Ministry of Education and Science of Ukraine
Sumy State University

A. Yu. Badalian,
I. I. Kozlova

MATHEMATICAL ANALYSIS

Lecture notes

Sumy
Sumy State University
2021

Ministry of Education and Science of Ukraine
Sumy State University

MATHEMATICAL ANALYSIS

Lecture notes

for students of specialty *113 "Applied Mathematics"*
full time course of study

Approved at the meeting
of the Department of Applied Mathematics and Modeling
of Complex Systems
as lecture notes on discipline "Mathematical Analysis".
Minutes № 2 of 25.02.2021.

Sumy
Sumy State University
2021

Lecture notes on the discipline "Mathematical Analysis" / compilers: A. Yu. Badalian, I. I. Kozlova. – Sumy : Sumy State University, 2021. – 58 p.

Department of Applied Mathematics and Modeling of Complex Systems

Content

P.

The Limit of the Numerical Continuity

1. Limit of a variable

A numerical continuity is an infinite set of numbers x_1, x_2, …, x_n, …, arranged in a specified order one after another, denoted by $\{x_n\}$.

The numbers that are part of a continuity are called its *members*.

In many cases, you can create a formula for a general member x_n continuity.

Let the variable x be given by its values x_1, x_2, …, x_n,… We can assume that the continuity $\{x_n\}$ is given. We give the values of the boundary of the continuity $\{x_n\}$ and the boundary of the variable x_n, $n = 1, 2, …$.

Definition. The constant number is called the limit of the variable x_n if any positive number ε is given in advance, which can be arbitrarily small, and there is a number N such that as soon as n is greater than N ($n > N$), then

$$|x_n - a| < \varepsilon.$$

If a is the limit of the variable x_n, then we can say that x_n tends to the limit a, when n tends to infinity, and write down

$$\lim_{n \to \infty} x_n = a.$$

Geometrically a constant number a is the boundary of the variable x_n if for any predetermined small circumference of point a of radius ε, there is such a value x_N that all points corresponding to the following values of the variable are in neighborhood (Fig. 1).

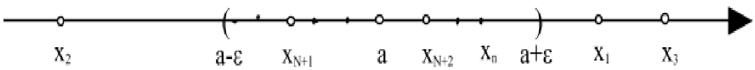


Figure 1 – Bounded order

Definition. The variable x_n tends to ∞ if any predetermined positive number M, which can be arbitrarily large, finds a number N such that as soon as n is greater than N ($n > N$), then $|x_n| > M$.

If the variable x_n tends to ∞, then it is called an infinitely large variable and is written as $x_n \to \infty$ or $\lim_{n \to \infty} x_n = \infty$.

Definition. The variable x_n tends to $+\infty$ if any predetermined number $M > 0$, which can be arbitrarily large.

There is such a number N that as soon as n is greater than N ($n > N$), then $x_n > M$.

Definition. The variable x_n tends to $-\infty$ if any positive number M, which can be arbitrarily large, finds a number N such that as soon as n is greater than N ($n > N$), then $x_n < -M$.

2. Boundary functions

Let function $y = f(x)$ be defined in some neighborhood of the point $x = a$ or in some points of this neighborhood.

Definition. A constant number b is called the boundary of the function $f(x)$ when x tends to a ($x \to a$) if any predetermined positive number ε can be arbitrarily small, and there will be such a positive number δ that as soon as $|x - a| < \delta$, then $|f(x) - b| < \varepsilon$.

If b is the boundary $f(x)$ for $x \to a$, then $\lim_{x \to a} f(x) =$ $= b$, or $f(x) \to b$ for $x \to a$. If $f(x) \to b$ for $x \to a$, then from a geometric point of view for all points x that are distant from the point a and are not further than δ, the points M of the graph of the function $= f(x)$ are placed in a band 2ε wide, which is bounded by lines $y = b - \varepsilon$ and $y = b + \varepsilon$ (Fig. 2).

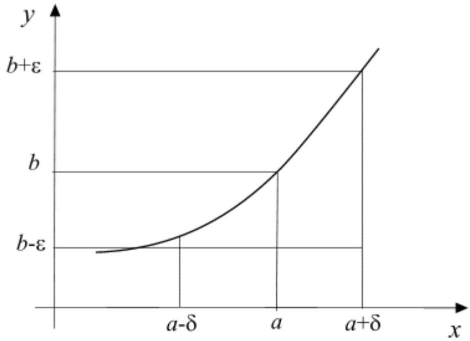


Figure 2 – Function limit

Definition. A constant number b_1 is called the boundary of the function $f(x)$ at the point $x = a$ on the left if no matter how a positive number ε is predetermined, which can be arbitrarily small and there is such a positive number δ that as soon as $a - x < \delta$, then $|f(x) - b_1| < \varepsilon$.

It can be written as follows: $\lim_{x \to a-0} f(x) = b_1$.

Definition. The constant number b_2 is called the boundary of the function $f(x)$ at the point $x = a$ on the right if any predetermined positive number ε, which can be arbitrarily small, finds such a positive number δ that as soon as $a - x < \delta$, then $|f(x) - b_2| < \varepsilon$, which can be written as $\lim_{x \to a+0} f(x) = b_2$.

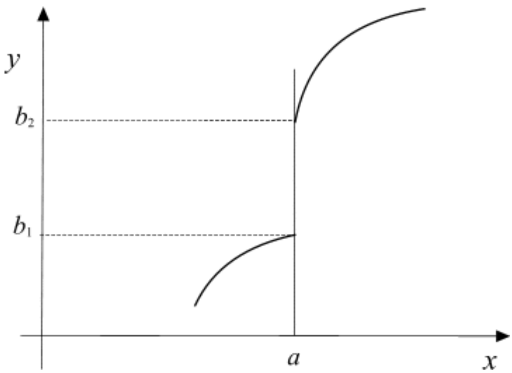


Figure 3 – One-sided border

We can prove that if the boundary on the right and the boundary on the left exist and are equal $b_1 = b_2 = b$, then b will be the boundary of the function $f(x)$ at the point $x = a$.

The existence of the boundary of the function for $x \to a$ does not require the function being defined at the point $x = a$.

Definition. A constant number b is called the boundary of the function $f(x)$ when x goes to infinity if for any positive number ε, which can be arbitrarily small, there is a positive number N such that only for all values of x that satisfy the inequality $|x| > N$ the inequality $|f(x) - b| < \varepsilon$ holds.

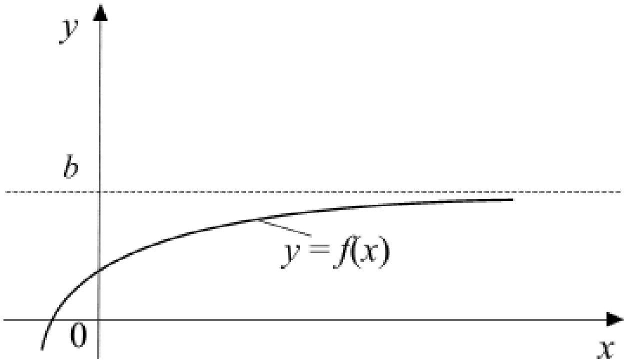


Figure 4 – The limit of the function is at infinity

Definition. The function $f(x)$ will be infinitely large for $x \to a$ if for each positive number M, which can be arbitrarily large, we can find such $\delta > 0$ that for all values of x other than a, satisfying the condition $|x - a| < \delta$, there is the inequality $|f(x)| > M$.

In this case, write $\lim_{x \to a} f(x) = \infty$ or $f(x) \to \infty$ for $x \to a$.

