ON SUBHARMONIC FUNCTIONS AND DIFFERENTIAL GEOMETRY IN THE LARGE

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We consider an open, two-dimensional Riemannian manifold M whose metric is defined by a positive definite quadratic form

$$ds^2 = E(\xi,\eta)d\xi^2 + 2F(\xi,\eta)d\xi d\eta + G(\xi,\eta)d\eta^2, \qquad (1.1)$$

ξ and η denoting local parameters. If E, F and G are sufficiently regular, then it is possible to introduce (local) *isothermic parameters*, i.e. there exists a coordinate transformation $x = x(\xi,\eta)$, $y = y(\xi,\eta)$ such that $E = G>0$, $F=0$ in the (x, y)-parameter system. Then we can write

$$ds^2 = e^{2u(x,y)}(dx^2 + dy^2) = e^{2u(z)}|dz|^2, \qquad (1.2)$$

putting $z = x + iy$. Such a transformation always exists, for example, when E, F and G are of class C^3, and in this case the corresponding function u is also of class C^3.

By the Theorema egregium the Gaussian curvature K can be calculated from the E, F, G and their partial derivatives up to the second order. In the isothermic parameter system (1.2) one obtains the particularly simple expression

$$K = -e^{-2u}\Delta u \quad \left(\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2\right). \qquad (1.3)$$

Hence, letting $dA=e^{2u}dxdy$ denote the area element on M we have

$$KdA = -\Delta u dxdy. \qquad (1.4)$$

Furthermore, one finds after some calculation the following expression for the geodesic curvature k of a curve on M

$$k = e^{-u}\left(k_e + \frac{\partial u}{\partial n}\right). \qquad (1.5)$$

Here k_e denotes the euclidean curvature of the corresponding curve $z=z(t)$ in the z-plane with the convention that $sign\ k_e \equiv sign\ \frac{d}{dt}\left[\arg\frac{dz}{dt}\right]$, and n designates the normal to $z(t)$ in the direction $\arg\left(-i\frac{dz}{dt}\right)$. (1.2) and (1.5) imply

$$kds = \left(k_e + \frac{\partial u}{\partial n}\right)|dz|. \qquad (1.6)$$

In general, isothermic parameters can only be introduced in the small. In order to be able to treat problems pertaining to differential geometry in the large we have to consider the Riemann surface *S* which is determined by the conformal *M*. The local uniformizers are then defined as functions which map a portion of M conformally onto a region in the *z*-plane. Hence their real and imaginary parts form parameters. Conversely, if *x* and *y* are local isothermic parameters a set of local isothermic, then either *x+iy* or *y+ix* constitutes a local uniformizer.

We thus are led to conceive of *M* as a Riemann surface on which a conformal metric

$$ds = e^{u(z)}|dz| \quad (z = \text{local uniformizer}) \qquad (1.7)$$

has been introduced. Thereby a change of uniformizers $z = \varphi(\zeta)$ implies the transformation

$$\tilde{u}(\zeta) = u(\varphi(\zeta)) + \log|\varphi'(\zeta)|, \qquad (1.8)$$

due to the conformal invariance of $ds = e^{u(z)}|dz| = e^{\tilde{u}(\zeta)}|d\zeta|$.

This issue is proffesor H. Hopf. He drew our attention to the connection between differential geometry and potential theory which is revealed by relations (1.3) and (1.4). For example, the function *u(x,y)* is subharmonic in a certain *(x,y)*-parameter region if and only if $K \leq 0$ in the corresponding domain on *M*. This fact had already been used by E. F. Beckenbach and T. Rado in their proof of the isoperimetric inequality on surfaces of negative curvature. Analogously, и is super-harmonic if and only if $K \geq 0$. Furthermore, (1.4) discloses an even deeper connection: The surface integral of К, considered as a set function, is essentially the measure associated with *u*. Consequently, results of differential geometry in the large involving the curvatura integra, such as those due to S. Cohn-Vossen, F. Fiala, Ch. Blanc and F. Fiala, have a potentialtheoretical meaning. It is therefore natural to apply functiontheoretical methods to this field in the hope that not only other (and eventually simpler) proofs of known results will be found, but also theorems which are new in both their differential geometrical and potential-theoretical aspects.

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