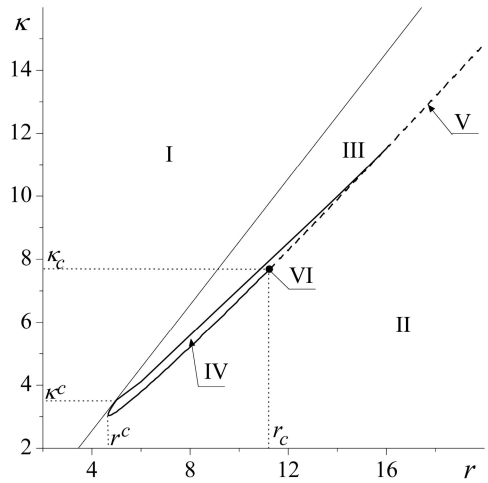


Fig. 3 *– Phase diagram at $A = 0{,}1$ and $C = 0$*

Here critical values of the parameters κ and r, which define the position of the bifurcation point of the stationary state generation, are shown by thin solid line: at the intersection of this line two non-zero stationary states (see curve 2, Fig. 1) abruptly appear in the system. During such transition under the condition $\kappa < \kappa^c$ the upper stationary state (see curve 2, Fig. 1) is a stable focus (region II) due to $\lambda < 0$ and $\varpi \neq 0$. Phase portrait is shown in Fig. 4a. Otherwise, at $\kappa > \kappa^c$ (region III), such state is unstable focus ($\lambda > 0$, $\varpi \neq 0$) (see phase portrait in Fig. 4b). Lower stationary state (see curve 2, Fig. 1) is unstable, and on the phase portrait it looks like hyperbolic point – saddle. Solid curve, which limits region IV, determines the critical values of the Andronov-Hopf bifurcation parameters: with the increase in the dissipation parameter κ during the transition from region II to region IV under the condition $r^c < r < r_c$ the time dissipative structure of the form of a stable limit cycle (see Fig. 4c) is realized in the system. Here all phase trajectories both inside and outside are attracted by the manifold. Further increase in the dissipation parameter leads to the annihilation of the stable limit cycle (transition to region III). During the transition from region II to region IV under the condition $r > r_c$ at the parameter values, which correspond to the hatched curve, bifurcation of generation of the semi-stable limit cycle, which disintegrates into the external stable and internal unstable cycles, occurs in the system. Further increase in the dissipation parameter leads to the destruction of unstable limit cycle (transition to region IV). Thus, narrow region V depicted by the hatched curve defines the values of the system parameters, at which unstable limit cycle exists together with the stable one. In this case, stationary behavior of the system essentially depends on the initial conditions, phase portrait is represented in Fig. 4d. If conditions $\kappa = \kappa^c$ and $r = r_c$ hold, phase portrait is characterized by the presence of embedded limit cycles with neutral stability. Such representation of the self-organization was studied in detail in [16].

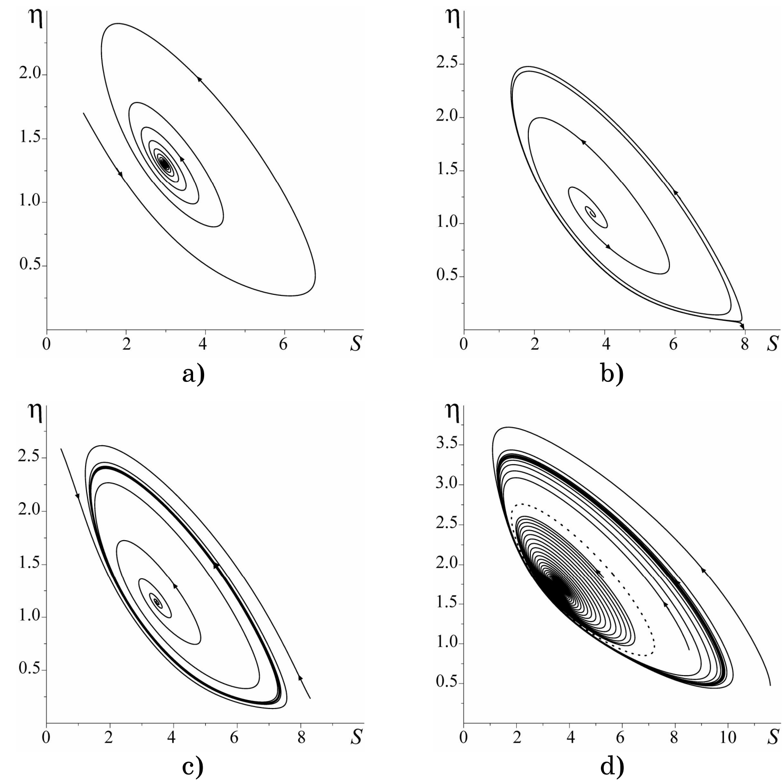
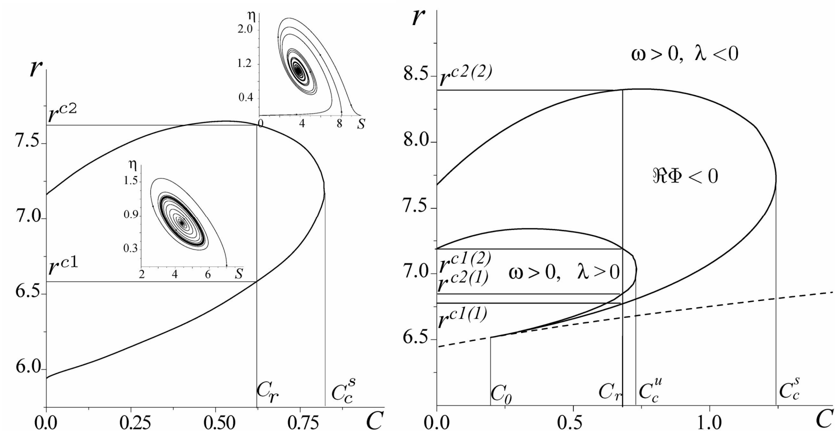