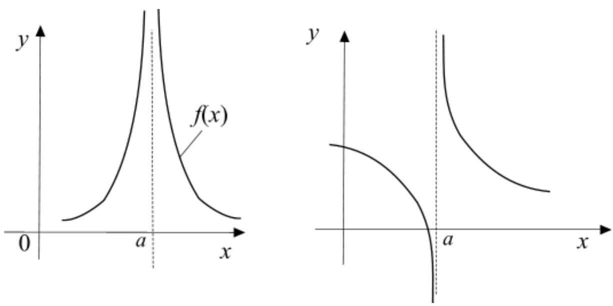


Figure 5 – Infinitely large function

If the function $f(x) \to \infty$ for $x \to \infty$, then write: $\lim_{x\to\infty} f(x) = \infty$.

Definition. The function $y = f(x)$ is called bounded in the domain where the argument x changes if there is a positive number M for which the inequality $|f(x)| \leq M$ will hold for all values of x belonging to this domain. If such a number M does not exist, then the function $f(x)$ is called unlimited in this area.

Definition. The function $f(x)$ is said to be bounded at $x \to a$ if there exists a circle centered at point a at which this function is bounded.

Definition. The function $y = f(x)$ is said to be bounded for $x \to \infty$ if there exists a number $N > 0$ that can be arbitrarily large such that for all values of x, satisfying the inequalities $|x| > N$, the function $f(x)$ is bounded.

Theorem 1. If $\lim_{x\to a} f(x) = b$ and b has a finite value, then the function $f(x)$ will be bounded for $x \to a$.

Theorem 2. If $\lim_{x\to a} f(x) \neq 0$, then the function $y = \frac{1}{f(x)}$ will be bounded for $x \to a$.

3. Infinitesimale and their basic properties

Definition. The function $\alpha = \alpha(x)$ is called infinitesimal for $x \to a$ or for $x \to \infty$ if $\lim_{x\to a} \alpha(x) = 0$ or $\lim_{x\to\infty} \alpha(x) = 0$.

From the definition of the boundary it follows that if $\lim_{x\to\infty} \alpha(x) = 0$, no matter how predetermined a positive number ε, which can be arbitrarily small, is and there is such a positive number δ that for all x, satisfying the inequalities $|x - a| < \delta$, the requirement holds $|\alpha(x)| < \varepsilon$.

Theorem 3. If the function $y = f(x)$ is represented as the sum of the constant number b and the infinitesimal α (for $x \to a$ or for $x \to \infty$)

$$y = a + b,$$

then write $\lim_{x\to a} y = b$ or $\lim_{x\to a} y = b$.

Theorem 4. If $\alpha = \alpha(x) \to 0$ for $x \to a$ (or for $x \to \infty$) and does not turn to zero, then $y = \frac{1}{\alpha} \to \infty$.

Theorem 5. The algebraic sum of the finite number of infinitesimals is a function of infinitesimal.

Theorem 6. The product $\alpha(x) \cdot Z(x)$, where $\alpha(x)$ is an infinitesimal and $Z(x)$ is a bounded function for $x \to a$ (or for $x \to \infty$), is an infinitesimal function.

Theorem 7. The fraction $\frac{\alpha(x)}{Z(x)}$, where $\lim\limits_{\substack{x\to a \\ (x\to\infty)}} \alpha(x) = 0$ and $\lim\limits_{\substack{x\to a \\ (x\to\infty)}} Z(x) \neq 0$, is an infinitesimal function.

4. Basic theorems on boundaries

Theorem 8. The boundary of the algebraic sum of a finite number of functions is equal to the algebraic sum of the boundaries of these functions

$$\lim_{x\to a} \sum_{k=1}^{n} U_k(x) = \sum_{k=1}^{n} \lim_{x\to a} U_k(x).$$

Theorem 9. The boundary of the product of a finite number of functions is equal to the product of the boundaries of these functions

$$\lim_{x\to a} \prod_{k=1}^{n} U_k(x) = \prod_{k=1}^{n} \lim_{x\to a} U_k(x).$$

Theorem 10. The boundary of the fraction of two functions is equal to the fraction of the boundaries of these functions if the limit of the denominator is nonzero

$$\lim_{x\to a} \frac{\lim_{x\to a} U(x)}{\lim_{x\to a} V(x)} \left(\lim_{x\to a} V(x) \neq 0\right).$$

Theorem 11. If between the corresponding values of three functions *U* (*x*), *V* (*x*), *Z* (*x*) the inequalities *U* (*x*) ≤ *Z* (*x*) ≤ *V* (*x*) holding *U* (*x*) and *V* (*x*) for *x* → *a* go to the same boundary b, then $Z(x)$ for $x \to a$ goes to the same boundary b.

Theorem 12. If for $x \to a$ the function $y(x)$ acquires non-negative values of $y(x) \geq 0$ and thus goes to the boundary b, then b is a non-negative number ($b \geq 0$).

Theorem 13. If the inequality $V(x) \geq U(x)$ holds between the corresponding values of two functions $U(x)$ and $V(x)$ that go to the boundaries at $x \to a$, then the inequality $\lim_{x \to a} V(x) \geq \lim_{x \to a} U(x)$ holds.

Theorem14. Each variable of limited increasing value has a limit.

Theorem 15. Each variable bounded by a decreasing value has a limit.

4.1. The First remarkable limit

Theorem 16. $\lim_{x \to 0} \frac{\sin x}{x} = 1$ is called the first remarkable boundary.

4.2. The Second remarkable limit

Theorem 17. The variable $\left(1 + \frac{1}{n}\right)^n$ for $n \to \infty$ has a boundary between the numbers of 2 and 3.

Definition. The limit of the variable $\left(1+\frac{1}{n}\right)^n$ for $n \to \infty$ is called number e:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

By theorem 17

$$2 \leq e \leq 3.$$

The number e is a transcendental number, first proved by S. Hermit in 1873. Therefore, e = 2,718 281 828 459 045 ...

Theorem 18. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ is called the second remarkable limit.

5. Infinitesimal differences

Let $\alpha(x)$ and $\beta(x)$ be infinitesimals for $x \to a$ $(x \to \infty)$.

Definition 1. If $\lim\limits_{\substack{x\to a \\ (x\to\infty)}} \frac{\alpha(x)}{\beta(x)} = A \neq 0$ (A is a finite number), then infinitely small $\alpha(x)$ and $\beta(x)$ are called infinitesimals of the same order.

Definition 2. If $\lim\limits_{\substack{x\to a\\(x\to\infty)}} \frac{\alpha(x)}{\beta(x)} = 0 \left(\lim\limits_{\substack{x\to a\\(x\to\infty)}} \frac{\beta(x)}{\alpha(x)} = \infty \right)$, then $\alpha(x)$ is called a small value of a higher order than the order of $\beta(x)$.

Definition 3. An infinitesimal quantity $\alpha(x)$ is called an infinitesimal of k-th order with respect to $\beta(x)$, if $\alpha(x)$ and $\beta^k(x)$ are infinitesimal quantities of the same order, then $\lim\limits_{\substack{x\to a\\(x\to\infty)}} \frac{\alpha(x)}{\beta(x)} = A \neq 0$.

Definition 4. If $\lim\limits_{\substack{x\to a\\(x\to\infty)}} \frac{\alpha(x)}{\beta(x)} = 1$, then $\alpha(x)$ and $\beta(x)$ are called equivalent to infinitesimal quantities and should be as written $\alpha(x) \sim \beta(x)$.

Theorem 19. If $\alpha(x) \sim \beta(x)$ for $x \to a$ or $x \to \infty$, then $(\alpha(x) - \beta(x))$ - infinitesimal value of higher order then $\alpha(x)$, and then $\beta(x)$.

Theorem 20. If $\lim\limits_{\substack{x\to a\\(x\to\infty)}} \alpha(x) = 0$, $\lim\limits_{\substack{x\to a\\(x\to\infty)}} b(x) = 0$ and $(\alpha(x) - \beta(x))$ is an infinite a small value of the order higher then $\alpha(x)$, or $\beta(x)$, then $\alpha(x) \sim \beta(x)$.

6. Table of equivalent infinitesimals

Let $\alpha(x) \to 0$ for $x \to 0$.

