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MATHEMATICAL MODELS IN PHYSICS

In Three Parts

Part 1. MATHEMATICAL MODELS IN MECHANICS

Lecture notes

Sumy
Sumy State University
2023

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Part 1. MATHEMATICAL MODELS IN MECHANICS

Lecture notes

For students of specialty 113 “Applied Mathematics”
Full-time course of study

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Department of Applied Mathematics and Complex Systems Modeling

CONTENTS

1 UNITS, PHYSICAL QUANTITIES, AND VECTORS

Physics is one of the most fundamental sciences. Scientists of all disciplines use the ideas of physics, including chemists who study the structure of molecules, paleontologists who try to reconstruct how dinosaurs walked, and climatologists who study how human activities affect the atmosphere and oceans. Physics is also the foundation of all engineering and technology. No engineer could design a flat-screen TV, a prosthetic leg, or even a better mousetrap without first understanding the basic laws of physics.

The study of physics is also an adventure. You'll find it challenging, sometimes frustrating, occasionally painful, and often richly rewarding. If you have ever wondered why the sky is blue, how radio waves can travel through empty space, or how a satellite stays in orbit, you can find the answers by using fundamental physics. You'll come to see physics as a towering achievement of the human intellect in its quest to understand our world and ourselves.

In this opening chapter, we'll go over some important preliminaries that we'll need throughout our study. We'll discuss the nature of physical theory and the use of idealized models to represent physical systems. We'll introduce the systems of units used to describe physical quantities and discuss ways to describe the accuracy of a number. We'll look at examples of problems for which we can't (or don't want to) find a precise answer, but for which rough estimates can be useful and interesting. Finally, we'll study several aspects of vectors and vector algebra. We'll need vectors throughout our study of physics to help us describe and analyze physical quantities, such as velocity and force, that have direction as well as magnitude.

1.1 The Nature of Physics

Physics is an *experimental* science. Physicists observe the phenomena of nature and try to find patterns that relate these phenomena. These patterns are called physical theories or, when they are very well established and widely used, physical laws or principles.

CAUSION! The meaning of "theory": A theory is *not* just a random thought or an unproven concept. Rather, a theory is an explanation of natural phenomena based on observation and accepted fundamental principles. An example is the well-established theory of biological evolution, which is the result of extensive research and observation by generations of biologists.

To develop a physical theory, a physicist has to ask appropriate questions, design experiments to try to answer the questions, and draw appropriate conclusions from the results.

The development of physical theories often takes an indirect path, with blind alleys, wrong guesses, and the discarding of unsuccessful theories in favor of more promising ones. Physics is not simply a collection of facts and principles; it is alsothe *process* by which we arrive at general principles that describe how the physical universe behaves.

No theory is ever regarded as the ultimate truth. It's always possible that new observations will require that a theory be revised or discarded. Note that we can disprove a theoryby finding behavior that is inconsistent with it, but we can never prove that a theory is always correct.

1.2 Solving Physics Problems

At some point in their studies, almost all students find themselves thinking, "I understand the concepts, but I just can't solve the problems". But in physics, truly understanding a concept *means* being able to apply it to a variety of problems. Learning how to solve problems is absolutely essential; you don't *know* physics unless you can *do* physics.

How do you learn to solve physics problems? In every chapter of this book, you'll find *Problem-Solving Strategies* that offer techniques for setting up and solving problems efficiently and accurately. Following each *Problem-Solving Strategy* are one or more worked *Examples* that show these techniques in action. The *Problem-Solving Strategies* will also steer you away from some *incorrect* techniques that you may be tempted to use. You'll also find additional examples that aren't associated with a particular

Problem-Solving Strategy. In addition, at the end of each chapter, you'll find a *Bridging Problem* that uses more than one of the key ideas from the chapter. Study these strategies and problems carefully, and work through each example for yourself on a piece of paper.

Different techniques are useful for solving different kinds of physics problems, which is why this book offers dozens of *Problem-Solving Strategies.* No matter what kind of problem you're dealing with, however, there are certain key steps that you'll always follow. These same steps are equally useful for problems in math, engineering, chemistry,and many other fields. In this book we've organized these steps into four stages of solving a problem.

All of the *Problem-Solving Strategies* and *Examples* in this book will follow these four steps. In some cases we'll combine the first two or three steps. We encourage you to follow these same steps when you solve problems yourself. You may find it useful to remember the acronym ***ISEE*** — short for *Identify, Set up, Execute,* and *Evaluate.*

PROBLEM-SOLVING STRATEGY
1.1 Solving Physics Problems

IDENTIFY *the relevant concepts:*

- Use the physical conditions stated in the problem to help you decide which physics concepts are relevant.
- Identify the **target variables** of the problem — that is, the quantities whose values you're trying to find, such as the speed at which a projectile hits the ground, the intensity of a sound made by a siren, or the size of an image made by a lens.
- Identify the known quantities, as stated or implied in the problem. This step is essential whether the problem asks for an algebraic expression or a numerical answer.

SET UP *the problem*:

- Given the concepts, known quantities, and target variables that you found in the IDENTIFY step, choose the equations that you'll use to solve the problem and decide how you'll use them. Study the worked examples in this book for tips on how to select the proper equations. If this seems challenging, don't worry — you will get better with practice!
- Make sure that the variables you have identified correlate exactly with those in the equations.
- If appropriate, draw a sketch of the situation described in the problem. (Graph paper and a ruler will help you make clear, useful sketches).

EXECUTE *the solution*:

- Here's where you'll "do the math" with the equations that you selected in the SET UP step to solve for the target variables that you found in the IDENTIFY step. Study the worked examples to see what's involved in this step.

EVALUATE *your answer:*

- Check your answer from the SOLVE step to see if it's reasonable. (If you're calculating how high a thrown baseball goes, an answer of 1.0 mm is unreasonably small and an answer of 100 km is unreasonably large). If your answer includes an algebraic expression, confirm that it correctly represents what would happen if the variables in it had very large or very small values.

•   For future reference, make note of any answer that represents a quantity of particular significance. Ask yourself how you might answer a more general or more difficult version of the problem you have just solved.

Idealized Models

In everyday conversation we use the word “model” to mean either a small-scale replica, such as a model railroad, or a person who displays articles of clothing (or the absence thereof). In physics a **model** is a simplified version of a physical system that would be too complicated to analyze in full detail.

For example, suppose we want to analyze the motion of a thrown baseball (**Fig. 1.1a**).How complicated is this problem? The ball is not a perfect sphere (it has raised seams),and it spins as it moves through the air. Air resistance and wind influence its motion, the ball’s weight varies a little as its altitude changes, and so on. If we try to include all these effects, the analysis gets hopelessly complicated. Instead, we invent a simplified version of the problem. We ignore the size, shape, and rotation of the ball by representing it as a point object, or **particle.** We ignore air resistance by making the ball move in a vacuum, and make the weight constant. Now we have a problem which is simple enough to deal with (Fig. 1.1b). We’ll analyze this model in detail in Chapter 3. We have to overlook quite a few minor effects to make an idealized model, but we must be careful not to neglect too much. If we ignore the effects of gravity completely, then our model predicts that when we throw the ball up, it will go in a straight line and disappear into space. A useful model simplifies a problem enough to make it manageable, yet keeps its essential features.

The validity of the predictions we make using a model is limited by the validity of the model. For example, Galileo’s prediction about falling objects (see Section 1.1) corresponds to an idealized model that does not include the effects of air resistance. This model works fairly well for a dropped cannonball, but not so well for a feather.

Idealized models play a crucial role throughout this book. Watch for them in discussions of physical theories and their applications to specific problems.

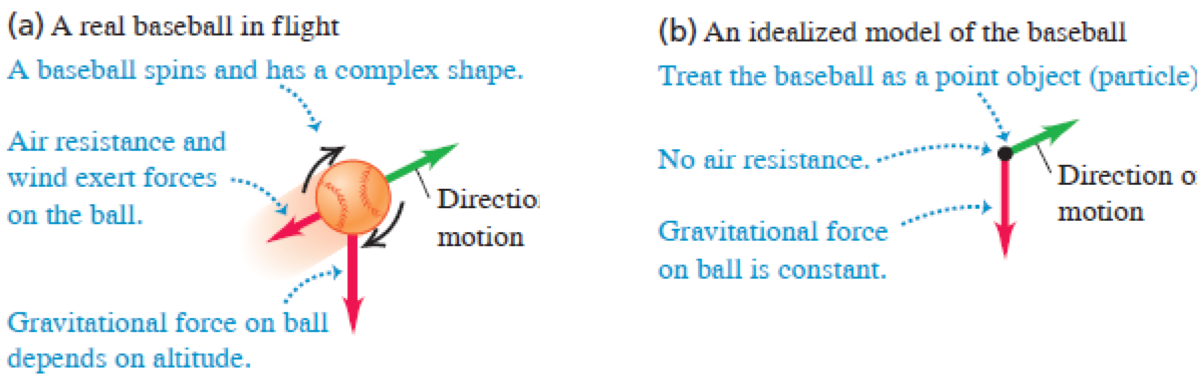


Figure 1.1 - To simplify the analysis of (a) a baseball in flight, we use (b) an idealized model

1.1 Standards and Units

As we learned in Section 1.1, physics is an experimental science. Experiments require measurements, and we generally use numbers to describe the results of measurements. Any number that is used to describe a physical phenomenon quantitatively is called a **physical quantity.** For example, two physical quantities that describe you are your weight and your height. Some physical quantities are so fundamental that we can define them only by describing how to measure them. Such a definition is called an **operational definition.** Two examples are measuring a distance by using a ruler and measuring a time interval by using a stopwatch. In other cases we define a physical quantity by describing how to calculate it from other quantities that we *can* measure. Thus we might define the average speed of a moving object as the distance traveled (measured with aruler) divided by the time of travel (measured with a stopwatch).

When we measure a quantity, we always compare it with some reference standard. When we say that a basketball hoop is 3.05 meters above the ground, we mean that this distance is 3.05 times as long as a meter stick, which we define to be 1 meter long. Such a standard defines a **unit** of the quantity. The meter is a unit of distance, and the second is a unit of time.When we use a number to describe a physical quantity, we must always specify the unit that we are using; to describe a distance as simply “3.05” wouldn’t mean anything.

To make accurate, reliable measurements, we need units of measurement that do not change and that can be duplicated by observers in various locations. The system of units used by scientists and engineers around the world is commonly called “the metric system,” but since 1960 it has been known officially as the **International System,** or **SI** (the abbreviationfor its French name, *Système International*).

Time

From 1889 until 1967, the unit of time was defined as a certain fraction of the mean solar day, the average time between successive arrivals of the sun at its highest point in the sky.The present standard, adopted in 1967, is much more precise. It is based on an atomic clock, which uses the energy difference between the two lowest energy states of the cesium atom (^{133}Cs). When bombarded by microwaves of precisely the proper frequency, cesium atoms undergo a transition from one of these states to the other. One **second** (abbreviated s) is defined as the time required for 9,192,631,770 cycles of this microwave radiation.

Length

In 1960 an atomic standard for the meter was also established, using the wavelength of theorange-red light emitted by excited atoms of krypton (^{186}Kr). From this length standard,the speed of light in vacuum was measured to be 299,792,458 m/s. In November 1983, the length standard was changed again so that the speed of light in vacuum was *defined* to be precisely 299,792,458 m/s. Hence the new definition of the **meter** (abbreviated m) is the distance that light travels in vacuum in 1/299,792,458 second. This modern definition provides a much more precise standard of length than the one based on a wavelength of light.

Mass

Until recently the unit of mass, the **kilogram** (abbreviated kg), was defined to be the mass of a metal cylinder kept at the International Bureau of Weights and Measures in France. This was a very inconvenient standard to use. Since 2018 the value of the kilogram has been based on a fundamental constant of nature called *Planck’s constant* (symbol h), whose defined value $h = 6.62607015 \times 10^{-34}$ kg·m^2/s is related to those of the kilogram, meter, and second. Given the values of the meter and the second, the masses of objects can be experimentally determined in terms of h. (We’ll explain the meaning of h in Chapter 28). The *gram* (which is not a fundamental unit) is 0.001 kilogram.

Other *derived units* can be formed from the fundamental units. For example, the units of speed are meters per second, or m/s; these are the units of length (m) divided by the units of time (s).

Unit Prefixes

Once we have defined the fundamental units, it is easy to introduce larger and smaller units for the same physical quantities. In the metric system these other units are related to the fundamental units (or, in the case of mass, to the gram) by multiples of 10 or 1/10. Thus one kilometer (1 km) is 1000 meters, and one centimeter (1 cm) is 1/100 meter. We usually express multiples of 10 or 1/10 in exponential notation: $1000 = 10^3$, $1/1000 = 10^{-3}$, and so on. With this notation, 1 km = 10^3 m and 1 cm = 10^{-2} m.

The names of the additional units are derived by adding a **prefix** to the name of the fundamental unit. For example, the prefix "kilo-," abbreviated k, always means a unit larger by a factor of 1000; thus

1 kilometer = 1 km = 10^3 meters = 10^3 m
1 kilogram = 1 kg = 10^3 grams = 10^3 g
1 kilowatt = 1 kW = 10^3 watts = 10^3 W

1.4 Using and Converting Units

We use equations to express relationships among physical quantities, represented by algebraic symbols. Each algebraic symbol always denotes both a number and a unit. For example, *d* might represent a distance of 10 m, *t* a time of 5 s, and υ a speed of 2 m/s.

An equation must always be **dimensionally consistent.** You can't add apples and automobiles; two terms may be added or equated only if they have the same units. For example, if an object moving with constant speed υ travels a distance *d* in a time *t*, these quantities are related by the equation

$$d = \upsilon t.$$

If *d* is measured in meters, then the product υt must also be expressed in meters. Using the above numbers as an example, we may write

$$10\,\mathrm{m} = \left(2\,\frac{\mathrm{m}}{\mathrm{s}}\right)(5\,s).$$

Because the unit s in the denominator of m/s cancels, the product has units of meters, as it must. In calculations, units are treated just like algebraic symbols with respect to multiplication and division.

CAUTION! **Always use units in calculations** Make it a habit to *always* write numbers with the correct units and carry the units through the calculation as in the example above. This provides a very useful check. If at some stage in a calculation you find that an equation or an expression has inconsistent units, you know you have made an error. In this book we'll *always* carry units through all calculations, and we strongly urge you to follow this practice when you solve problems.