Fig. 4 *– Phase portraits in different regions of the diagram from Fig. 3 at* $A = 0{,}1$, $C = 0$: $\kappa = 5{,}1$, $r = 8$ *(a);* $\kappa = 5{,}6$, $r = 8$ *(b);* $\kappa = 5{,}5$, $r = 8$ *(c);* $\kappa = 9{,}86$, $r = 14$ *(d)*

In Fig. 5 we show the regions of different system behavior at positive and negative values of the parameter A and fixed value of the dissipation parameter (corresponding dependences of the order parameter stationary value on the dissipation parameter are presented by curves 2 and 2′ in Fig. 1).



a) b)

Fig. 5 *– Phase diagram of the Andronov-Hopf bifurcation at* $\kappa = 5{,}0$*:* $A = -0{,}1$ *(a);* $A = 0{,}1$ *(b)*

In the case of $A = 0{,}1$ the system is characterized by a single stationary value (see curve 2′, Fig. 1). On the phase portrait (S, η) it is a stable focus ($\lambda < 0$, $\varpi \neq 0$) outside the region limited by the solid curve (see the insert in Fig. 5a). At the system parameter values inside the region the time dissipative structure (see insert inside the limited region) is realized in the system. Region of the time dissipative structure realization shown in Fig. 5a exists up to the critical value $C = C_c^s$. At $C = C_r < C_c^s$ increase in the value of the external pump parameter r leads to the reversible self-organization picture: at $r = r^{c1}$ time dissipative structure appears in the system by the Andronov-Hopf bifurcation scenario, and at $r = r^{c2}$ it disappears by the same scenario. Change in the system behavior is more complicated in the case of $A > 0$. The phase diagram is represented in Fig. 5b. At the values of the external pump intensity less than the critical value, which is defined by the hatched curve, $\eta_0 = 0$ (Fig. 1). At small values of the parameter $C < C_0$ the ordered phase is characterized by unstable focus. The appropriate stationary value corresponds to the upper branch of the curve 2 in Fig. 1. At large values of $C_c^u < C < C_c^s$ picture of the dynamic mode reconstruction is similar to the above-considered case for $C < C_c^s$ (see Fig. 5a). At the parameter value of $C = C_r$: $C_0 < C_r < C_c^u$ we have a complicated self-organization picture. Here with the increase in the external pump parameter r one can observe the following chain of transformations of the dynamic modes: 1) system moves to the mode characterized by two stationary states (here stable focus corresponds to the ordered phase); 2) dissipative structure, which exists until $r < r^{c2(1)}$, appears at $r = r^{c1(1)}$ due to the self-organization by the Andronov-Hopf bifurcation scenario; 3) further increase in the external pump parameter r leads to the disorganization (at $r = r^{c2(1)}$), i.e. the dissipative structure destruction, and system moves to unstable mode characterized by unstable focus; 4) in the case of overcoming the energy barrier $r = r^{c1(2)}$ system is self-organized again by the Andronov-Hopf bifurcation scenario with the time dissipative structure formation – stable limit cycle; 5) further increase in the external pump parameter leads to the dissipative structure destruction at $r = r^{c2(2)}$, and stable focus corresponds to the ordered phase.

5. CONCLUSIONS

The self-organization processes of synergetic system taking into account the nonlinear dependence of the order parameter relaxation time on the its value and the presence of additional external perturbations, which are modeled by the potential corresponding to the catastrophe A_2, are considered It is shown that depending on the parameters non-equilibrium phase transitions occurring in the system can be both the first- and the second-order transitions. Using the algorithm of bifurcation of the limit cycle generation, the region of the time dissipative structure existence is found. It is revealed that changing the parameters of the external potential, increase in the external pump leads to the reversible self-organization picture.

The performed theoretical investigations can be applied for the study of the formation conditions of stable periodic pulsed radiation in optically bistable systems. The considered case of the relaxation time ratio of the fundamental modes corresponds to the systems of solid-state lasers. For such systems additional perturbations model the action of optical modulator, which specifies the processes of additional interphoton interaction in resonator, and used dependence **(4)**, which takes into account the intensity losses, models the influence of nonlinear filter absorbing a weak signal **[1]**.

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