- $\sin\alpha(x) \sim \alpha(x)$
- $\tan\alpha(x) \sim \alpha(x)$
- $1-\cos\alpha(x) \sim \frac{1}{2}(\alpha(x))^2$
- $\arccos\alpha(x) \sim \alpha(x)$
- $\arctan\alpha(x) \sim \alpha(x)$
- $\ln(1+\alpha(x)) \sim \alpha(x)$
- $a^{\alpha(x)} - 1 \sim \alpha(x)\ln a \quad (a > 0)$
- $e^{\alpha(x)} - 1 \sim \alpha(x)$ (special case 7°, when $a = e$)
- $\left(1+\alpha(x)\right)^p - 1 \sim p\alpha(x)$
- $\sqrt[n]{1+\alpha(x)} - 1 \sim \frac{\alpha(X)}{n}\left(\text{special case } 9^\circ, \text{ when } p = \frac{1}{n}\right)$

7. Continuity of functions

Let a function $y = f(x)$ be defined at some point $x = x_0$ and in its vicinity. Let us denote $y_0 = f(x_0)$. We give an increment Δx of the argument x and obtain $x = x_0+\Delta x$. Then the function $y = f(x)$ will increase $\Delta y = f(x_0+\Delta x) - f(x_0)$.

Definition. The function $y = f(x)$ is said to be continuous at the point x_0 (or at a value $x = x_0$) if it is defined in some neighborhood, then

$$\lim_{\Delta x \to 0} \Delta y = 0,$$

or

$$\lim_{\Delta x \to 0} (f\ (x0 + \Delta x) - f\ (x0)) = 0.$$

The condition of continuity can be written as follows:

$$\lim_{\Delta x \to 0} f\ (x0 + \Delta x) = f\ (x0).$$

Let $x = x_0 + \Delta x$. Then, $x \to x_0$ when $\Delta x \to 0$ that is,

$$x_0 = \lim_{\Delta x \to 0} x = \lim_{x \to \lim\limits_{x \to x_0}} x.$$

Thus, it is possible to rewrite the condition of continuity in a new form:

$$\lim_{x \to x_0} f(x) = f\left(\lim_{x \to x_0}\right).$$

Therefore, if the function $f(x)$ is continuous at $x \to x_0$, the notations *lim* and *f* may be swapped.

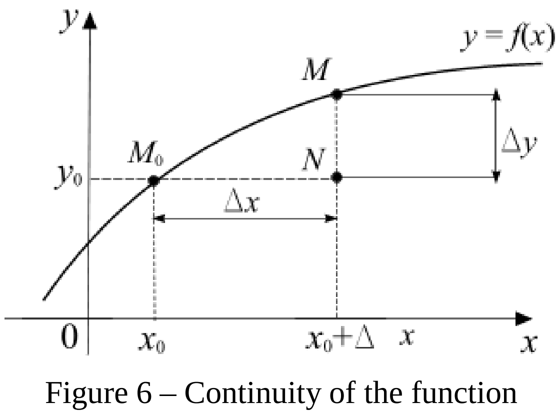


Figure 6 – Continuity of the function

Geometrically, the continuity of the function at a given point means that the difference of the ordinates in the graph of the function $y = f(x)$ at the points $x_0+\Delta x$ and $x_0\Delta y$ will be arbitrarily small in absolute value if $|\Delta x|$ is small enough (Fig. 6).

Theorem 21. If the functions $f_1(x)$ and $f_2(x)$ are continuous at the point x_0, then the function $\varphi(x) = f_1(x) + f_2(x)$ is also continuous at the point x_0.

Theorem 22. The product of two continuous functions at the point x_0 is a continuous function at the point x_0.

Theorem 23. The fraction of two continuous functions at the point x_0 is a function continuous at the point x_0 if the denominator at the point x_0 is not zero.

Theorem 24. If a function $u = \varphi(x)$ is continuous at the point $x = x_0$ and a function $y = f(u)$ is continuous at the point $u = u_0$ and $u_0 = \varphi(x)$, then a complex function $y = f(\varphi(x))$ is continuous at the point $x = x_0$.

Definition. If the function $f(x)$ is continuous at each point of the interval (a, b), where $a < b$, then the function $f(x)$ is called to be continuous in the interval (a, b).

Definition. If the function $f(x)$ is defined for $x = a$ and $\lim_{x \to a+0} f(x) = f(a)$, then the function $f(x)$ is right continuous at the point $x = a$.

Definition. If the function $f(x)$ is defined for $x = b$ and $\lim_{x \to b-0} f(x) = f(b)$, then the function $f(x)$ is left continuous at the point $x = b$.

Definition. If the function $f(x)$ is continuous in the interval (a, b), right continuous on the at the point $x = a$ and left continuous at the point $x = b$, then the function $f(x)$ is called to be continuous in the segment $[a, b]$.

Definition. If at any point $x = x_*$ for the function $y = f(x)$ at least one of the conditions of continuity is not satisfied, that is if 1) the function is indefinite for $x = x_*$; 2) the function $f(x)$ is defined for $x = x_*$, but there is no $\lim_{x \to x_*} f(x)$; and 3) the function $f(x)$ is defined for $x = x_*$, there is $\lim_{x \to x_*} f(x)$, but $\lim_{x \to x_*} f(x) \neq \lim_{x \to x_*} f(x_*)$, than for $x = x_*$. The function $f(x)$ is called discontinuous and the point $x = x_*$ is called the breakpoint of the function $f(x)$.

8. Properties of continuous functions in closed interval

Property 1. If the function $f(x)$ is continuous in the segment $[a, b]$ where $a < b$, then there is at least one point

$x = \bar{x}$ on this segment that the value of the function $f(x)$ at this point will not exceed the value of $f(x)$ at other points of the segment $[a, b]$, that is $f(\bar{x}) \leq f(x), x \in [a, b]$, and there is at least one point $x = \bar{\bar{x}}$ that the value of $f(x)$ at this point will be not less than the value of $f(x)$ at other points of the segment $[a, b]$, that is $f(\bar{\bar{x}}) \geq f(x), x \in [a, b]$.

Definition. The value of $f(\bar{x})$ is called the smallest value of the function $f(x)$ in the segment $[a, b]$ and is denoted by m, the value of $f(\bar{\bar{x}})$ is called the largest value of the function $f(x)$ in the segment $[a, b]$ and is denoted by M.

Then property 1 can be formulated as follows:
The function $f(x)$, continuous in the interval $[a, b]$, reaches its smallest and largest values on this segment.

Conclusion from property 1: If the function $f(x)$ is continuous in the interval $[a, b]$, then it is bounded on this segment.

Property 2. If the function $f(x)$ is continuous in the segment $[a, b]$ and at its ends the function has the values with different signs, that is $f(a) \cdot f(b) < 0$, then inside the segment there is at least one-point $x = x_0$ $(a < x_0 < b)$ that the value of the function at this point is zero, that is $f(x_0) = 0$.

Geometrically, this means that the graph of the function $y = f(x)$ intersects the axis Ox if the points of the graph of the function $y = f(x)$ lie on different sides of the axis *Ox* (Fig. 7).

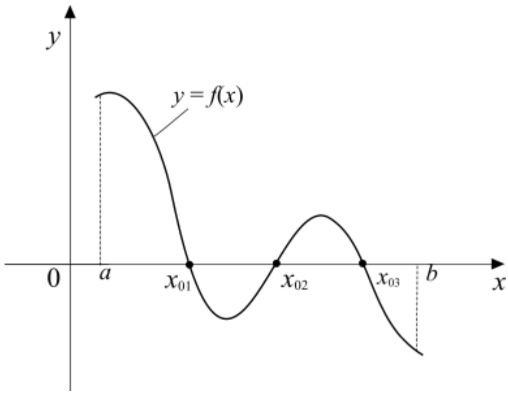


Figure 7 – The zeros of a function

Figure 7 shows three such points: x_{01}, x_{02}, x_{03}.