1.5 Uncertainty and Significant Figures

Measurements always have uncertainties. If you measure the thickness of the cover of a hardbound version of this book using an ordinary ruler, your measurement is reliable to only the nearest millimeter, and your result will be 3 mm. It would be *wrong* to state this result as 3.00 mm; given the limitations of the measuring device, you can't tell whether the actual thickness is 3.00 mm, 2.85 mm, or 3.11 mm. But if you use a micrometer caliper, a device that measures distances reliably to the nearest 0.01 mm, the result will be 2.91 mm. The distinction between the measurements with a ruler and with a caliper is in their **uncertainty**; the measurement with a caliper has a smaller uncertainty. The uncertainty is also called the **error** because it indicates the maximum difference there is likely to be between the measured value and the true value. The uncertainty or error of a measured value depends on the measurement technique used.

We often indicate the **accuracy** of a measured value – that is, how close it is likely to be to the true value – by writing the number, the symbol $\pm$, and a second number indicating the uncertainty of the measurement. If the diameter of a steel rod is given as 56.47 $\pm$ 0.02 mm, this means that the true value is likely to be within the range from 56.45 mm to 56.49 mm. In a commonly used shorthand notation, the number 1.6454(21) means 1.6454 $\pm$ 0.0021. The numbers in parentheses show the uncertainty in the final digits of the main number.

We can also express accuracy in terms of the maximum likely **fractional error** or **percent error** (also called *fractional uncertainty* and *percent uncertainty*). A resistor labeled "47 ohms $\pm$ 10 %" probably has a true resistance that differs from 47 ohms by no more than 10 % of 47 ohms -that is, by about 5 ohms. The resistance is probably between 42 and 52 ohms. For the diameter of the steel rod given above, the fractional error is (0.02 mm)/(56.47 mm), or about 0.0004; the percent error is (0.0004)(100 %), or about 0.04 %. Even small percent errors can be very significant (**Fig. 1.2**).

In many cases the uncertainty of a number is not stated explicitly. Instead, the uncertainty is indicated by the number of meaningful digits, or **significant figures,** in the measured value. We gave the thickness of the cover of the book as 2.91 mm, which has three significant figures. By this we mean that the first two digits are known to be correct, while the third digit is uncertain. The last digit is in the hundredths place, so the uncertainty is about 0.01 mm. Two values with the *same* number of significant figures may have *different* uncertainties; a distance given as 137 km also has three significant figures, but the uncertainty is about 1 km. A distance given as 0.25 km has two significant figures (the zero to the left of the decimal point doesn't count); if given as 0.250 km, it has three significant figures. When you use numbers that have uncertainties to compute other numbers, the computed numbers are also uncertain. When numbers are multiplied or divided, the result can have no more significant figures than the factor with the fewest significant figures has. For example, $3.1416 \times 2.34 \times 0.58 = 4.3$. When we add and subtract numbers, it is the location of the decimal point that matters, not the number of significant figures. For example, 123.62 + 8.9 = 132.5. Although 123.62 has an uncertainty of about 0.01, 8.9 has an uncertainty of about 0.1. So their sum has an uncertainty of about 0.1 and should be written as 132.5, not 132.52. **Table 1.2** summarizes these rules for significant figures.

Figure 1.2 - This spectacular mishap was the result of a very small percent error - traveling a few meters too far at the end of a journey of hundreds of thousands of meters

TABLE 1.1 - Using Significant Figures

Multiplication or division:
Result can have no more significant figures than the factor with the fewest significant figures:

$$\frac{0.745 \times 2.2}{3.885} = 0.42$$

$$1.32578 \times 10^7 \times 4.11 \times 10^{-3} = 5.45 \times 10^4$$

Addition or subtraction:
Number of significant figures is determined by the term with the largest uncertainty (i.e., fewest digits to the right of the decimal point):

$$27.153 + 138.2 - 11.74 = 153.6$$

To apply these ideas, suppose you want to verify the value of π, the ratio of the circumference of a circle to its diameter. The true value of this ratio to ten digits is 3.141592654. To test this, you draw a large circle and measure its circumference and diameter to the nearest millimeter, obtaining the values 424 mm and 135 mm. You enter these into your calculator and obtain the quotient (424 mm)/(135 mm) = 3.140740741. This may seem to disagree with the true value of π, but keep in mind that each of your measurements has three significant figures, so your measured value of π can have only three significant figures. It should be stated simply as 3.14. Within the limit of three significant figures, your value does agree with the true value.

In the examples and problems in this book we usually give numerical values with three significant figures, so your answers should usually have no more than three significant figures. (Many numbers in the real world have even less accuracy. The speedometer in a car, for example, usually gives only two significant figures). Even if you do the arithmetic with a calculator that displays ten digits, a ten-digit answer would misrepresent the accuracy of the results. Always round your final answer to keep only the correct number of significant figures or, in doubtful cases, one more at most.

Here's a special note about calculations that involve multiple steps: As you work, it's helpful to keep extra significant figures in your calculations. Once you have your final answer, round it to the correct number of significant figures. This will give you the most accurate results.

When we work with very large or very small numbers, we can show significant figures much more easily by using **scientific notation,** sometimes called **powers-of-10 notation.** The distance from the earth to the moon is about 384,000,000 m, but writing the number in this form doesn't indicate the number of significant figures. Instead, we move the decimal point eight places to the left (corresponding to dividing by 10^8) and multiply by 10^8; that is,

$$384{,}000{,}000 \text{ m} = 3.84 \times 10^8 \text{ m}.$$

In this form, it is clear that we have three significant figures. The number 4.00×10^{-7} also has three significant figures, even though two of them are zeros. Note that in scientific notation the usual practice is to express the quantity as a number between 1 and 10 multiplied by the appropriate power of 10.

When an integer or a fraction occurs in an algebraic equation, we treat that number as having no uncertainty at all. For example, in the equation $\upsilon_x^2 = \upsilon_{0x}^2 + 2a_x(x - x_0)$, which is Eq. (2.13) in Chapter 2, the coefficient 2 is *exactly* 2. We can consider this coefficient as having an infinite number of significant figures (2.000000…). The same is true of the exponent 2 in υ_x^2 and υ_{0x}^2.

Finally, let's note that **precision** is not the same as *accuracy.* A cheap digital watch that gives the time as 10:35:17 a.m. is very *precise* (the time is given to the second), but if the watch runs several minutes slow, then this value isn't very *accurate.* On the other hand, a grandfather clock might be very accurate (that is, display the correct time), but if the clock has no second hand, it isn't very precise. A high-quality measurement is both precise *and* accurate.

1.6 Estimates and Orders of Magnitude

We have stressed the importance of knowing the accuracy of numbers that represent physical quantities. But even a very crude estimate of a quantity often gives us useful information. Sometimes we know how to calculate a certain quantity, but we have to guess at the data we need for the calculation. Or the calculation might be too complicated to carry out exactly, so we make rough approximations. In either case our result is also a guess, but such a guess can be useful even if it is uncertain by a factor of two, ten, or more. Such calculations are called **order-of-magnitude estimates.** The great Italian-American nuclear physicist Enrico Fermi (1901–1954) called them “back-of-the-envelope calculations.” Even when they are off by a factor of ten, the results can be useful and interesting.

1.7 Vectors and Vector Addition

Some physical quantities, such as time, temperature, mass, and density, can be described completely by a single number with a unit. But many other important quantities in physics have a *direction* associated with them and cannot be described by a single number. A simple example is the motion of an airplane: We must say not only how fast the plane is moving but also in what direction. The speed of the airplane combined with its direction of motion constitute a quantity called *velocity.* Another example is *force,* which in physics means a push or pull exerted on an object. Giving a complete description of a force means describing both how hard the force pushes or pulls on the object and the direction of the push or pull. When a physical quantity is described by a single number, we call it a **scalar quantity.** In contrast, a **vector quantity** has both a **magnitude** (the “how much” or “how big” part) and a direction in space. Calculations that combine scalar quantities use the operations of ordinary arithmetic.

For example, 6 kg + 3 kg = 9 kg, or 4 × 2 s = 8 s. However, combining vectors requires a different set of operations.

To understand more about vectors and how they combine, we start with the simplest vector quantity, **displacement.** Displacement is a change in the position of an object. Displacement is a vector quantity because we must state not only how far the object moves but also in what direction. Walking 3 km north from your front door doesn't get you to the same place as walking 3 km southeast; these two displacements have the same magnitude but different directions.

We usually represent a vector quantity such as displacement by a single letter, such as $\vec{A}$ in**Fig. 1.3a**. In this book we always print vector symbols in ***italic type with an arrow above them.*** We do this to remind you that vector quantities have different properties from scalar quantities; the arrow is a reminder that vectors have direction. When you handwrite a symbol for a vector, *always* write it with an arrow on top. If you don't distinguish between scalar and vector quantities in your notation, you probably won't make the distinction in your thinking either, and confusion will result.

We always *draw* a vector as a line with an arrowhead at its tip. The length of the line shows the vector's magnitude, and the direction of the arrowhead shows the vector's direction. Displacement is always a straight-line segment directed from the starting point to the ending point, even though the object's actual path may be curved (Fig. 1.3b). Note that displacement is not related directly to the total *distance* traveled. If the object were to continue past P_2 and then return to P_1, the displacement for the entire trip would be *zero* (Fig. 1.3c). If two vectors have the same direction, they are **parallel.** If they have the same magnitude *and* the same direction, they are *equal*, no matter where they are located in space. The vector $\vec{A}'$ from point P_3 to point P_4 in **Fig. 1.4** has the same length and direction as the vector $\vec{A}$ from P_1 to P_2. These two displacements are equal, even though they start at different points. We write this as $\vec{A}' = \vec{A}$ in Fig. 1.4. Two vector quantities are equal only when they have the same magnitude *and* the same direction.

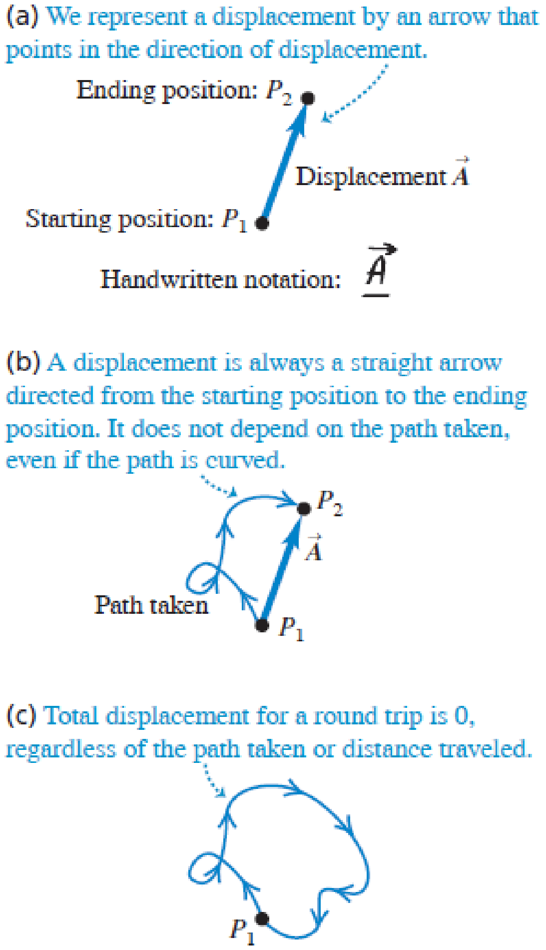


Figure 1.3 - Displacement as a vector quantity

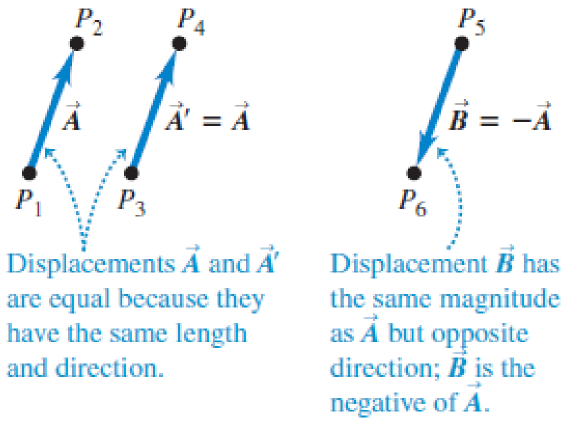


Figure 1.4 - The meaning of vectors that have the same magnitude and the same or opposite direction

Vector $\vec{B}$ in Fig. 1.4, however, is not equal to $\vec{A}$ because its direction is *opposite* that of $\vec{A}$. We define the **negative of a vector** as a vector having the same magnitude as the original vector but the *opposite* direction. The negative of vector quantity $\vec{A}$ is denoted as $-\vec{A}$. If $\vec{A}$ is 87 m south, then $-\vec{A}$ is 87 m north. Thus we can write the relationship between $-\vec{A}$ and $\vec{B}$ in Fig. 1.4 as $\vec{A} = -\vec{B}$ or $\vec{B} = -\vec{A}$. When two vectors $\vec{A}$ and $\vec{B}$ have opposite directions, whether their magnitudes are the same or not, we say that they are **antiparallel.**

We usually represent the *magnitude* of a vector quantity by the same letter used for the vector, but in *lightface italic type* with *no* arrow on top. For example, if displacement vector $\vec{A}$ is 87 m south, then A = 87 m. An alternative notation is the vector symbol with vertical bars on both sides:

$$\text{(Magnitude of } \vec{A}\text{)} = A = |\vec{A}|. \tag{1.1}$$

The magnitude of a vector quantity is a scalar quantity (a number) and is *always positive.* Note that a vector can never be equal to a scalar because they are different kinds of quantities. The expression "$\vec{A}$ = 6 m" is just as wrong as "2 oranges = 3 apples"!

When we draw diagrams with vectors, it’s best to use a scale similar to those used for maps. For example, a displacement of 5 km might be represented in a diagram by a vector 1 cm long, and a displacement of 10 km by a vector 2 cm long.

Vector Addition and Subtraction

Suppose a particle undergoes a displacement $\vec{A}$ followed by a second displacement $\vec{B}$. Thefinal result is the same as if the particle had started at the same initial point and undergonea single displacement $\vec{C}$ (**Fig. 1.5a**). We call displacement $\vec{C}$ the **vector sum,** or **resultant,** of displacements $\vec{A}$ and $\vec{B}$. We express this relationship symbolically as

$$\vec{C} = \vec{A} + \vec{B}. \tag{1.2}$$

The boldface plus sign emphasizes that adding two vector quantities requires a geometrical process and is not the same operation as adding two scalar quantities such as 2 + 3 = 5. In vector addition we usually place the *tail* of the *second* vector at the *head*,or tip, of the *first* vector (Fig. 1.5a).