Property 3. If the function $f(x)$ is continuous in the intervals $[a, b]$ and m and M are the smallest and largest values of the function $f(x)$, respectively, then for any number μ that satisfies the inequality

$$m < \mu < M$$

in the segment $[a, b]$, there is at least one point $x = x_*$ that satisfies $f(x_*) = \mu$.

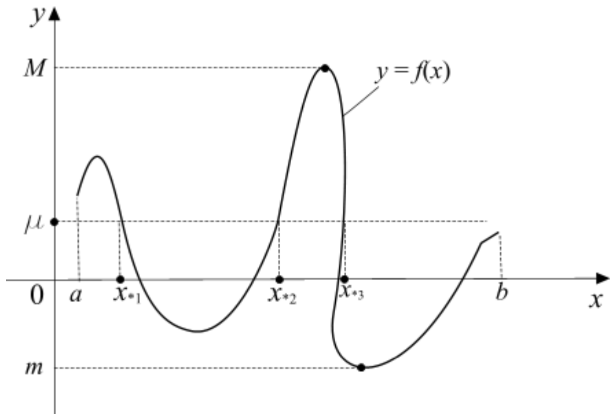


Figure 8 – The largest and smallest values of the function on the segment

There are three such points in Fig. 8: x_{*1}, x_{*2}, x_{*3}.

Differential Calculus:
Functions of a Single Variable

9. Definition of a derivative. Geometrical and mechanical interpretation of the derivative

Suppose that the function $y = b\,(x)$ is given on some interval $(a;\, b)$. Take any point $x\ (a;\, b)$ and give x an arbitrary increment $\Delta\, x$ such that the point $x + \Delta\, x$ also belongs to the interval $(a;\, b)$. Find the derivative of the function: $\Delta\, y = f\,(x + \Delta\, x) - f\,(x)$.

Definition. The derivative of the function $y = f\,(x)$ at the point x is the limit of the ratio of the increment of the function $\Delta\, y$ at this point to the increment of the argument $\Delta\, x$, when the increment of the argument goes to zero in any way.

The derivative of the function $y = f\,(x)$ at the point x is denoted by one of the following symbols: $f'(x);\ y';\ \frac{df}{dx};\ \frac{dy}{dx};\ y'_x$. Thus, by definition

$$f'_x = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x\Delta)-f(x)}{\Delta x}.$$

For each value x, the derivative $f'(x)$ is a complete value, i.e the derivative is also a function depending on x.

If the limit $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ at some point x does not exist, then derivative $f'(x)$ does not exist at this point either.

The operation of finding the derivative of the function $f(x)$ is called differentiation of this function.

In general, if the function $y = f(x)$ describes some process, then the derivative $y' = f'(x)$ is the rate of change of this process. Now define unilateral derivatives. Let the function $y = f(x)$ be defined around x.

Definition. If there is a limit

$$\lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} \frac{\Delta y}{\Delta x} = \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} \frac{f(x+\Delta x) - f(x)}{\Delta x},$$

then it is called the right derivative of $f(x)$ at the point x and it is denoted by

$$f'_+ = \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} \frac{\Delta y}{\Delta x} = \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Similarly, the left derivative is determined:

$$f'_- = \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} \frac{\Delta y}{\Delta x} = \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

If the function $f(x)$ is given on the segment $[a, b]$, then the derivative at point a means the right derivative, and at point b–the left derivative.

10. Differentiation of functions

Definition. The function $y = f(x)$ is said to be differentiated when $x = x_0$ if the function $f(x)$ has a derivative at the point $x = x_0$, i.e if it exists:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x)}{\Delta x} = f'(x_0).$$

Definition. The function $y = f(x)$ is said to be differentiated on the segment $[a, b]$ (intervals (a, b)) if it is differentiated at each point of this segment (interval).

Theorem 25. If the function $y = f(x)$ is differentiated at the point x_0, then it is continuous at this point.

It follows from this theorem that at breakpoints the function has no derivative.

The inverse statement is incorrect, i.e there are continuous functions that at some points are not differentiated.

11. Rules for differentiation of functions

Theorem 26. If $y = C$ where C is a constant number, then $y' = C' = 0$.

Theorem 27. If $y = x$, then $y' = x' = 1$.

Theorem 28. If the functions $U = U(x)$ and $V = V(x)$ are differentiated at the point x, then the function $U + V$ is also differentiated at this point and the formula is valid:

$$(U' + V') = U' + V'.$$

Theorem 29. If the functions $U = U(x)$ and $V = V(x)$ are differentiated at the point x, the function $U(x) \cdot V(x)$ is also differentiated at this point and is the formula valid:

$$(U' \cdot V') = U'V \cdot V'U.$$

Theorem 30. The constant factor can be taken out as a sign of the derivative, i.e

$$(C \cdot U') = C \cdot U'.$$

Theorem 31. If the functions $U = U(x)$ and $V = V(x)$ are differentiated at the point x, then the function $\frac{U(x)}{V(x)}$ $(V(x) \neq$ $\neq 0)$ is also differenced at this point and the formula is valid:

$$\left(\frac{U}{V}\right)' = \frac{U' \cdot V - U \cdot V'}{V^2}.$$

Theorem 32. (Derivative of a compound function). If the function $U = \varphi(x)$ has the derivative U'_x at the point x and the function $y = f(U)$ has the derivative y'_U at the corresponding point U, then the composite function $y = f(\varphi(x))$ has the derivative y'_x at the point x and the valid formula $y'_x = y'_U \cdot U'_x$ is valid.

12. Differentiation of trigonometric functions

Theorem 33. Derivatives of trigonometric functions are found by the formulas:

$$(\sin x)' = \cos x;$$

$$(\cos x)' = -\sin x;$$

$$(\mathrm{tg}\, x)' = \frac{1}{\cos^2 x}, \qquad x \neq \frac{\pi}{2} + \pi n, \quad n \in Z;$$

$$(\mathrm{ctg}\, x)' = -\frac{1}{\sin^2 x}, \qquad x \neq \pi k, \quad k \in Z.$$

13. Derivative of a logarithmic function

$$y = \log_a x \quad (a > 0, \quad a \neq 1) \ \text{ and } \ y = \ln|x|.$$

Theorem 34. Valid formulas:

$$(\log_a x)' = \frac{1}{x}\log_a e \quad \text{or} \quad (\log_a x)' = \frac{1}{x \ln a}\ ,$$

$$x > 0, \quad a > 0, \quad a \neq 1;$$

$$(\ln|x|)', \quad x \neq 0.$$

14. Differentiation of inverse functions

Theorem 35. If there is an inverse function $y = y\,(x)$ for the function $x = x\,(y)$ (i.e both functions are strictly monotonic at some intervals) and the function $x = x\,(y)$ has a nonzero derivative x'_y at the point y, then y corresponds to the point x, the function $y = y\,(x)$ has a derivative y'_x, and

$$y'_x = \frac{1}{x'_y}.$$

Now consider the function $y = \arcsin x$ $\left(|x| < 1, |y| < \frac{\pi}{2}\right)$.

This is the inverse function for the function $x = \sin y$. By theorem we have

$$(\arcsin x)'_x = y'_x = \frac{1}{(\sin y)'_y} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} =$$

$$= \frac{1}{\sqrt{1-x^2}},$$

because cos y = 0 for $|y| < \frac{\pi}{2}$.

Thus, we obtain the formula:

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \quad \text{for} \quad |x| < 1.$$

Now consider the function $y = \arccos x$ ($|x| < 1$, $0 < y < \pi$) – the inverse function for the function x = cos y.

According to theorem

$$(\arccos x)'_x = y'_x = \frac{1}{(\cos y)'_y} = \frac{1}{-\sin x} = -\frac{1}{\sqrt{1-\cos^2 y}} =$$

$$= -\frac{1}{\sqrt{1-x^2}},$$

because sin$y$ > 0 for 0 < y < π.

Another equality is proved:

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}, \quad \text{for} \quad |x| < 1.$$

Function $y = arctg\, x$ $(-\infty < x < +\infty, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$

is an inverse function for the function x = tg x. Then

$$(arctg\, x)'_x = \frac{1}{(tg\, y)'_y} = cos^2 x = \frac{1}{1+x^2}.$$

So, $(arctg\, x)' = \frac{1}{1+x^2}$.

Function $y = arctg\,x\ (-\infty < x < +\infty, \quad 0 < y < \pi)$ is an inverse function for the function $x = ctg\,x$.

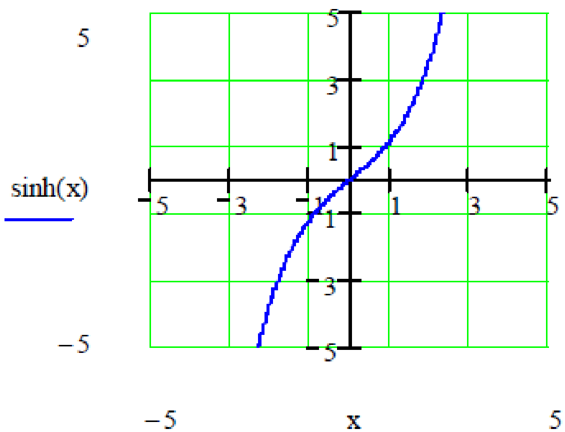
$$(arcctg\,x)'_x = \frac{1}{(ctg\,y)'_y} = -sin^2 x = -\frac{1}{1+x^2}.$$

We have proved that $(arcctg\,x)' = -\frac{1}{1+x^2}$.

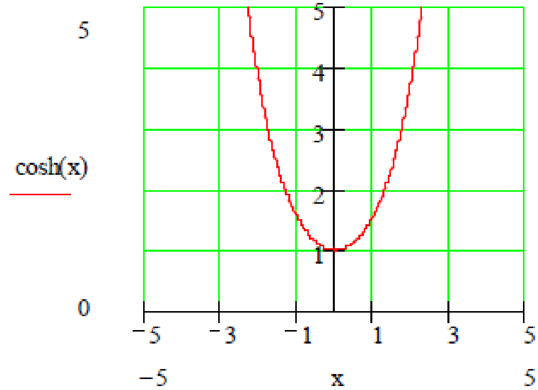
15. Differentiation of hyperbolic functions

Definition. Hyperbolic sine *sh x*, cosine *ch x*, tangent *thx* and cotangent *cth x* are functions that are determined by the following formulas:

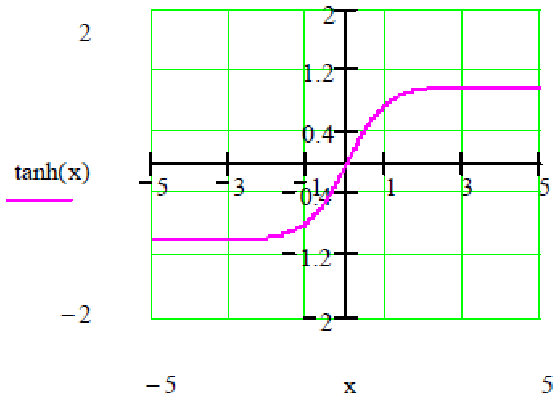
$$shx = \frac{e^x - e^{-x}}{2}$$



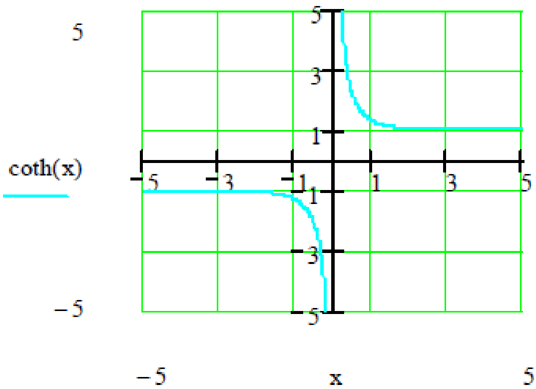
$$chx = \frac{e^x + e^{-x}}{2}$$



$$thx = \frac{shx}{chx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



$$cthx = \frac{chx}{shx} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \quad x \neq 0$$



The functions $y = sh\,x$, $y = ch\,x$, $y = th\,x$ are defined for all real x, and the function $y = cth\,x$ for all real x except $x = 0$.

Hyperbolic functions are related by as follows:

1. $ch^2x - sh^2x = 1;$

2. $sh(a+b) = sha \cdot chb + chb \cdot sha;$

3. $sh(a-b) = sha \cdot chb - chb \cdot sha;$

4. $ch(a+b) = cha \cdot chb + sha \cdot shb;$

5. $ch(a-b) = cha \cdot chb - sha \cdot shb;$

6. $sh2x = 2shx \cdot chx;$

7. $ch2x = ch^2x + sh^2x;$

8. $th(a+b) = \frac{tha+thb}{1+tha \cdot thb};$

9. $th(a-b) = \frac{tha-thb}{1-tha \cdot thb};$

10. $ch(a+b) = \frac{1+cha \cdot chb}{cha+chb};$

11. $ch(a-b) = \frac{1-cha \cdot chb}{cha-chb};$

12. $1 - th^2x = \frac{1}{ch^2x};$

13. $1 - ch^2x = -\frac{1}{sh^2x}.$

Theorem 36. Derivatives of hyperbolic functions are determined by formulas:

$$(sh\ x)' = ch\ x;$$

$$(ch\ x)' = sh\ x;$$

$$(th\ x)' = \frac{1}{ch^{\ 2}x};$$

$$(ch\ x)' = -\frac{1}{sh^{\ 2}x}, \quad x \neq 0.$$

16. Table of derivatives

The previous paragraphs provide formulas by which you can find derivatives without using the definitions of the derivative.

We assume that $U = U\ (x)$ and $V = V\,(x)$ are differentiated functions, and C – a constant value.

We write down the rules of differentiation and derivatives of basic elementary functions in the table:

1. $(u+v)' = u' + v'$;

2. $(u \cdot v)' = u'v + uv'$;

3. $(C \cdot u)' = C \cdot (u)'$;

4. $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}, \quad v \neq 0$;

5. $y'_x = y'_u \cdot u'_x,$ if $y = f(u)$ and $u = u(x)$;

6. $C' = 0$;

7. $x' = 1$;

8. $(u^\alpha)' = \alpha \cdot u^{\alpha-1} \cdot u'$;

9. $(a^u)' = a^u \cdot \ln a \cdot u', \quad a > 0, \ a \neq 1$;

10. $(e^u)' = e^u \cdot u'$;

11. $(\log_a u)' = \frac{1}{u \ln a} \cdot u', \ a > 0, \ \ a \neq 1$;

12. $(\ln u)' = \frac{1}{u} \cdot u'$;

13. $(\sin u)' = \cos u \cdot u'$;

14. $(\cos u)' = -\sin u \cdot u'$;

15. $(\operatorname{tg} u)' = \frac{1}{\cos^2 u} \cdot u'$;

16. $(\operatorname{ctg} u)' = -\frac{1}{\sin^2 u} \cdot u'$;

17. $(\operatorname{sh} u)' = \operatorname{ch} u \cdot u'$;

18. $(\operatorname{ch} u)' = \operatorname{sh} u \cdot u'$;

19. $(\operatorname{th} u)' = \frac{1}{\operatorname{ch}^2 u} \cdot u'$;

20. $(\operatorname{cth} u)' = -\frac{1}{\operatorname{sh}^2 u} \cdot u'$;

21. $(\arcsin u)' = \frac{1}{\sqrt{1-u^2}} \cdot u'$;

22. $(\arccos u)' = -\frac{1}{\sqrt{1-u^2}} \cdot u'$;

23. $(\operatorname{arctg} u)' = \frac{1}{1+u^2} \cdot u'$;

24. $(\operatorname{arcctg} u)' = -\frac{1}{1+u^2} \cdot u'$.