If we make the displacements $\vec{A}$ and $\vec{B}$ in reverse order, with $\vec{B}$ first and $\vec{A}$ second, the result is the same (Fig. 1.5b). Thus

$$\vec{C} = \vec{B} + \vec{A} \text{ and } \vec{A} + \vec{B} = \vec{B} + \vec{A}. \tag{1.3}$$

This shows that the order of terms in a vector sum doesn’t matter. In other words, vector addition obeys the *commutative* law. Figure 1.5c shows another way to represent the vector sum: If we draw vectors $\vec{A}$ and $\vec{B}$ with their tails at the same point, vector $\vec{C}$ is the diagonal of a parallelogram constructed with $\vec{A}$ and $\vec{B}$ as two adjacent sides.

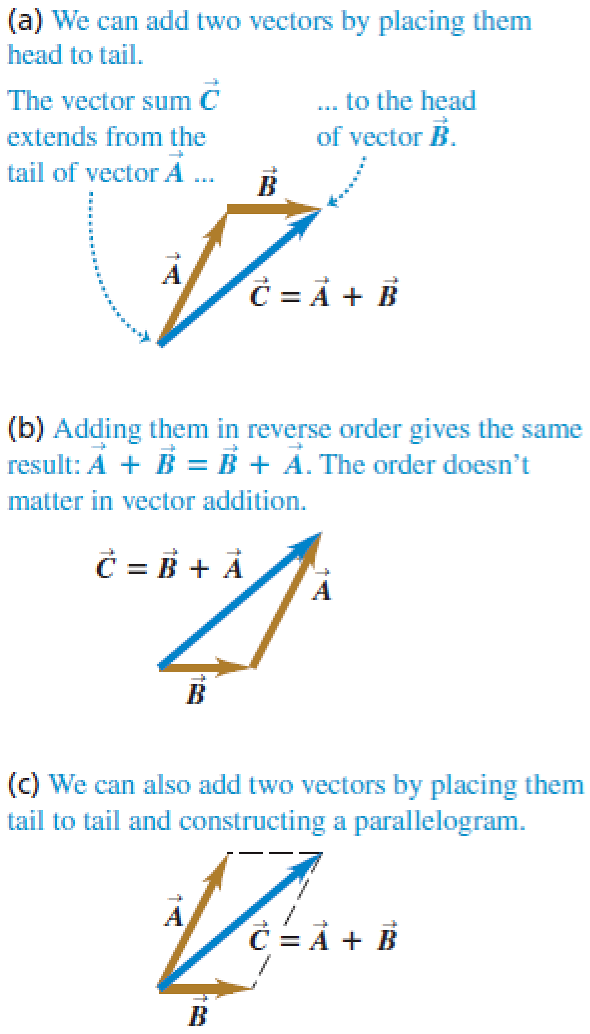


Figure 1.5 - Three ways to add two vectors

CAUTION! **Magnitudes in vector addition.** It's a common error to conclude that if $\vec{C}=\vec{A}+\vec{B}$, then magnitude C equals magnitude A plus magnitude B. In general, this conclusion is *wrong;* for the vectors shown in Fig. 1.5, $C<A+B$. The magnitude of $\vec{A}+\vec{B}$ depends on the magnitudes of $\vec{A}$ and $\vec{B}$ *and* on the angle between $\vec{A}$ and $\vec{B}$. Only in the special case in which $\vec{A}$ and $\vec{B}$ are *parallel* is the magnitude of $\vec{C}=\vec{A}+\vec{B}$ equal to the sum of the magnitudes of $\vec{A}$ and $\vec{B}$ (**Fig. 1.6a**). When the vectors are *antiparallel* (Fig. 1.6b), the magnitude of $\vec{C}$ equals the *difference* of the magnitudesof $\vec{A}$ and $\vec{B}$. Be careful to distinguish between scalar and vector quantities, and you'll avoid making errors about the magnitude of a vector sum.

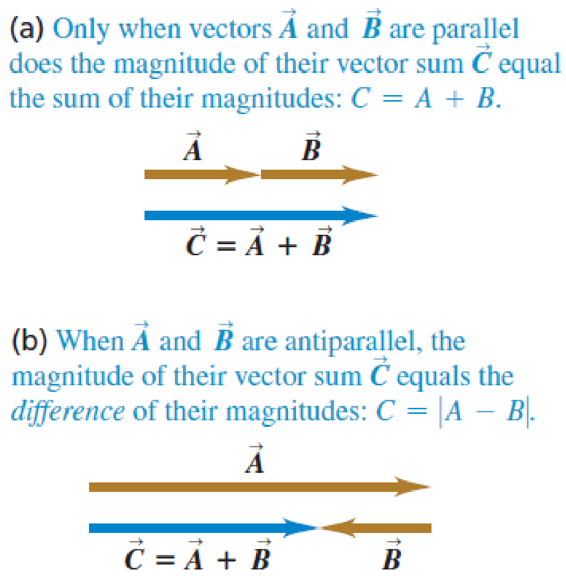


Figure 1.6 - Adding vectors that are (a) parallel and (b) antiparallel

Figure 1.7a shows *three* vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$. To find the vector sum of all three, in Fig. 1.7b we first add $\vec{A}$ and $\vec{B}$ to give a vector sum $\vec{D}$; we then add vectors $\vec{C}$ and $\vec{D}$ by the same process to obtain the vector sum $\vec{R}$:

$$\vec{R}=(\vec{A}+\vec{B})+\vec{C}=\vec{D}+\vec{C}.$$

Alternatively, we can first add $\vec{B}$ and $\vec{C}$ to obtain vector $\vec{E}$ (Fig. 1.7c), and then add $\vec{A}$ and $\vec{E}$ to obtain $\vec{R}$:

$$\vec{R}=\vec{A}+(\vec{B}+\vec{C})=\vec{A}+\vec{E}.$$

We don't even need to draw vectors $\vec{D}$ and $\vec{E}$; all we need to do is draw $\vec{A}$, $\vec{B}$, and $\vec{C}$ in succession,with the tail of each at the head of the one preceding it. The sum vector $\vec{R}$ extendsfrom the tail of the first vector to the head of the last vector (Fig. 1.7d). The order makes no difference; Fig. 1.7e shows a different order, and you should try others. Vector addition obeys the *associative* law.

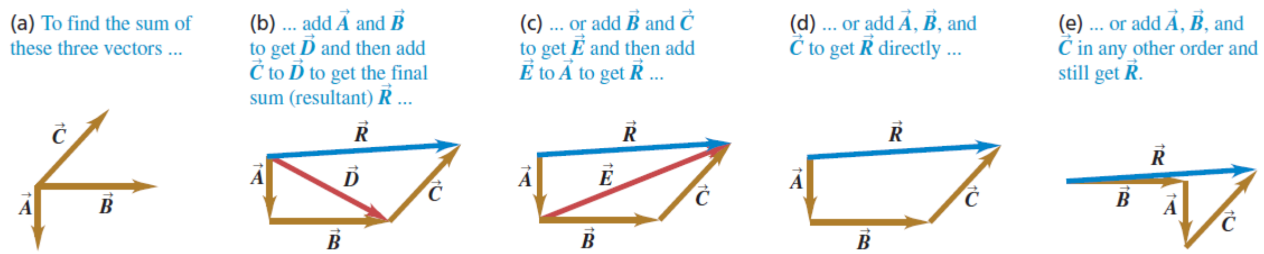


Figure 1.7 - Several constructions for finding the vector sum $\vec{A}+\vec{B}+\vec{C}$

We can *subtract* vectors as well as add them. To see how, recall that vector $-\vec{A}$ has the same magnitude as $\vec{A}$ but the opposite direction. We define the difference $\vec{A}-\vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$ to be the vector sum of $\vec{A}$ and $-\vec{B}$:

$$\vec{A}-\vec{B}=\vec{A}+(-\vec{B}). \tag{1.4}$$

Figure 1.8 shows an example of vector subtraction.

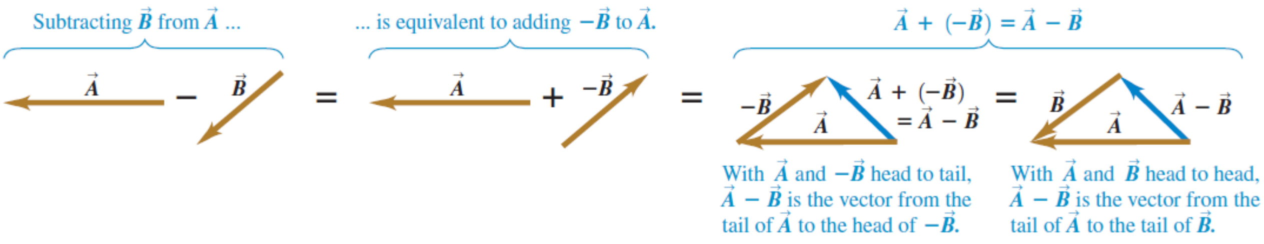


Figure 1.8 - To construct the vector difference $\vec{A} - \vec{B}$, you can either place the tail of $-\vec{B}$ at the head of $\vec{A}$ or place the two vectors $\vec{A}$ and $\vec{B}$ head to head

A vector quantity such as a displacement can be multiplied by a scalar quantity (an ordinary number). The displacement $2\vec{A}$ is a displacement (vector quantity) in the same direction as vector $\vec{A}$ but twice as long; this is the same as adding $\vec{A}$ to itself (**Fig. 1.9a**). In general, when we multiply vector $\vec{A}$ by a scalar *c*, the result $c\vec{A}$ has magnitude $|c|A$ (the absolute value of *c* multiplied by the magnitude of vector $\vec{A}$). If *c* is positive, $c\vec{A}$ is in the same direction as $\vec{A}$; if *c* is negative, $c\vec{A}$ is in the direction opposite to $\vec{A}$. Thus $3\vec{A}$ is parallel to $\vec{A}$, while $-3\vec{A}$ is antiparallel to $\vec{A}$ (Fig. 1.9b).

A scalar used to multiply a vector can also be a physical quantity. For example, you may be familiar with the relationship $\vec{F} = m\vec{a}$; the net force $\vec{F}$ (a vector quantity) that acts on an object is equal to the product of the object's mass *m* (a scalar quantity) and its acceleration $\vec{a}$ (a vector quantity). The direction of $\vec{F}$ is the same as that of $\vec{a}$ because *m* is positive, and the magnitude of $\vec{F}$ is equal to the mass *m* multiplied by the magnitude of $\vec{a}$. The unit of force is the unit of mass multiplied by the unit of acceleration.

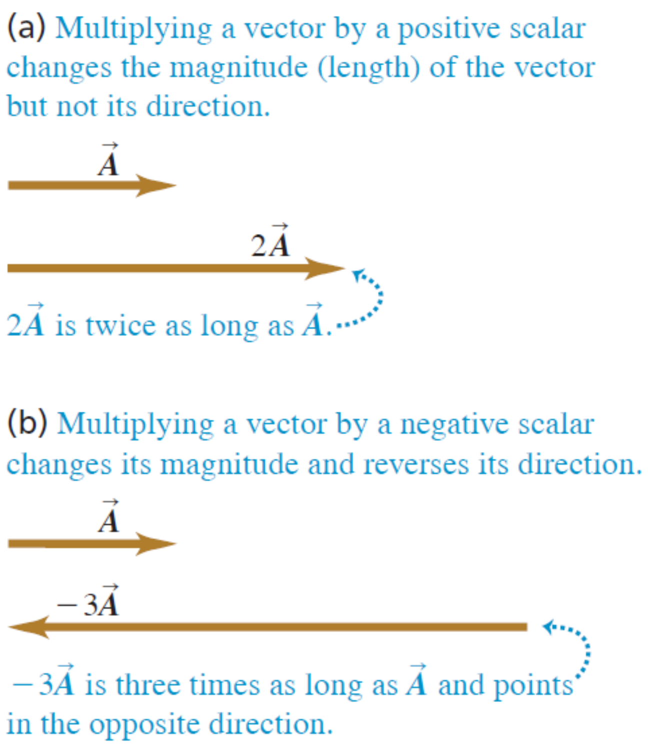


Figure 1.9 - Multiplying a vector by a scalar

1.8 Components of Vectors

In Section 1.7 we added vectors by using a scale diagram and properties of right triangles. But calculations with right triangles work only when the two vectors are perpendicular. So we need a simple but general method for adding vectors. This is called the method of *components*.

To define what we mean by the components of a vector $\vec{A}$, we begin with a rectangular (Cartesian) coordinate system of axes (**Fig. 1.10**). If we think of $\vec{A}$ as a displacement vector, we can regard $\vec{A}$ as the sum of a displacement parallel to the *x*-axis and a displacementparallel to the *y*-axis. We use the numbers A_x and A_y to tell us how much displacement there is parallel to the *x*-axis and how much there is parallel to the *y*-axis, respectively. For example, if the +*x*-axis points east and the +*y*-axis points north, $\vec{A}$ in Fig. 1.10 could be the sum of a 2.00 m displacement to the east and a 1.00 m displacement to the north. Then A_x = +2.00 m and A_y = +1.00 m. We can use the same idea for any vectors, not just displacement vectors. The two numbers A_x and A_y are called the **components** of $\vec{A}$.

CAUTION! **Components are not vectors.** The components A_x and A_y of a vector $\vec{A}$ are numbers; they are *not* vectors themselves. This is why we print the symbols for components in lightface italic type with *no* arrow on top instead of in boldface italic with an arrow, which is reserved for vectors.