17. Implicit differentiation

Let the differentiated function $y(x)$ be given implicitly by the equation:

$$F(x, y) = 0.$$

To differentiate an implicit function, we need to take the derivative of x from both parts of this equation, assuming that y is a function of x, and solve the resulting equation with respect to y'.

18. Parametric differentiation

Let the differentiated function y from the variable x be given by parametric equations:

$$\begin{cases} x = \varphi(t) \\ y = \psi(t), \end{cases} \quad \alpha < t < \beta \, .$$

Suppose that the functions $x = \varphi(t)$ and $y = \psi(t)$ have derivatives of the variable t, and the function $x = \varphi(t)$ has an inverse function $t = \Phi(x)$ which also has a derivative (of the variable x). Then the parametrically given function y of x can be considered as a composite function $y = \Psi(t)$, where $t = \Phi(x)$ and $y'_x = y'_t \cdot t'_x = y'_t \cdot \frac{1}{x'_t} = \frac{y'_t}{x'_t}$. Therefore $y'_x = \frac{y'_t}{x'_t}$.

19. Differential of a function. Properties of the differential

Let the function $y = f(x)$ be differentiated at the point $x \in [a, b]$, i.e at this point it has a derivative:

$$y'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x},$$

where

$$\frac{\Delta y}{\Delta x} = y'(x) + \alpha, \quad \alpha \to 0 \quad \text{at} \quad \Delta x \to 0,$$

or

$$\Delta y = y'(x) + \Delta x + \alpha \cdot \Delta x.$$

The first of the terms is linear with respect to Δx and for $\Delta x \to 0$ and $f'(x) \neq 0$ is infinitesimally small of the same order of Δx, because $\lim_{\Delta x \to 0} \frac{f'(x)}{\Delta x} = f'(x)$; and the second term is infinitely small of a higher order than Δx, because $\lim_{\Delta x \to 0} \frac{\alpha \cdot \Delta x}{\Delta x} = \lim_{\Delta x \to 0} 2 = 0$. Thus, the first term is the main part of the increment of the function, linear with respect to the increment of the argument.

Definition. The differential dy of the function $y = y(x)$ at the point x is called the principal, linear with respect to Δx, part of the increment of the function $y(x)$ at this point:

$$dy = y'(x) \cdot \Delta x.$$

The differential dy is also called a first-order differential.

If $y = x$, then $y' = x' = 1$, so $dy = dx = \Delta x$, i.e the differential dx of the independent variable x coincides with its increment Δx. Therefore, the differential dy can be written as: $dy = y'(x)\,dx$.

Let's find out the geometric interpretation of the differential

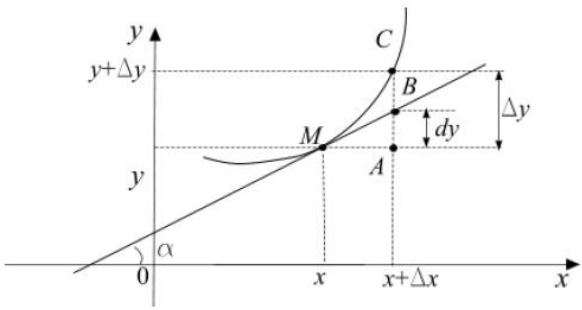


Figure 9 – Differential of a function

$$AC = \Delta y, \qquad \text{from } \Delta AMB : \frac{AB}{AM} = tg\alpha$$

$$\Longrightarrow AB = AM \cdot tg\alpha = \Delta x \cdot y'(x) = y'(x)dx = dy,$$

when it is clear that the differential of the function *y* (*x*) at given values of *x* is equal to the increment of the ordinate tangent to the curve *y* = *y* (*x*) at the point *x*. The increment of the function is equal to the increment of the ordinate of the curve (Fig. 9).

Since the differential of the function is equal to the product of its derivative and the differential of the independent variable, the properties of the differential immediately follow from the corresponding properties of the derivative (differentiation rules). If *u* and *v* are differentiated functions from *x*, *C* is a constant, then we write down the rules for finding differentials:

1. $dC = 0$;

2. $d(u + v) = du + dv$;

3. $d(u \cdot v) = v \cdot du + u \cdot dv$;

4. $d(C \cdot u) = C \cdot du$;

5. $d\left(\frac{u}{v}\right) = \frac{v \cdot du - u \cdot dv}{v^2}$.

If *y* = *y* (*u*) is a complex function, where *u* = *u* (*x*), and *y* (*u*), *u* (*x*) are differentiated at points *u* and *x*, then there is a derivative $y'_x = y'_u \cdot u'_x$. Hence, there is a differential $dy = y'_x \cdot dx = y'_u \cdot u'_x \cdot dx = y'_u \cdot du$, i.e $dy = y'_u \cdot du$.

We see that the first-order differential of the function is determined by the same formula, regardless of whether the variable function is an independent variable or it is a function of another variable. This property of the differential is called the form invariance (invariance) of differential equation.

Now consider the application of the differential in approximate calculations. Since the differential *dy* of the function $y = y(x)$ at the point x is the principal, linear with respect to Δx, then the part of the increment of the function $y(x)$ at this point:

$$\Delta y \approx dy,$$

or

$$y(x + \Delta x) - y(x) \approx y'(x) \cdot dx,$$

where

$$y(x + \Delta x) \approx y(x) + y'(x) \cdot \Delta x.$$

This is the formula for approximate calculations.

20. Higher–order derivatives and differentials

Let the function $y = y(x)$ be differentiated on the interval $(a; b)$. Then its derivative $y'(x)$ (first-order derivative) is also a function of x. If the function $y'(x)$ also has a derivative in the interval $(a; b)$ or at some point x is $(a;$

b), then this last derivative is called a second-order derivative and is denoted as follows:

$$y'', \qquad \frac{d^2y}{dx^2}, \qquad \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

A derivative of a second-order derivative, if it exists, is called of derivative of the third order and it is denoted as follows:

$$y''', \qquad \frac{d^3y}{dx^3}, \qquad \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right).$$

The derivative of the *n*–th order of the function *y* = *y* (*x*) is called the derivative of the first order, if it exists, from the derivative of the (*n*–1) – th order and is denoted as follows:

$$y^{(n)}, \qquad \frac{d^{(n)}y}{dx^{(n)}}, \qquad \frac{d}{dx}\left(\frac{d^{(n-1)}y}{dx^{(n-1)}}\right).$$

Derivatives of the order above the first order are called the derivatives of the higher order.

For *n*-th derivatives there are formulas:

1. $(u+v)^{(n)} = u^{(n)} + v^{(n)}$;

2. $(C \cdot u)^{(n)} = C \cdot u^{(n)}$

3. $(u \cdot v)^{(n)} = u^{(n)} \cdot v + n \cdot u^{(n-1)} \cdot v' + \frac{n(n-1)}{2!} u^{(n-2)} \cdot v'' + \frac{n(n-1)(n-2)}{3!} u^{(n-3)} \cdot v''' + \cdots + nu'v^{(n-1)} + u \cdot v^{(n)}$.

Formula (3) is called the Leibniz formula.

If the function $y = y(x)$ is implicitly given by the equality $F(x; y) = 0$, differentiating this equality by x and solving the obtained equation with respect to y', we find a derivative of the first order. To find the derivative of the second order, it is necessary to differentiate by x the derivative of the first order and to substitute its value in the received relation. Continuing the differentiation, you can find one after another successive derivative of any order. All of them will be expressed through the independent variable x and the function y itself.

If the function $y = y(x)$ is given parametrically by equations $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$, $t \in (\alpha; \beta)$ and has a derivative of the first order $\frac{dy}{dx} = \frac{y'_t}{x'_t}$, then the derivative of the second order from the function $y = y(x)$, if it exists, is determined by the formula $\frac{d^2y}{dx^2} = \frac{\left(\frac{dy}{dx}\right)'_t}{x'_t}$

$$\left(\text{really, } \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)'_x = \left(\frac{dy}{dx}\right)'_t \cdot t'_x = \frac{\left(\frac{dy}{dx}\right)'_t}{x'_t}\right).$$

Similarly, find the derivative of the *n*-th order ($n > 2$);

$$\frac{d^ny}{dx^n} = \frac{\left(\frac{d^{(n-1)}y}{dx^{(n-1)}}\right)'_t}{x'_t}.$$

The differential of the second order d^2y of the twice differentiated function $y = y(x)$ is called the differential from the differential of the first order:

$$d^2y = d(dy).$$

Since dx does not depend on x, then when differentiating the first-order differential dx can be taken as the sign of the derivative. Therefore, we have

$$d^2y = d(dy) = d(y'(x)dx) = (y'(x)dx)'_x \cdot dx$$
$$= y''(x)dxdx = y''(x)dx^2.$$

that is

$$d^2y = y''(x)dx^2.$$

A third-order differential d^3y, if it exists, is the differential from a second-order differential:

$$d^3y = d(d^2y) = d(y''(x)dx^2) = y'''(x)dx^3.$$

A differential of the nth order d n y, if it exists, is called the differential from differential (n–1)-th order:

$$d^n y = d\left(d^{(n-1)} y\right) = y^{(n)}(x) dx^n.$$

Higher-order differentials do not have invariant properties. We show this by the example of a second-order differential. Suppose that a doubly differentiated compound function $y = y(u)$ is given, where $u = u(x)$. Find for it a second-order differential:

$$\begin{aligned} d^2 y = d(dy) = d(y'_u \cdot du) =. \, d(y'_u) \cdot du + y'_u d(du) \\ = y''_{uu} \cdot dudu + y'_u \cdot d^2 u \\ = y''_{uu} \cdot du^2 + y'_u \cdot u''_{xx} \cdot dx^2, \end{aligned}$$

that is

$$d^2 y = y''_{uu} \cdot du^2 + y'_u \cdot u''_{xx} \cdot dx^2.$$

Therefore, the second-order differential has no invariant properties.

The Study of Functions

21. Increasing and decreasing functions. Extreme points

Let the function $f(x)$ be defined on some interval (a, b), and x_0 is the inner point of this interval.

Definition. The function $f(x)$ is called ascending at the point x_0 if there is an environment $(x_0 - \delta; x_0 + \delta)$, $\delta > 0$, point x_0, which is contained in the interval (a, b), and such that $f(x) < f(x_0)$ for all $x \in (x_0 - \delta; x_0)$ and $f(x) > f(x_0)$ for all $x \in (x_0; x_0 + \delta)$.

Definition. If there is such an environment $(x_0 - \delta; x_0 + \delta)$, $\delta > 0$ of the point x_0 in the interval (a, b) and $f(x) < f(x_0)$ for all $x \in (x_0 - \delta) \cup (x_0 + \delta)$, the point x_0 is called the point of maximum of the function $f(x)$, and the number $f(x_0)$ is called the maximum function $f(x)$.

Definition. If there is such a circle $(x_0 - \delta; x_0 + \delta)$, $\delta > 0$, point x_0, which is contained in the interval (a, b), and $f(x) < f(x_0)$ for all $x \in (x_0 - \delta; x_0)$, and $f(x) > f(x_0)$, $x \neq x_0$, then the point x_0 is called the minimum point of the function $f(x)$, and the number $f(x_0)$ is called the minimum of the function $f(x)$.

Note that the points of maximum and minimum of the function are called extreme points or the maximum and minimum extremes of the function.

Definition. If a function is increasing (decreasing) at each internal point of the interval (a, b), then it is called increasing (decreasing) on this interval.

Theorem 37. (Sufficient signs of growth (decline) of the function at the point). If the function $f(x)$ at the interior point x_0 of the interval (a, b) has a derivative $f'(x_0)$ and $f'(x_0) > 0$ ($f'(x_0) < 0$), then the function $f(x)$ at the point x_0 increases (decreases).

22. Finding the largest and smallest values of the function on the segment

Suppose that a continuous function $f(x)$ is given on the interval $(a; b)$. Then, according to the Weierstrass theorem, the function on this segment reaches its largest and smallest values. However, the Weierstrass theorem does not give a way to define those points of the segment $(a; b)$ in which the function reaches its largest (smallest) value. The theorem only states that such points exist. This can be both the inner points of the segment and its end points. Fig.10 shows a graph of a continuous function, which at the inner point c_1 of the segment (a; b) acquires the largest value, and at the inner point c_2 – the smallest value.

Fig. 10.2 shows a graph of the function, which at the ends of the segment acquires the smallest and largest values.

However, it may be that one of the values of the function acquires inside the segment, and the other – at one end. Thus, Figure 10.3 shows a graph of a continuous function, which at the left end of the segment (point a) acquires the smallest value, and at the inner point (point c) – the largest value.

If the function acquires the largest (smallest) value within the segment, then this largest (smallest) value is both the local maximum (minimum) of the given function. Hence the way to find the points at which the function acquires the largest (smallest) value.

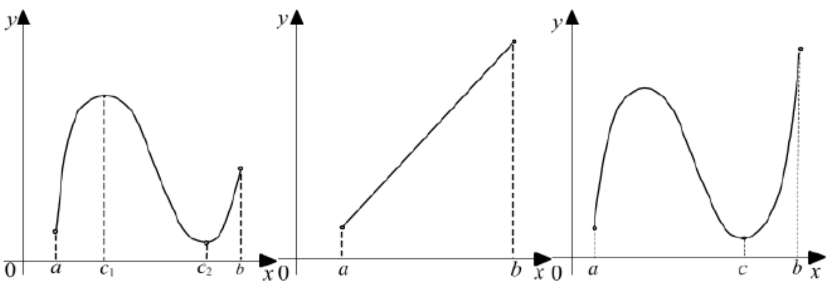


Figure 10 – Function graphs

To find the largest (smallest) value of a continuous function on the segment (a; b), it is necessary to find all local maxima (minima) and compare them with the values of the function, which it acquires at the ends of the segment. The largest (smallest) number among the found numbers will be the largest (smallest) value of the function on the segment (a; b).

23. Conditional extremum

Let the function $z = f\,(x\,;y)$ be definite and continuous in a closed domain D.

Then at some points in this area, it reaches its greatest and least importance.

These values are achieved by the function at the inner points of the segment or at points lying on the boundary of the region.

Rule of finding the largest and smallest values of the function:

1) Find all the critical points of the function belonging to a given area, and calculate the value of the function in them.

2) Find the largest and smallest values of the function at the boundaries of the domain.

3) Compare all the found values of the function and choose from them the largest and smallest values.

Conditional Extremum of the Function of Several Variables

The conditional extremum of the function $z = f(x; y)$ is the extremum of this function, reached under the condition that the arguments x and y are connected by the equation $g(x; y) = C$.

The equation $g(x; y) = C$ is called the *coupling equation*.

Geometric interpretation: the choice of the largest (smallest) value among the points, lying on the line defined by the connection equation.

To find a conditional extremum, it is necessary to express one variable by another from the connection equation: $y = \varphi(x)$.

Substitute this expression by a function of two variables and obtain the function of one variable:

$$z = f(x, y) = f(x, \varphi(x)).$$

Its extremum will be the conditional extremum of the function $z = f(x; y)$.

24. Convexity and concavity of graphs. Inflection points

Let the curve be given by the equation $y = f(x)$, where $f(x)$ is a continuous function that has a continuous derivative $f'(x)$ on some interval $[a; b]$. Then at each point of such curve you can draw a tangent. Such curves are called smooth.

Take an arbitrary point on the curve $M_0(x_0; y_0)$, where $x_0 \in \langle a; b \rangle$, $y_0 = f(x_0)$.

Definition 1. If there is a circle around $(x_0 - \delta;\ x_0 + \delta) \subset \langle a; b \rangle$ the point x_0 such that for all $(x_0 - \delta;\ x_0 + \delta)(x \neq x_0)$ the corresponding points of the curve lie above the tangent drawn to the curve at the point M_0, then the curve at the point M_0 is called concave upwards (Fig. 11).