We can calculate the components of vector $\vec{A}$ if we know its magnitude A and its direction. We'll describe the direction of a vector by its angle relative to some reference direction. In Fig. 1.10 this reference direction is the positive x-axis, and the angle between vector $\vec{A}$ and the positive x-axis is θ (the Greek letter theta). Imagine that vector $\vec{A}$ originally lies along the $+x$-axis and that you then rotate it to its true direction, as indicated by the arrow in Fig. 1.10 on the arc for angle θ. If this rotation is from the $+x$-axis toward the $+y$-axis, as is the case in Fig. 1.10, then θ is *positive;* if the rotation is from the $+x$-axis toward the $-y$-axis, then θ is *negative.* Thus the $+y$-axis is at an angle of 90°, the $-x$-axis is at 180°, and the $-y$-axis is at 270° (or –90°). If θ is measured in this way, then, from the definition of the trigonometric functions,

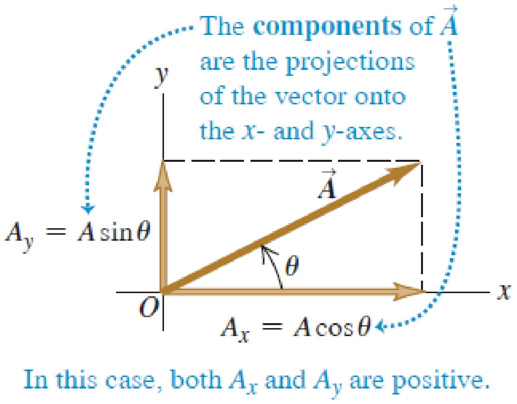


Figure 1.10 - Representing a vector $\vec{A}$ in terms of its components A_x and A_y

$$\frac{A_x}{A} = \cos\theta \text{ and } \frac{A_y}{A} = \sin\theta,$$

$$A_x = A\cdot\cos\theta \text{ and } A_y = A\cdot\sin\theta \tag{1.5}$$

(θ measured from the $+x$-axis, rotating toward the $+y$-axis).

In Fig. 1.10 A_x and A_y are positive. This is consistent with Eqs. (1.5); θ is in the first quadrant (between 0° and 90°), and both the cosine and the sine of an angle in this quadrant are positive. But in **Fig. 1.11a** the component B_x is negative and the component B_y is positive. (If the $+x$-axis points east and the $+y$-axis points north, $\vec{B}$ could represent a displacement of 2.00 m west and 1.00 m north. Since west is in the $-x$-direction and north is in the $+y$-direction, $B_x = -2.00$ m is negative and $B_y = +1.00$ m is positive). Again, this is consistent with Eqs. (1.5); now θ is in the second quadrant, so $\cos\theta$ is negative and $\sin\theta$ is positive. In Fig. 1.11b both C_x and C_y are negative (both $\cos\theta$ and $\sin\theta$ are negative in the third quadrant).

CAUTION! **Relating a vector's magnitude and direction to its components.** Equations (1.5) are correct *only* when the angle θ is measured from the positive x-axis. If the angle of the vector is given from a different reference direction or you use a different rotation direction, the relationships are different!

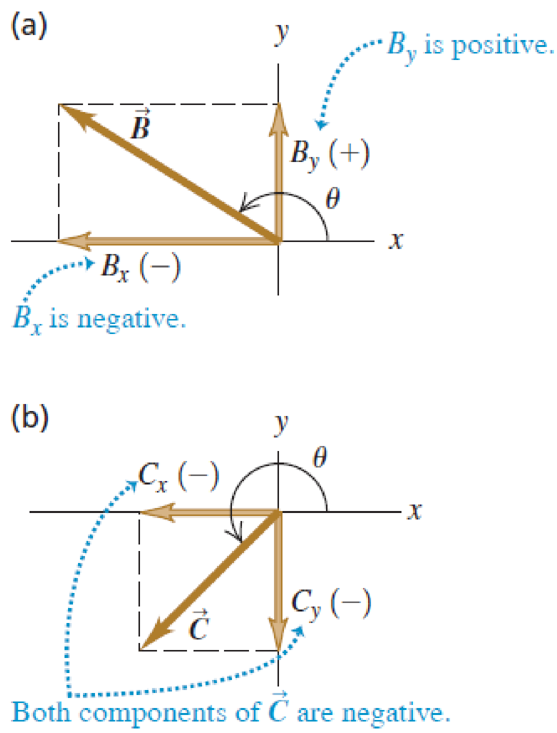


Figure 1.11 - The components of a vector may be positive or negative numbers

Using Components to Do Vector Calculations

Using components makes it relatively easy to do various calculations involving vectors. Let's look at three important examples: finding a vector's magnitude and direction, multiplying a vector by a scalar, and calculating the vector sum of two or more vectors.

1 Finding a vector's magnitude and direction from its components. We can describe a vector completely by giving either its magnitude and direction or its *x*- and *y*-components. Equations (1.5) show how to find the components if we know the magnitude and direction. We can also reverse the process: We can find the magnitude and direction if we know the components. By applying the Pythagorean theorem to Fig. 1.10, we find that the magnitude of vector $\vec{A}$:

$$A = \sqrt{A_x^2 + A_y^2}\,. \tag{1.6}$$

(We always take the positive root).

Equation (1.6) is valid for any choice of *x*-axis and *y*-axis as long as they are mutually perpendicular. The expression for the vector direction comes from the definition of the tangent of an angle. If θ is measured from the positive *x*-axis, and a positive angle is measured toward the positive *y*-axis (as in Fig. 1.16), then

$$\tan\theta = \frac{A_y}{A_x} \text{ and } \theta = \arctan\left(\frac{A_y}{A_x}\right). \tag{1.7}$$

We'll always use the notation arctan for the inverse tangent function. The notation $\tan^{-1}$ is also commonly used, and your calculator may have an INV or 2ND button to be used wih the TAN button.

CAUTION! **Finding the direction of a vector from its components.** There's one complication in using Eqs. (1.7) to find θ: Any two angles that differ by 180° have the same tangent. For example, in **Fig. 1.12** the tangent of the angle θ is $\tan\theta = A_x / A_y = +1$. A calculator will tell you that $\theta = \tan^{-1}(+1) = 45°$. But the tangent of 180° + 45° = 225° is also equal to +1, so θ could also be 225° (which is actually the case in Fig. 1.12). *Always* draw a sketch like Fig. 1.12 to determine which of the two possibilities is correct.

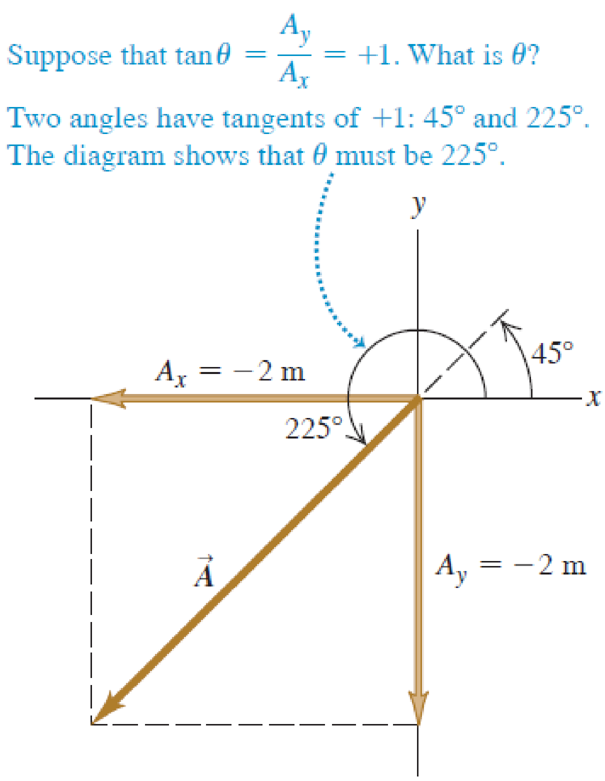


Figure 1.12 - Drawing a sketch of a vector reveals the signs of its *x*- and *y*-components

2 Multiplying a vector by a scalar. If we multiply a vector $\vec{A}$ by a scalar *c*, each component of the product $\vec{D} = c\vec{A}$ is the product of *c* and the corresponding component of $\vec{A}$:

$D_x = cA_x$, $D_y = cA_y$ (components of $\vec{D} = c\vec{A}$).

For example, Eqs. (1.8) say that each component of the vector $2\vec{A}$ is twice as great as the corresponding component of $\vec{A}$, so $2\vec{A}$ is in the same direction as $\vec{A}$ but has twice the magnitude. Each component of the vector $-3\vec{A}$ is three times as great as the corresponding component of $\vec{A}$ but has the opposite sign, so $-3\vec{A}$ is in the opposite direction from $\vec{A}$ and has three times the magnitude. Hence Eqs. (1.8) are consistent with our discussion in Section 1.7 of multiplying a vector by a scalar (see Fig. 1.9).

3 Using components to calculate the vector sum (resultant) of two or more vectors. Figure 1.13 shows two vectors $\vec{A}$ and $\vec{B}$ and their vector sum $\vec{R}$, along with the *x*- and *y*-components of all three vectors. The *x*-component $R_x$of the vector sum is simply the sum (A_x+ B_x) of the *x*-components of the vectors being added. The same is true for the *y*-components. In symbols

Each **component of $\vec{R} = \vec{A} + \vec{B}$** ...

$$R_x = A_x + B_x, \qquad R_y = A_y + B_y \tag{1.9}$$

... is the sum of the corresponding components of $\vec{A}$ and $\vec{B}$.

Figure 1.13 shows this result for the case in which the components A_x, A_y, B_x, and B_y are all positive. Draw additional diagrams to verify for yourself that Eqs. (1.9) are valid for *any* signs of the components of $\vec{A}$ and $\vec{B}$.

If we know the components of any two vectors $\vec{A}$ and $\vec{B}$, perhaps by using Eqs. (1.5), we can compute the components of the vector sum $\vec{R}$. Then if we need the magnitude and direction of $\vec{R}$, we can obtain them from Eqs. (1.6) and (1.7) with the *A*'s replaced by *R*'s.

We can use the same procedure to find the sum of any number of vectors. If $\vec{R}$ is the vector sum of $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$, $\vec{E}$, …, the components of $\vec{R}$ are

$$R_x = A_x + B_x + C_x + D_x + E_x + \ldots,$$

$$R_y = A_y + B_y + C_y + D_y + E_y + \ldots. \tag{1.10}$$

We have talked about vectors that lie in the *xy*-plane only, but the component method works just as well for vectors having any direction in space. We can introduce a *z*-axis perpendicular to the *xy*-plane; then in general a vector $\vec{A}$ has components A_x, A_y, and A_z in the three coordinate directions. Its magnitude *A* is

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}\,. \tag{1.11}$$

Again, we always take the positive root. Also, Eqs. (1.10) for the vector sum $\vec{R}$ have a third component:

$$R_z = A_z + B_z + C_z + D_z + E_z + \ldots.$$

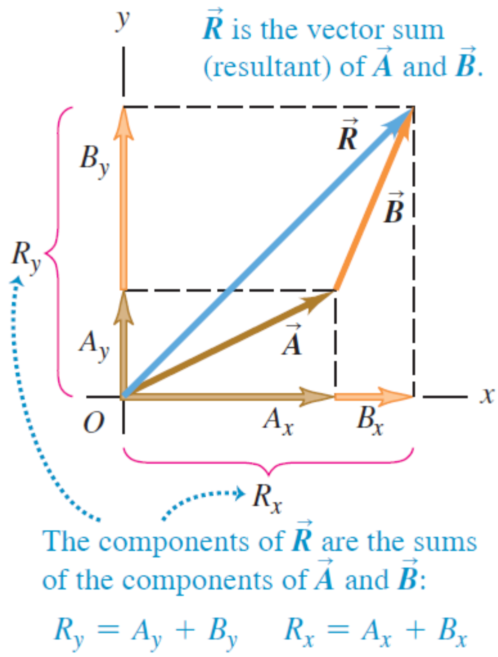


Figure 1.13 - Finding the vector sum (resultant) of $\vec{A}$ and $\vec{B}$ using components

We've focused on adding *displacement* vectors, but the method is applicable to all vector quantities. When we study the concept of force in Chapter 4, we'll find that forces are vectors that obey the same rules of vector addition.

1.9 Unit Vectors

A **unit vector** is a vector that has a magnitude of 1, with no units. Its only purpose is to *point*—that is, to describe a direction in space. Unit vectors provide a convenient notation for many expressions involving components of vectors. We'll always include a caret, or "hat" (^), in the symbol for a unit vector to distinguish it from ordinary vectors whose magnitude may or may not be equal to 1.

In an *xy*-coordinate system we can define a unit vector dn that points in the direction of the positive *x*-axis and a unit vector en that points in the direction of the positive *y*-axis (**Fig. 1.14a**). Then we can write a vector $\vec{A}$ in terms of its components as

$$\vec{A} = A_x\hat{i} + A_y\hat{j}\,. \qquad (1.12)$$

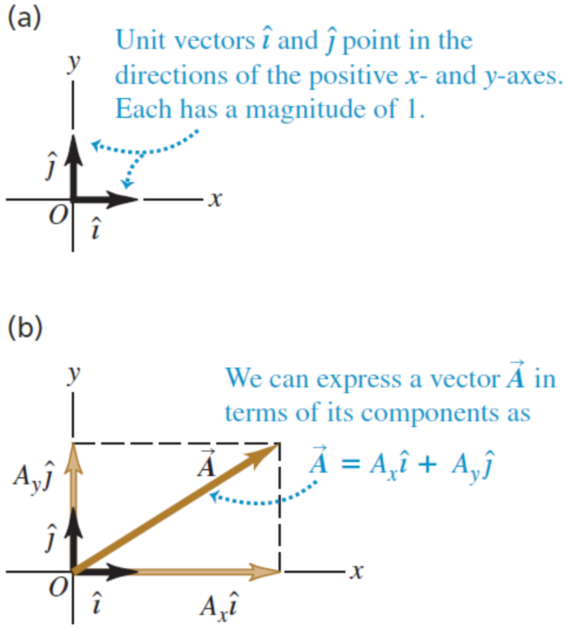


Figure 1.14 - (a) The unit vectors dn and en. (b) Expressing a vector $\vec{A}$ in terms of its components

Equation (1.12) is a vector equation; each term, such as $A_x\hat{i}$, is a vector quantity (Fig. 1.14b). Using unit vectors, we can express the vector sum $\vec{R}$ of two vectors $\vec{A}$ and $\vec{B}$ as follows:

$$\vec{A} = A_x\hat{i} + A_y\hat{j}\,,\ \vec{B} = B_x\hat{i} + B_y\hat{j}\,,$$

$$\vec{R} = \vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} = R_x\hat{i} + R_y\hat{j}. \qquad (1.13)$$

Equation (1.13) restates the content of Eqs. (1.9) in the form of a single vector equation rather than two component equations.

If not all of the vectors lie in the *xy*-plane, then we need a third component. We introduce a third unit vector $\hat{k}$ that points in the direction of the positive *z*-axis (**Fig. 1.15**). Then Eqs. (1.12) and (1.13) become

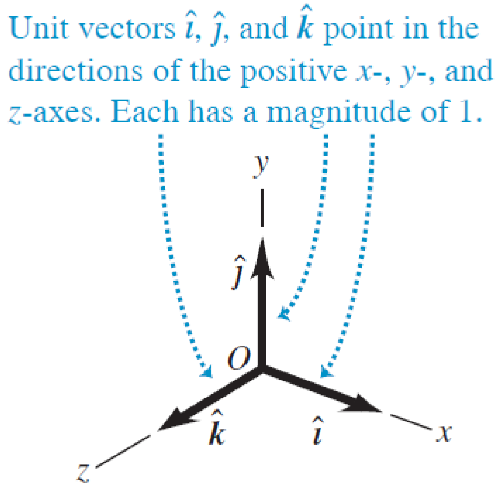


Figure 1.15 - The unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\,,\ \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\,, \qquad (1.14)$$

$$\vec{R} = \vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k} = R_x\hat{i} + R_y\hat{j} + R_z\hat{k}\,. \qquad (1.15)$$

1.10 Products of Vectors

We saw how vector addition develops naturally from the problem of combining displacements. It will prove useful for calculations with many other vector quantities. We can also express many physical relationships by using *products* of vectors. Vectors are not ordinary numbers, so we can't directly apply ordinary multiplication to vectors. We'll define two different kinds of products of vectors. The first, called the *scalar product,* yields a result that is a scalar quantity. The second, the *vector product,* yields another vector.

Scalar Product

We denote the **scalar product** of two vectors $\vec{A}$ and $\vec{B}$ by $\vec{A}\cdot\vec{B}$. Because of this notation, the scalar product is also called the **dot product.** Although $\vec{A}$ and $\vec{B}$ are vectors, the quantity $\vec{A}\cdot\vec{B}$ is a scalar.

To define the scalar product $\vec{A}\cdot\vec{B}$ we draw the two vectors $\vec{A}$ and $\vec{B}$ with their tails at the same point (**Fig. 1.16a**). The angle ϕ (the Greek letter phi) between their directions ranges from 0° to 180°. Figure 1.16b shows the projection of vector $\vec{B}$ onto the direction of $\vec{A}$; this projection is the component of $\vec{B}$ in the direction of $\vec{A}$ and is equal to $B\cos\phi$. (We can take components along *any* direction that's convenient, not just the *x*- and *y*-axes). We define $\vec{A}\cdot\vec{B}$ to be the magnitude of $\vec{A}$ multiplied by the component of $\vec{B}$ in the direction of $\vec{A}$, or

Scalar (dot) product of vectors $\vec{A}$ and $\vec{B}$ | Magnitudes of $\vec{A}$ and $\vec{B}$

$$\vec{A}\cdot\vec{B} = AB\cos\phi = |\vec{A}||\vec{B}|\cos\phi \qquad (1.16)$$

Angle between $\vec{A}$ and $\vec{B}$ when placed tail to tail

Alternatively, we can define $\vec{A}\cdot\vec{B}$ to be the magnitude of $\vec{B}$ multiplied by the component of $\vec{A}$ in the direction of $\vec{B}$, as in Fig. 1.16c. Hence $\vec{A}\cdot\vec{B} = B\cdot(A\cos\phi) = AB\cos\phi$, which is the same as Eq. (1.16).

The scalar product is a scalar quantity, not a vector, and it may be positive, negative, or zero. When ϕ is between 0° and 90°, $\cos\phi > 0$ and the scalar product is positive. When ϕ is between 90° and 180° so $\cos\phi < 0$, the component of $\vec{B}$ in the direction of $\vec{A}$ is negative, and $\vec{A}\cdot\vec{B}$ is negative. Finally, when ϕ = 90°, $\vec{A}\cdot\vec{B} = 0$. *The scalar product of two perpendicular vectors is always zero.* For any two vectors $\vec{A}$ and $\vec{B}$, $AB\cos\phi = BA\cos\phi$. This means that $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$. The scalar product obeys the commutative law of multiplication; the order of the two vectors does not matter.

We'll use the scalar product in Chapter 6 to describe work done by a force. In later chapters we'll use the scalar product for a variety of purposes, from calculating electric potential to determining the effects that varying magnetic fields have on electric circuits.

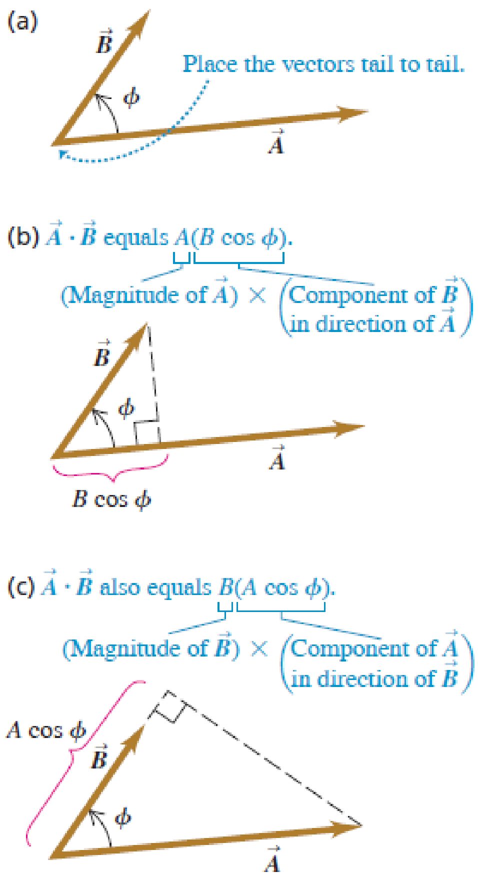


Figure 1.16 - Calculating the scalar product of two vectors, $\vec{A}\cdot\vec{B} = AB\cos\phi$

Using Components to Calculate the Scalar Product

We can calculate the scalar product $\vec{A}\cdot\vec{B}$ directly if we know the *x*-, *y*-, and *z*-components of $\vec{A}$ and $\vec{B}$. To see how this is done, let's first work out the scalar products of the unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$. All unit vectors have magnitude 1 and are perpendicular to each other. Using Eq. (1.16), we find

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1\cdot 1\cdot\cos 0° = 1,$$

$$\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 1\cdot 1\cdot\cos 90° = 0. \qquad (1.17)$$

Now we express $\vec{A}$ and $\vec{B}$ in terms of their components, expand the product, and use these products of unit vectors:

$$\begin{aligned}\vec{A}\cdot\vec{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) = \\ &= A_x\hat{i}B_x\hat{i} + A_x\hat{i}B_y\hat{j} + A_x\hat{i}B_z\hat{k} + \\ &+ A_y\hat{j}B_x\hat{i} + A_y\hat{j}B_y\hat{j} + A_y\hat{j}B_z\hat{k} + \\ &+ A_z\hat{k}B_x\hat{i} + A_z\hat{k}B_y\hat{j} + A_z\hat{k}B_z\hat{k}.\end{aligned} \qquad (1.18)$$

From Eqs. (1.17) you can see that six of these nine terms are zero. The three that survive give

Scalar (dot) product of vectors $\vec{A}$ and $\vec{B}$

Components of $\vec{A}$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z \qquad (1.19)$$

Components of $\vec{B}$

Thus *the scalar product of two vectors is the sum of the products of their respective components.* The scalar product gives a straightforward way to find the angle ϕ between any two vectors $\vec{A}$ and $\vec{B}$ whose components are known. In this case we can use Eq. (1.19) to find the scalar product of $\vec{A}$ and $\vec{B}$.

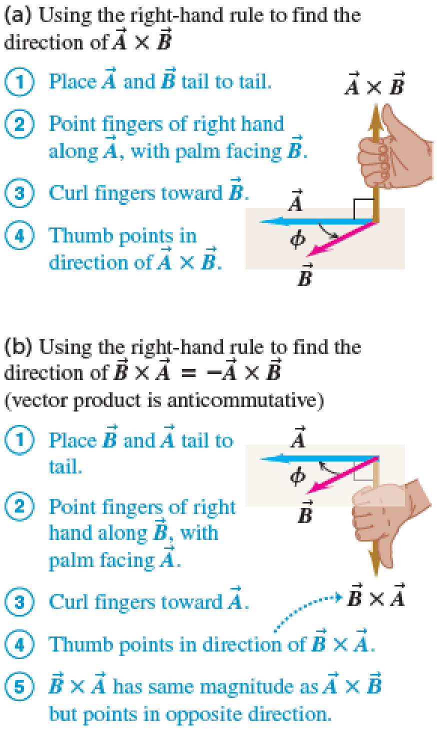


Figure 1.17 - The vector product of (a) $\vec{A}\times\vec{B}$ and (b) $\vec{B}\times\vec{A}$

Vector Product

We denote the **vector product** of two vectors $\vec{A}$ and $\vec{B}$, also called the **cross product,** by $\vec{A}\times\vec{B}$. As the name suggests, the vector product is itself a vector. We'll use this product in Chapter 10 to describe torque and angular momentum; in Chapters 27 and 28 we'll use it to describe magnetic fields and forces.

To define the vector product $\vec{A}\times\vec{B}$, we again draw the two vectors $\vec{A}$ and $\vec{B}$ with their tails at the same point (**Fig. 1.17a**). The two vectors then lie in a plane. We define the vector product to be a vector quantity with a direction perpendicular to this plane (that is, perpendicular to both $\vec{A}$ and $\vec{B}$) and a magnitude equal to *AB* sin ϕ. That is, if $\vec{C} = \vec{A}\times\vec{B}$, then

$$C = AB\sin\phi. \qquad (1.20)$$

We measure the angle ϕ from $\vec{A}$ toward $\vec{B}$ and take it to be the smaller of the two possible angles, so f ranges from 0° to 180°. Then $\sin\phi \geq 0$ and C in Eq. (1.20) is never negative, as must be the case for a vector magnitude. Note that when $\vec{A}$ and $\vec{B}$ are parallel or antiparallel, ϕ = 0° or 180° and C = 0. That is, *the vector product of two parallel or antiparallel vectors is always zero.* In particular, *the vector product of any vector with itself is zero.*

CAUTION! **Vector product vs. scalar product.** Don't confuse the expression $AB\sin\phi$ for the magnitude of the vector product $\vec{A}\times\vec{B}$ with the similar expression $AB\cos\phi$ for the scalar product $\vec{A}\cdot\vec{B}$. To see the difference between these two expressions, imagine that we vary the angle between $\vec{A}$ and $\vec{B}$ while keeping their magnitudes constant. When $\vec{A}$ and $\vec{B}$ are parallel, the magnitude of the vector product will be zero and the scalar product will be maximum. When $\vec{A}$ and $\vec{B}$ are perpendicular, the magnitude of the vector product will be maximum and the scalar product will be zero.

There are always *two* directions perpendicular to a given plane, one on each side of the plane. We choose which of these is the direction of $\vec{A}\times\vec{B}$ as follows. Imagine rotating vector $\vec{A}$ about the perpendicular line until $\vec{A}$ is aligned with $\vec{B}$, choosing the smaller of the two possible angles between $\vec{A}$ and $\vec{B}$. Curl the fingers of your right hand around the perpendicular line so that your fingertips point in the direction of rotation; your thumb will then point in the direction of $\vec{A}\times\vec{B}$. Figure 1.17a shows this **right-hand rule** and describes a second way to think about this rule.

Similarly, we determine the direction of $\vec{B}\times\vec{A}$ by rotating $\vec{B}$ into $\vec{A}$ as in Fig. 1.17b. The result is a vector that is *opposite* to the vector $\vec{A}\times\vec{B}$. The vector product is *not* commutative but instead is *anticommutative:* For any two vectors $\vec{A}$ and $\vec{B}$,

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}. \qquad (1.21)$$

Using Components to Calculate the Vector Product

If we know the components of $\vec{A}$ and $\vec{B}$, we can calculate the components of the vector product by using a procedure similar to that for the scalar product. First we work out the multiplication table for unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$, all three of which are perpendicular to each other (**Fig. 1.18a**). The vector product of any vector with itself is zero, so

$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \mathbf{0}.$$

The boldface zero is a reminder that each product is a zero *vector* - that is, one with all components equal to zero and an undefined direction. Using Eqs. (1.20) and (1.21) and the right-hand rule, we find

$$\hat{i}\times\hat{j} = -\hat{j}\times\hat{i} = \hat{k},$$

$$\hat{j}\times\hat{k} = -\hat{k}\times\hat{j} = \hat{i}, \qquad (1.22)$$

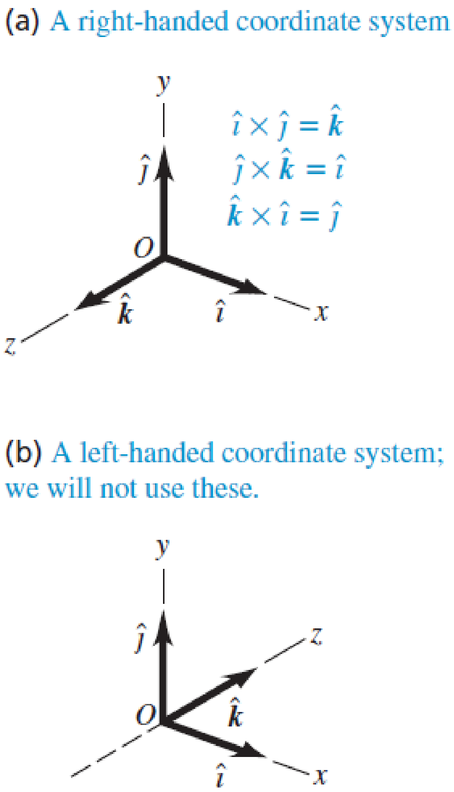


Figure 1.18 - (a) We'll always use a right-handed coordinate system, like this one. (b) We'll never use a lefthanded coordinate system (in which $\hat{i}\times\hat{j} = -\hat{k}$, and so on)

$$\hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j}.$$

Next we express $\vec{A}$ and $\vec{B}$ in terms of their components and the corresponding unit vectors, and we expand the expression for the vector product:

$$\begin{aligned}\vec{A}\times\vec{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) = \\ &= A_x\hat{i}\times B_x\hat{i} + A_x\hat{i}\times B_y\hat{j} + A_x\hat{i}\times B_z\hat{k} + A_y\hat{j}\times B_x\hat{i} + A_y\hat{j}\times B_y\hat{j} + A_y\hat{j}\times B_z\hat{k} + \\ &+ A_z\hat{k}\times B_x\hat{i} + A_z\hat{k}\times B_y\hat{j} + A_z\hat{k}\times B_z\hat{k}.\end{aligned} \qquad (1.23)$$