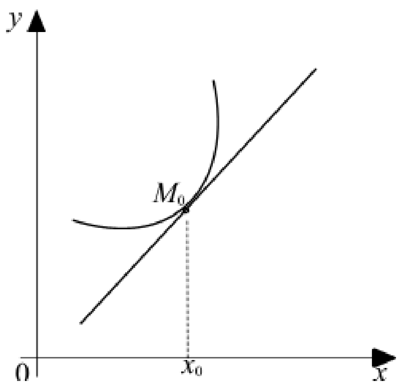


Figure 11 – Concave upwards

Definition 2. If there is a circle around $(x_0 - \delta;\ x_0 + \delta) \subset \langle a; b\rangle$ the point x_0 such that for all $(x_0 - \delta;\ x_0 + \delta)(x \neq x_0)$ the corresponding points of the curve lie below the tangent drawn to the curve at the point M_0, the curve at the point M_0 is called concave downwards (Fig. 12).

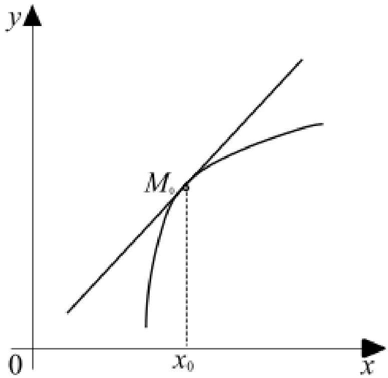


Figure 12 – Concave downwards

Definition 3. A point M_0 is called the point of inflection of the curve if there is a circle around $(x_0 - \delta;\ x_0 + \delta) \subset (a; b)$ the point X_0 such that for all $x \in (x_0\ ;\ x_0 + \delta)$ curve concave downwards (upwards) (Fig. 13).

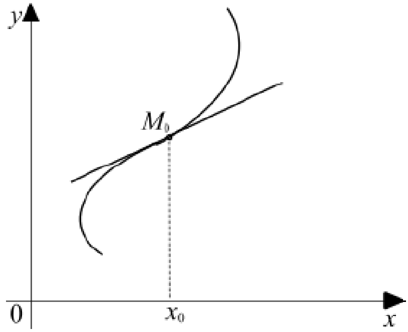


Figure 13 – Concave upwards

If the curve is given by the equation $y = f(x)$ and is concave upwards at each point of some interval, then it is called concave on this interval. If the curve at each point of the gap is concave downwards, it is called convex at this interval.

Therefore, the curve shown in Fig. 14 is concave. The curve shown in Fig. 15 is convex.

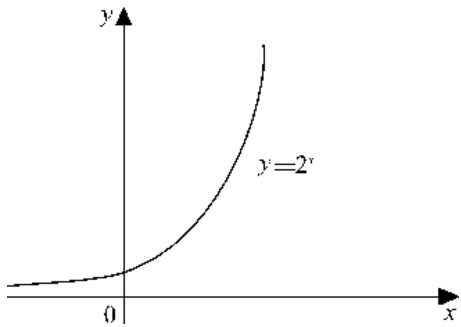


Figure 14 – Concave curve

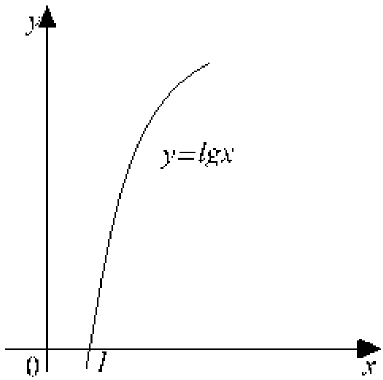


Figure 15 – Convex curve

The inflection point of the curve is still defined as the point at which the curve changes its type of concavity.

Theorem. Let the curve be given by the equation $y = f(x)$ and let there be a circle around $(x_0 - \delta;\ x_0 + \delta) \subset \subset \langle a; b \rangle$ the point x_0 such that the function $f(x)$ for each $x \in \in (x_0 - \delta;\ x_0 + \delta)$ has derivatives up to and including the second order, and $f'(x)$ at the point x_0 is a continuous function. Then, if $f''(x_0) > 0$, then the curve at the point $M_0(x_0; f(x_0))$ is concave upwards; if $f''(x_0) < 0$, then the curve at the point $M_0(x_0; f(x_0))$ is concave downwards.

So we have the following rule for finding the inflection points of the curve given by the equation $y = f(x)$.

In order to find the inflection points of the curve given by the equation $y = f(x)$, we must:

1) find the derivative of the second order $f''(x)$ and equate this derivative to zero;

2) among the roots of the equation $f''(x) = 0$ choose only the real roots and those that belong to the domain of the function; in the vicinity of each selected root determine the sign of the derivative of the second order $f''(x)$, first at values of x less than the considered root and then at values of x greater than this root. If at the transition x through the selected root x_0 the derivative $f''(x)$ changes sign, then the point $M_0(x_0; f(x_0))$ is the inflection point of a given curve. If the sign of the second-order derivative does not change when x passes through x_0, then $M_0(x_0; f(x_0))$ is not the inflection point of the curve.

Note that the rule for finding inflection points is similar to the first rule for finding extreme points. The only difference is that when finding extreme points, the change in the sign of the first-order derivative is checked, whereas when the inflection points are found, the change in the sign of the second-order derivative is checked.

References

1. Atanasyan L. S. Geometry : tutorial / L. S. Atanasyan, V. T. Bazylev. – Moscow : Education, 1986. – 336 p.
2. Davydov N. A. Course of mathematical analysi: tutorial / N. A. Davydov. – Moscow : Education, 1973. – 351 p.
3. Mathematical analysis in examples : textbook / L. I. Dyuzhenkova, T. V. Kolesnik, M. Ya. Lyashchenko et al. – Kyiv : Higher School, 2003. – Vol. 1. – 470 p.
4. Zhaldak M. I. Computer in mathematics lessons : guide for teachers / M. I. Zhaldak. – Kyiv : RNNC "Dinit", 2003. – 324 p.
5. Shkil M. I. Mathematical analysis : textbook : in 2 hours / M. I. Shkil. – Kyiv : Higher School, 1995. – Part 1. – 510 p.
6. Oak V. P. Higher Mathematics : textbook / V. P. Oak, I. I. Yurik. – Kyiv : А.С.К., 2006. – 648 p.
7. Lytvyn I. I. Higher mathematics : teaching / I. I. Lytvyn, O. M. Konopchuk, G. O. Zheliznyak. – Kyiv : Center for Educational Literature, 2004. – 368 p.
8. Shinkarika M. I. Higher mathematics : textbook / M. I. Shinkarika. – Ternopil, 2003. – 480 p.

9. Klepko V. Y. Higher mathematics in examples and problems : textbook / V. Y. Klepko, V. L Golets. – Kyiv : Center for Educational Literature, 2009. – 590 p.
10. Arkhipova O. S. A guide to solving typical problems in higher mathematics / O. S. Arkhipova, V. P. Protopopova, E. S. Pakhomova. – Kharkiv : KNAMG, 2008. – 210 p.

Навчальне видання

Бадалян Анна Юріївна,
Козлова Ірина Іванівна

МАТЕМАТИЧНИЙ АНАЛІЗ

Конспект лекцій

для студентів спеціальності *113 «Прикладна математика»*
денної форми навчання

(Англійською мовою)

Відповідальний за випуск І. В. Коплик
Редактор С. В. Чечоткіна
Комп'ютерне верстання А. Ю. Бадалян

Підписано до друку 11.03.2021, поз. 39.
Формат 60×84/16. Ум. друк. арк. 3,49. Обл.-вид. арк. 3,13. Тираж 5 пр. Зам. №

Видавець і виготовлювач
Сумський державний університет,
вул. Римського-Корсакова, 2, м. Суми, 40007
Свідоцтво суб'єкта видавничої справи ДК № 3062 від 17.12.2007.