We can also rewrite the individual terms in Eq. (1.23) as $A_x\hat{i}\times B_y\hat{j} = A_xB_y(\hat{i}\times\hat{j})$, and so on. Evaluating these by using the multiplication table for the unit vectors in Eqs. (1.22) and then grouping the terms, we get

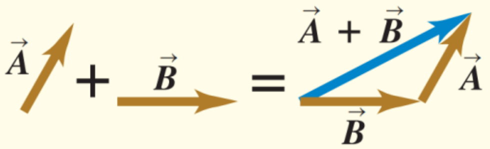
$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}\,. \qquad (1.24)$$

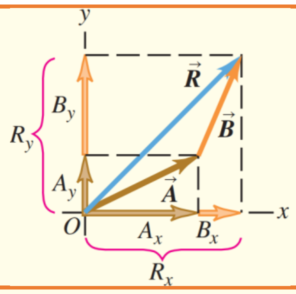
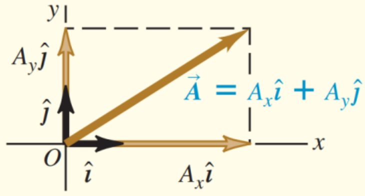
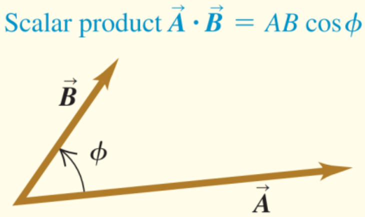
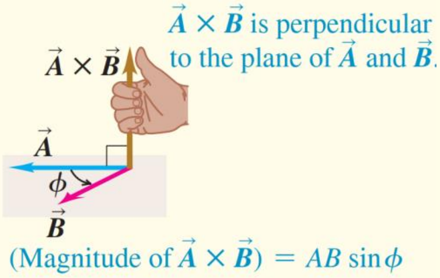
If you compare Eq. (1.24) with Eq. (1.14), you'll see that the components of $\vec{C} = \vec{A}\times\vec{B}$ are

Components of vector (cross) product $\vec{A}\times\vec{B}$

$$C_x = A_yB_z - A_zB_y \qquad C_y = A_zB_x - A_xB_z \qquad C_z = A_xB_y - A_yB_x \qquad (1.25)$$

$A_x, A_y, A_z =$ components of $\vec{A}$ $\qquad$ $B_x, B_y, B_z =$ components of $\vec{B}$

CHAPTER 1: SUMMARY	
Physical quantities and units: Three fundamental physical quantities are mass, length, and time. The corresponding fundamental SI units are the kilogram, the meter, and the second. Derived units for other physical quantities are products or quotients of the basic units. Equations must be dimensionally consistent; two terms can be added only when they have the same units.	
Significant figures: The accuracy of a measurement can be indicated by the number of significant figures or by a stated uncertainty. The significant figures in the result of a calculation are determined by the rules summarized in Table 1.2. When only crude estimates are available for input data, we can often make useful order-of-magnitude estimates	Significant figures in magenta $\pi = \dfrac{C}{2r} = \dfrac{0.424\text{ m}}{2(0.06750\text{ m})} = 3.14$ $123.62 + 8.9 = 132.5$
Scalars, vectors, and vector addition: Scalar quantities are numbers and combine according to the usual rules of arithmetic. Vector quantities have direction as well as magnitude and combine according to the rules of vector addition. The negative of a vector has the same magnitude but points in the opposite direction	

Vector components and vector addition: Vectors can be added by using components of vectors. The x-component of $\vec{R}=\vec{A}+\vec{B}$ is the sum of the x-components of $\vec{A}$ and $\vec{B}$, and likewise for the y- and z-components	$R_x = A_x + B_x$ $R_y = A_y + B_y$ $R_z = A_z + B_z$	
Unit vectors: Unit vectors describe directions in space. A unit vector has a magnitude of 1, with no units. The unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$, aligned with the x-, y-, and z-axes of a rectangular coordinate system, are especially useful	$\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$	
Scalar product: The scalar product $C=\vec{A}\cdot\vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$ is a scalar quantity. It can be expressed in terms of the magnitudes of $\vec{A}$ and $\vec{B}$ and the angle ϕ between the two vectors, or in terms of the components of $\vec{A}$ and $\vec{B}$. The scalar product is commutative; $\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}$. The scalar product of two perpendicular vectors is zero	$\vec{A}\cdot\vec{B}=AB\cos\phi=\lvert\vec{A}\rvert\lvert\vec{B}\rvert\cos\phi$ $\vec{A}\cdot\vec{B}=A_xB_x+A_yB_y+A_zB_z$	
Vector product: The vector product $\vec{C}=\vec{A}\times\vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$ is a third vector $\vec{C}$. The magnitude of $\vec{A}\times\vec{B}$ depends on the magnitudes of $\vec{A}$ and $\vec{B}$ and the angle ϕ between the two vectors. The direction of $\vec{A}\times\vec{B}$ is perpendicular to the plane of the two vectors being multiplied, as given by the right-hand rule. The components of $\vec{C}=\vec{A}\times\vec{B}$ can be expressed in terms of the components of $\vec{A}$ and $\vec{B}$. The vector product is not commutative; $\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$. The vector product of two parallel or antiparallel vectors is zero	$C=AB\sin\phi$ $C_x=A_yB_z-A_zB_y$ $C_y=A_zB_x-A_xB_z$ $C_z=A_xB_y-A_yB_x$	

2 MOTION ALONG A STRAIGHT LINE

What distance must an airliner travel down a runway before it reaches take-off speed? When you throw a ball straight up in the air, how high does it go? Whe na glass slips from your hand, how much time do you have to catch it before it hits the floor? These are the kinds of questions you'll learn to answer in this chapter. Mechanics is the study of the relationships among force, matter, and motion. In this chapter and the next we'll study kinematics, the part of mechanics that enables us to describe motion. Later we'll study dynamics, which helps us understand why objects move in different ways.

In this chapter we'll concentrate on the simplest kind of motion: an object moving along a straight line. To describe this motion, we introduce the physical quantities velocity and acceleration. In physics these quantities have definitions that are more precise and slightly different from the ones used in everyday language. Both velocity and acceleration are vectors: As you learned in Chapter 1, this means that they have both magnitude and direction. Our concern in this chapter is with motion along a straight line only, so we won't need the full mathematics of vectors just yet. But using vectors will be essential in Chapter 3 when we consider motion in two or three dimensions.

We'll develop simple equations to describe straight-line motion in the important special case when acceleration is constant. An example is the motion of a freely falling object. We'll also consider situations in which acceleration varies during the motion; in this case, it's necessary to use integration to describe the motion.

2.1 Displacement, Time, and Average Velocity

Suppose a drag racer drives her dragster along a straight track (**Fig. 2.1**). To study the dragster's motion, we need a coordinate system. We choose the *x*-axis to lie along the dragster's straight-line path, with the origin *O* at the starting line. We also choose a point on the dragster, such as its front end, and represent the entire dragster by that point. Hence we treat the dragster as a **particle.**

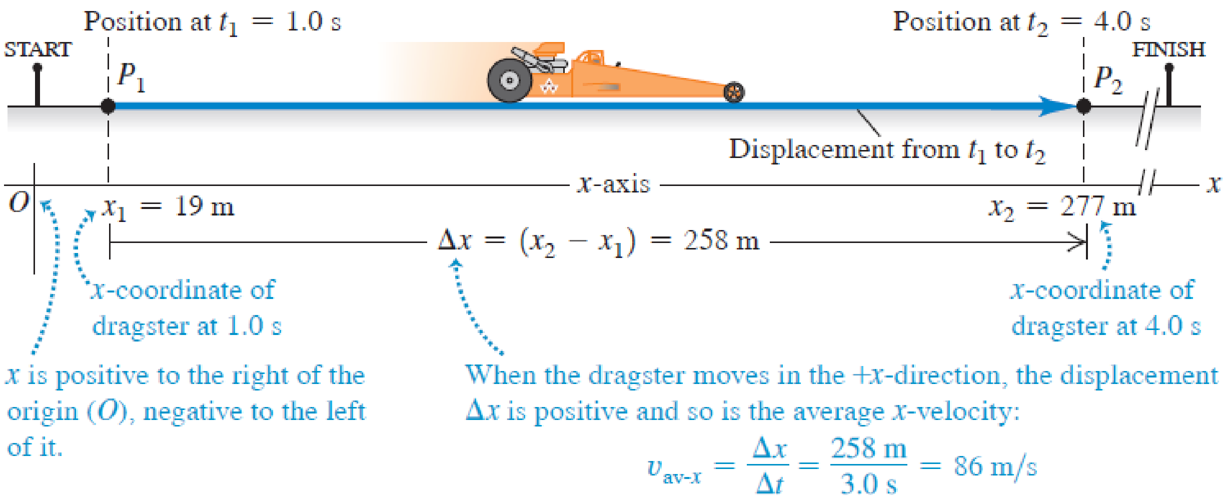


Figure 2.1 - Positions of a dragster at two times during its run

A useful way to describe the motion of this particle is in terms of the change in its coordinate *x* over a time interval. Suppose that 1.0 s after the start the front part of the dragster is at the point P_1, 19 m from the origin, and 4.0 s after the start - at the point P_2, at a distance of 277 m from the origin. The *displacement* of the particle is a vector that points from P_1 to P_2 (see Section 1.7). Figure 2.1 shows that this vector points along the *x*-axis. The *x*-component (see Section 1.8) of the displacement is the change in the value of *x*, (277 m – 19 m) = 258 m, that took place during the time interval of (4.0 s - 1.0 s) = 3.0 s. We define the dragster's **average velocity** during this time interval as a *vector* whose *x*-component is the change in *x* divided by the time interval: (258 m)/(3.0 s) = 86 m/s.

In general, the average velocity depends on the particular time interval chosen. For a 3.0 s time interval *before* the start of the race, the dragster is at rest at the starting line and has zero displacement, so its average velocity for this time interval is zero.

Let's generalize the concept of average velocity. At time t_1 the dragster is at point P_1, with coordinate x_1, and at time t_2 it is at point P_2, with coordinate x_2. The displacement of the dragster during the time interval from t_1 to t_2 is the vector from P_1 to P_2. The x-component of the displacement, denoted Δx, is the change in the coordinate x:

$$\Delta x = x_2 - x_1. \tag{2.1}$$

The dragster moves along the x-axis only, so the y- and z-components of the displacement are equal to zero.

The x-component of average velocity, or the **average x-velocity,** is the x-component of displacement, Δx, divided by the time interval Δt during which the displacement occurs. We use the symbol υ_{av-x} for average x-velocity (the subscript "av" signifies average value, and the subscript x indicates that this is the x-component):

Average x-velocity of a particle in straight-line motion during time interval from t_1 to t_2

x-component of the particle's displacement

$$\upsilon_{av\text{-}x} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \tag{2.2}$$

Final x-coordinate minus initial x-coordinate

Time interval

Final time minus initial time

As an example, for the dragster in Fig. 2.1, x_1 = 19 m, x_2 = 277 m, t_1 = 1.0 s, and t_2 = 4.0 s. So Eq. (2.2) gives

$$\upsilon_{av-x} = \frac{277\,\text{m} - 19\,\text{m}}{4.0\,\text{s} - 1.0\,\text{s}} = 86\,\text{m/s}.$$

The average x-velocity of the dragster is positive. This means that during the time interval, the coordinate x increased and the dragster moved in the positive x-direction (to the right in Fig. 2.1).

If a particle moves in the *negative* x-direction during a time interval, its average velocity for that time interval is negative. For example, suppose an official's truck moves to the left along the track (**Fig. 2.2**). The truck is at x_1 = 277 m at t_1 = 16.0 s and is at x_2 = 19 m at t_2 = 25.0 s. Then Δx = (19 m – 277 m) = –258 m and Δt = (25.0 s – 16.0 s) = 9.0 s. The x-component of average velocity is $\upsilon_{av-x} = \Delta x/\Delta t$ = (–258 m)/(9.0 s) = –29 m/s.

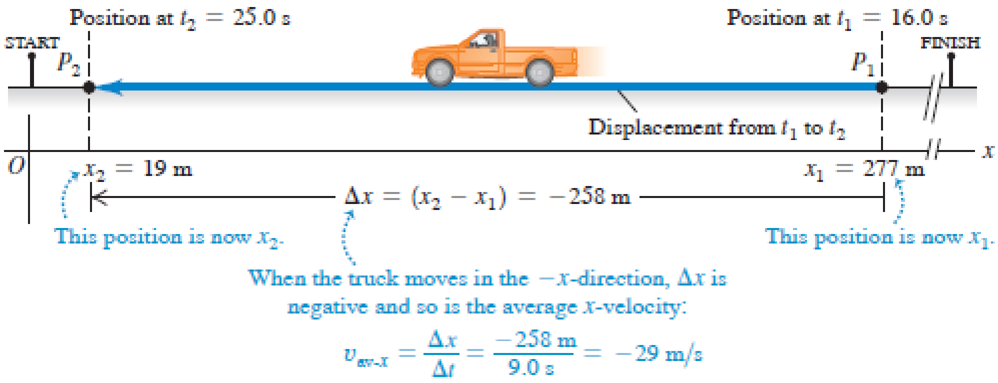


Figure 2.2 - Positions of an official's truck at two times during its motion. The points $P1$ and $P2$ now indicate the positions of the truck, not the dragster, and so are the reverse of Fig. 2.1

CAUTION! **The sign of average *x*-velocity.** In our example positive υ_{av-x} means motion to the right, as in Fig. 2.1, and negative υ_{av-x} means motion to the left. But that's *only* because we chose the +*x*-direction to be to the right. Had we chosen the +*x*-direction to be to the left, the average *x*-velocity υ_{av-x} would have been negative for the dragster moving to the right. In many problems the direction of the coordinate axis is yours to choose. Once you've made your choice, you *must* take it into account when interpreting the signs of υ_{av-x} and other quantities that describe motion.

With straight-line motion we sometimes call Δx simply the displacement and υ_{av-x} - simply the average velocity. But remember that these are the *x*-components of vector quantities that, in this special case, have *only* *x*-components. In Chapter 3, displacement, velocity, and acceleration vectors will have two or three nonzero components.

Figure 2.3 is a graph of the dragster's position as a function of time - that is, an ***x-t* graph.** The curve in the figure *does not* represent the dragster's path; as Fig. 2.1 shows, the path is a straight line. Rather, the graph represents how the dragster's position changes with time. The points p_1 and p_2 on the graph correspond to the points P_1 and P_2 along the dragster's path. Line $p1\ p2$ is the hypotenuse of a right triangle with vertical side $\Delta x = x_2 – x_1$ and horizontal side $\Delta t = t_2 – t_1$. The average *x*-velocity $\upsilon_{av-x} = \Delta x / \Delta t$ of the dragster equals the *slope* of the line $p1\ p2$ - that is, the ratio of the triangle's vertical side Δx to its horizontal side Δt. (The slope has units of meters divided by seconds, or m/s, the correct units for average *x*-velocity).

The average *x*-velocity depends on only the total displacement $\Delta x = x_2 – x_1$ that occurs during the time interval $\Delta t = t_2 – t_1$, not on what happens during the time interval. At time t_1 a motorcycle might have raced past the dragster at point P_1 in Fig. 2.1, then slowed down to pass through point P_2 at the same time t_2 as the dragster. Both vehicles have the same displacement during the same time interval and so have the same average *x*-velocity.

If distance is given in meters and time in seconds, average velocity is measured in meters per second, or m/s. Other common units of velocity are kilometers per hour (km/h), miles per hour (1 mi/h = 1.609 km/h), and knots (1 knot =1 nautical mile/h = 1.852 km/h).

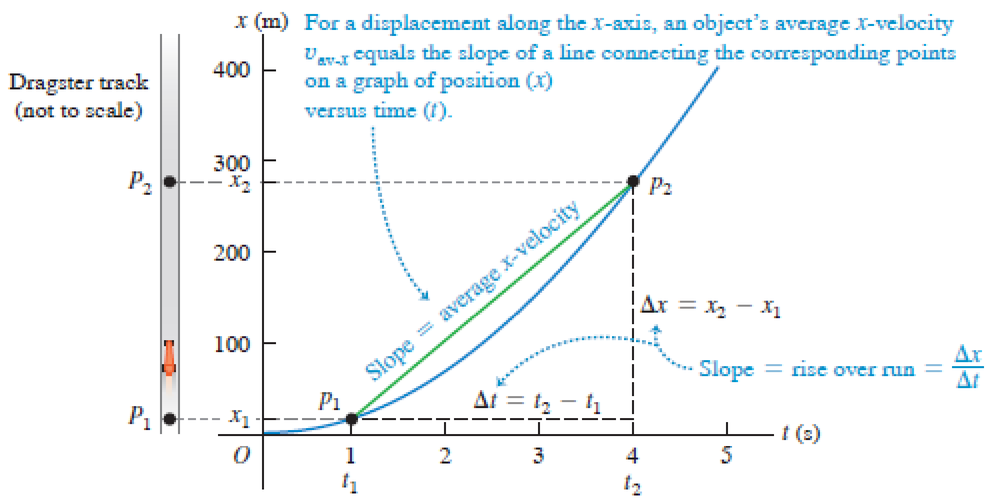


Figure 2.3 - A graph of the position of a dragster as a function of time

2.2 Instantaneous Velocity

Sometimes average velocity is all you need to know about a particle's motion. For example,a race along a straight line is really a competition to see whose average velocity, υ_{av-x}, has the greatest magnitude. The prize goes to the competitor who can travelthe displacement Δx from the start to the finish line in the shortest time interval, Δt.

But the average velocity of a particle during a time interval can't tell us how fast or in what direction the particle was moving at any given time during the interval. For that we need to know the **instantaneous velocity,** or the velocity at a specific instant of time or specific point along the path.

CAUTION! **How long is an instant**? You might use the phrase "It lasted just an instant" to refer to something that spanned a very short time interval. But in physics an instant has no duration at all; It refers to a single value of time.

To find the instantaneous velocity of the dragster in Fig. 2.1 at point P_1, we move point P_2 closer and closer to point P_1 and compute the average velocity $\upsilon_{av-x} = \Delta x/\Delta t$ over the ever-shorter displacement and time interval. Both Δx and Δt become very small, but their ratio does not necessarily become small. In the language of calculus, the limit of $\Delta x/\Delta t$ as Δt approaches zero is called the **derivative** of x with respect to t and is written dx/dt. We use the symbol υ_x, with no "av" subscript, for the instantaneous velocity along the x-axis,or the **instantaneous** ***x*****-velocity:**

The **instantaneous** ***x*****-velocity** of a particle in **straight-line motion** ...

$$\upsilon_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \qquad (2.3)$$

... equals the limit of the particle's average x-velocity as the time interval approaches zero ...

... and equals the instantaneous rate of change of the particle's x-coordinate.

The time interval Δt is always positive, so vx has the same algebraic sign as Δx. A positive value of υ_x means that x is increasing and the motion is in the positive x-direction; a negative value of υ_x means that x is decreasing and the motion is in the negative x-direction. An object can have positive x and negative υ_x, or the reverse; x tells us where the object is, while υ_x tells us how it's moving.

Instantaneous velocity, like average velocity, is a vector; Eq. (2.3) defines its x-component. In straight-line motion, all other components of instantaneous velocity are zero. In this case we often call υ_x simply the instantaneous velocity. (In Chapter 3 we'll deal with the general case in which the instantaneous velocity can have nonzero x-, y-, and z-components). When we use the term "velocity," we'll always mean instantaneous rather than average velocity.

"Velocity" and "speed" are used interchangeably in everyday language, but they have distinct definitions in physics. We use the term **speed** to denote distance traveled divided by time, on either an average or an instantaneous basis. Instantaneous *speed*, for which we use the symbol υ with *no* subscripts, measures how fast a particle is moving; instantaneous *velocity* measures how fast *and* in what direction it's moving. Instantaneous speed is the magnitude of instantaneous velocity and so can never be negative. For example, a particle with instantaneous velocity $\upsilon_x = 25$ m/s and a second particle with $\upsilon_x = -25$ m/s are moving in opposite directions at the same instantaneous speed 25 m/s.

CAUTION! **Average speed and average velocity.** Average speed is *not* the magnitude of average velocity. When Cesar Cielo set a world record in 2009 by swimming 100.0 m in 46.91 s, his average speed was (100.0 m)/(46.91 s) = 2.132 m/s. But because he swam two lengths in a 50 m pool, he started and ended at the same point and so had zero total displacement and zero average *velocity!* Both average speed and instantaneous speed are scalars, not vectors, because these quantities contain no information about direction.

Finding Velocity on an *x-t* Graph

We can also find the x-velocity of a particle from the graph of its position as a functionof time. Suppose we want to find the x-velocity of the dragster in Fig. 2.1 at point P_1. As point P_2 in Fig. 2.1 approaches point P_1, point p_2 in the x-t graphs of **Figs. 2.4a** and **2.4b** approaches point p_1 and the average x-velocity is calculated over shorter time intervals Δt. In the limit that $\Delta t \to 0$, shown in Fig. 2.4c, the slope of the line $p_1 p_2$ equals the slope of the line tangent to the curve at point p_1. Thus, *on a graph of position as* *a function of time for straight-line motion, the instantaneous x-velocity at any point is equal to the slope of the tangent to the curve at that point.*

If the tangent to the *x-t* curve slopes upward to the right, as in Fig. 2.4c, then its slope is positive, the *x*-velocity is positive, and the motion is in the positive *x*-direction. If the tangent slopes downward to the right, the slopes of the *x-t* graph and the *x*-velocity are negative, and the motion is in the negative *x*-direction. When the tangent is horizontal, the slope and the *x*-velocity are zero. **Figure 2.5** illustrates these three possibilities.

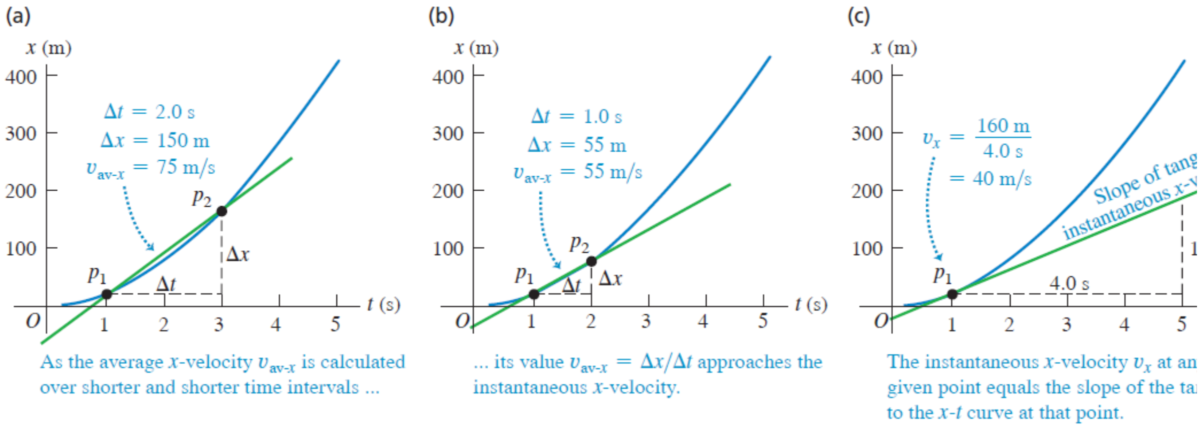


Figure 2.7 - Using an *x-t* graph to go from (a), (b) average *x*-velocity to (c) instantaneous *x*-velocity *vx*. In (c) we find the slope of the tangent to the *x-t* curve by dividing any vertical interval (with distance units) along the tangent by the corresponding horizontal interval (with time units)

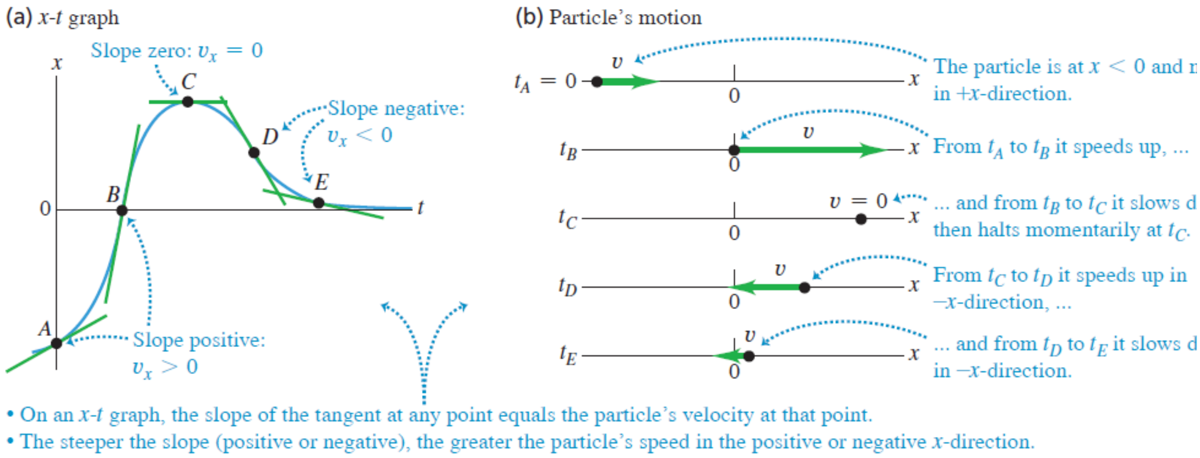


Figure 2.8 - (a) The *x-t* graph of the motion of a particular particle. (b) A motion diagram showing the position and velocity of the particle at each of the times labeled on the *x-t* graph

Figure 2.8 depicts the motion of a particle in two ways: as (a) an *x-t* graph and (b) a **motion diagram** that shows the particle's position at various instants (like frames from a video of the particle's motion) as well as arrows to represent the particle's velocity at each instant. We'll use both *x-t* graphs and motion diagrams in this chapter to represent motion. You'll find it helpful to draw *both* an *x-t* graph and a motion diagram when you solve any problem involving motion.

2.3 Average and Instantaneous Acceleration

Just as velocity describes the rate of change of position with time, *acceleration* describes the rate of change of velocity with time. Like velocity, acceleration is a vector quantity. When the motion is along a straight line, its only nonzero component is along that line. In everyday language, acceleration refers only to speeding up; in physics, acceleration refers to *any* kind of velocity change, so we say an object accelerates if it is either speeding up or slowing down.

Average Acceleration

Let's consider again a particle moving along the x-axis. Suppose that at time t_1 the particle is at point P_1 and has x-component of (instantaneous) velocity υ_{1x}, and at a later time t_2 it is at point P_2 and has x-component of velocity υ_{2x}. So the x-component of velocity changes by an amount $\Delta\upsilon_x = \upsilon_{2x} - \upsilon_{2x}$ during the time interval $\Delta t = t_2 – t_1$. As the particle moves from P_1 to P_2, its **average acceleration** is a vector quantity whose x-component a_{av-x} (called the **average *x*-acceleration**) equals $\Delta\upsilon_x$, the change in the x-component of velocity, divided by the time interval Δt:

Average *x*-acceleration of a particle in **straight-line motion** during time interval from t_1 to t_2

Change in x-component of the particle's velocity

$$a_{\text{av-}x} = \frac{\Delta v_x}{\Delta t} = \frac{v_{2x} - v_{1x}}{t_2 - t_1} \qquad (2.4)$$

Final x-velocity minus initial x-velocity

Time interval

Final time minus initial time

For straight-line motion along the x-axis we'll often call a_{av-x} simply the average acceleration. (We'll encounter the other components of the average acceleration vector in Chapter 3). If we express velocity in meters per second and time in seconds, then average acceleration is in meters per second per second. This is usually written as m>s^2 and is read "meters per second squared".

CAUTION! **Don't confuse velocity and acceleration.** Velocity describes how an object's position changes with time; it tells us how fast and in what direction the object moves. Acceleration describes how the velocity changes with time; it tells us how the speed and direction of motion change. Another difference is that you can *feel* acceleration but you can't feel velocity. If you're a passenger in a car that accelerates forward and gains speed, you feel pushed backward in your seat; if it accelerates backward and loses speed, you feel pushed forward. If the velocity is constant and there's no acceleration, you feel neither sensation. (We'll explain these sensations in Chapter 4).

Instantaneous Acceleration

We can now define **instantaneous acceleration** by following the same procedure that we used to define instantaneous velocity. Suppose a race car driver is driving along a straightaway as shown in **Fig. 2.6**. To define the instantaneous acceleration at point P_1 ,we take point P_2 in Fig. 2.6 to be closer and closer to P_1 so that the average acceleration is computed over shorter and shorter time intervals. Thus

The **instantaneous *x*-acceleration** of a particle in **straight-line motion** ...

$$a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt} \qquad (2.5)$$

... equals the limit of the particle's average x-acceleration as the time interval approaches zero ...

... and equals the instantaneous rate of change of the particle's x-velocity.

Figure 2.6 - A Grand Prix car at two points on the straightaway

In Eq. (2.5) a_x is the x-component of the acceleration vector, which we call the **instantaneous *x*-acceleration;** in straight-line motion, all other components of this vector are zero. From now on, when we use the term "acceleration," we'll always mean instantaneous acceleration, not average acceleration.

Finding Acceleration on a *vx-t* Graph or an *x-t* Graph

In Section 2.2 we interpreted average and instantaneous x-velocity in terms of the slope of a graph of position versus time. In the same way, we can interpret average and instantaneous x-acceleration by using a graph of instantaneous velocity υ_x versus time t - that is, a υ_x**-*t* graph** (**Fig. 2.7**). Points p_1 and p_2 on the graph correspond to points P_1 and P_2 in Fig. 2.6. The average x-acceleration $a_{\text{av}-x} = \Delta\upsilon_x / \Delta t$ during this interval is the slope of the line p_1p_2.

As point P_2 in Fig. 2.6 approaches point P_1, point p_2 in the υ_x-t graph of Fig. 2.7 approaches point p_1 , and the slope of the line p_1p_2 approaches the slope of the line tangent to the curve at point p_1. Thus, *on a graph of x-velocity as a function of time, the instantaneous x-acceleration at any point is equal to the slope of the tangent to the curve at that point.* Tangents drawn at different points along the curve in Fig. 2.7 have different slopes, so the instantaneous x-acceleration varies with time.

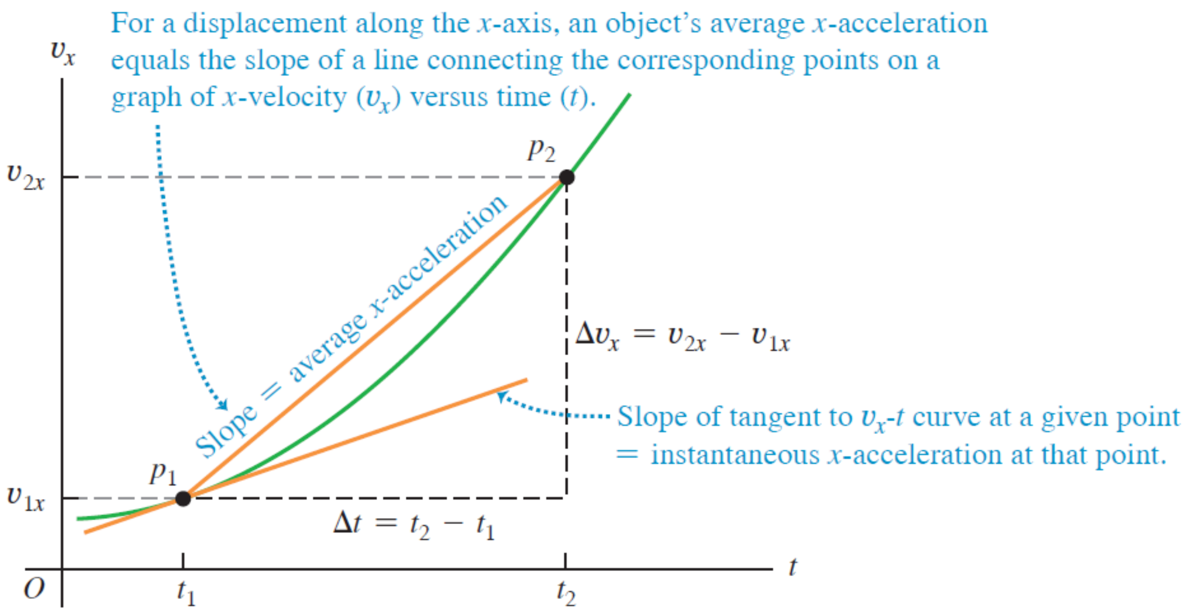


Figure 2.7 - A υ_x-t graph of the motion in Fig. 2.6

CAUTION! **Signs of *x*-acceleration and *x*-velocity.** The algebraic sign of the x-acceleration does *not* tell you whether an object is speeding up or slowing down. You must compare the signs of the x-velocity and the x-acceleration.

The term "deceleration" is sometimes used for a decrease in speed. Because it may mean positive or negative a_x, depending on the sign of υ_x, we avoid this term. We can also learn about the acceleration of an object from a graph of its *position* versus time. Because $a_x = d\upsilon_x / dt$ and $\upsilon_x = dx/dt$, we can write

$$a_x = \frac{d\upsilon_x}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}. \tag{2.6}$$

That is, a_x is the second derivative of x with respect to t. The second derivative of any function is directly related to the *concavity* or *curvature* of the graph of that function (**Fig. 2.8)**). At a point where the x-t graph is concave up (curved upward), such as point A or E in Fig. 2.8a, the x-acceleration is positive and vx is increasing. At a point where the *x*-*t* graph is concave down (curved downward), such as point *C* in Fig. 2.5a, the *x*-acceleration is negative and υ_x is decreasing. At a point where the *x*-*t* graph has no curvature, such as the inflection points *B* and *D* in Fig. 2.8a, the *x*-acceleration is zero and the velocity is not changing.

Examining the curvature of an *x*-*t* graph is an easy way to identify the *sign* of acceleration. This technique is less helpful for determining numerical values of acceleration because the curvature of a graph is hard to measure accurately.

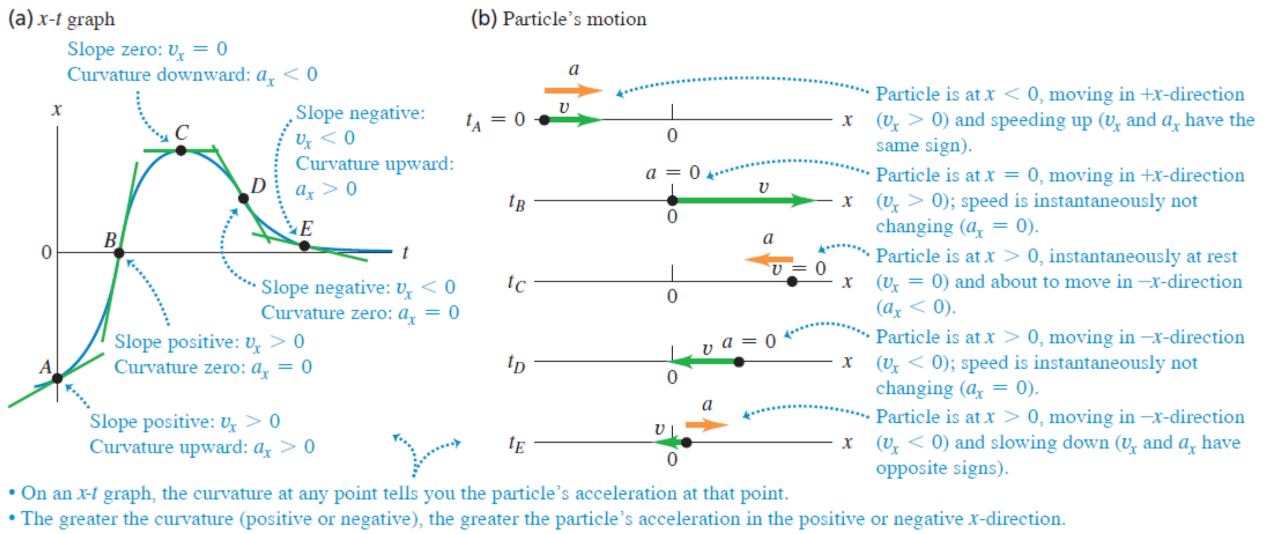


Figure 2.8 - (a) The same x-t graph as shown in Fig. 2.5a. (b) A motion diagram showing the position, velocity, and acceleration of the particle at each of the times labeled on the x-t graph

2.4 Motion with Constant Acceleration

The simplest kind of accelerated motion is straight-line motion with *constant* acceleration. In this case the velocity changes at the same rate throughout the motion. As an example, a falling object has a constant acceleration if the effects of the air are not important. The same is true for an object sliding on an incline or along a rough horizontal surface, or for an airplane being catapulted from the deck of an aircraft carrier.

Figures 2.9 and **2.10** depict a particle moving with constant acceleration in the form of graphs. Since the *x*-acceleration is constant, the $a_x - t$ **graph** (graph of *x*-acceleration versus time) in Fig. 2.9 is a horizontal line. The graph of *x*-velocity versus time, or $\upsilon_x - t$ graph, has a constant *slope* because the acceleration is constant, so this graph is a straight line (Fig. 2.10).

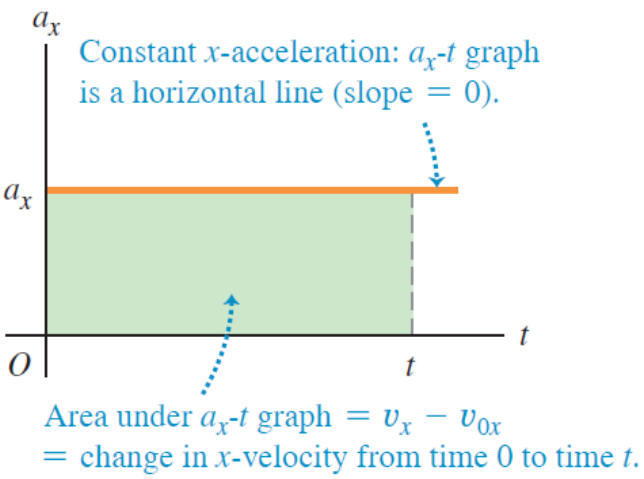


Figure 2.9 - An acceleration-time $(a_x - t)$ graph of straight-line motion with constant positive *x*-acceleration a_x

When the *x*-acceleration a_x is constant, the average *x*-acceleration $a_{\text{av}-x}$ for any time interval is the same as a_x. This makes it easy to derive equations for the position *x* and the *x*-velocity υ_x as functions of time. To find an equation for υ_x, we first replace $a_{\text{av}-x}$ in Eq. (2.4) by a_x:

$$a_x = \frac{\upsilon_{2x} - \upsilon_{1x}}{t_2 - t_1}. \tag{2.7}$$

Now we let $t_1 = 0$ and let t_2 be any later time t. We use the symbol υ_{0x} for the *initial x-velocity* at time $t = 0$; the x-velocity at the later time t is υ_x. Then Eq. (2.7) becomes

$$a_x = \frac{\upsilon_x - \upsilon_{0x}}{t - 0} \quad \text{or}$$

x-velocity at time t of a particle with **constant x-acceleration**

x-velocity of the particle at time 0

$$\upsilon_x = \upsilon_{0x} + a_x t \tag{2.8}$$

Constant x-acceleration of the particle

Time

In Eq. (2.8) the term $a_x t$ is the product of the constant rate of change of x-velocity, a_x, and the time interval t. Therefore, it equals the *total* change in x-velocity from $t = 0$ to time t. The x-velocity υ_x at any time t then equals the initial x-velocity υ_{0x} (at $t = 0$) plus the change in x-velocity $a_x t$. (Fig. 2.10).

Equation (2.8) also says that the change in x-velocity $\upsilon_x - \upsilon_{0x}$ of the particle between $t = 0$ and any later time t equals the *area* under the $a_x - t$ graph between those two times. You can verify this from Fig. 2.9: Under this graph is a rectangle of vertical side a_x, horizontal side t, and area $a_x t$. From Eq. (2.8) the area $a_x t$ is indeed equal to the change in velocity $\upsilon_x - \upsilon_{0x}$. In Section 2.6 we'll show that even if the x-acceleration is not constant, the change in x-velocity during a time interval is still equal to the area under the $a_x - t$ curve, although then Eq. (2.8) does not apply.

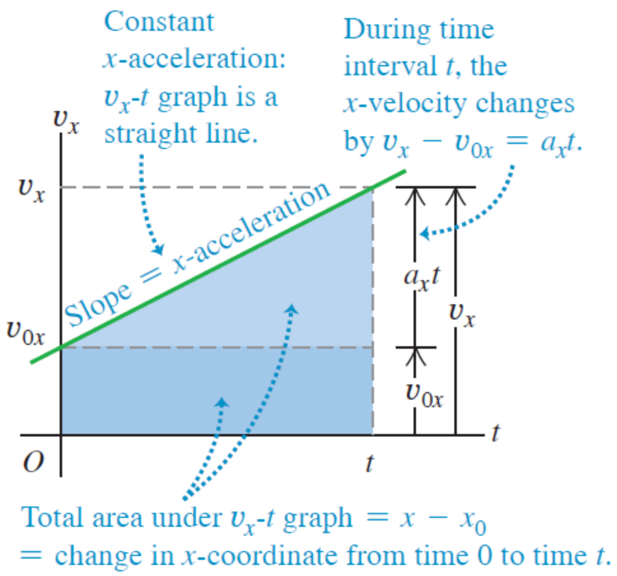


Figure 2.10 - A velocity-time $(\upsilon_x - t)$ graph of straight-line motion with constant positive x-acceleration a_x. The initial x-velocity υ_{0x} is also positive in this case

Next we'll derive an equation for the position x as a function of time when the x-acceleration is constant. To do this, we use two different expressions for the average x-velocity υ_{av-x} during the interval from $t = 0$ to any later time t. The first expression comes from the definition of υ_{av-x}, Eq. (2.2), which is true whether or not the acceleration is constant. The position at time $t = 0$, called the *initial position*, is x_0. The position at time t is simply x. Thus for the time interval $\Delta t = t - 0$ the displacement is $\Delta x = x - 0$, and Eq. (2.2) gives

$$\upsilon_{av-x} = \frac{x - x_0}{t}. \tag{2.9}$$

To find a second expression for υ_{av-x}, note that the x-velocity changes at a constant rate if the x-acceleration is constant. In this case the average x-velocity for the time interval from 0 to t is simply the average of the x-velocities at the beginning and end of the interval:

$$\upsilon_{av-x} = \frac{1}{2}(\upsilon_{0x} + \upsilon_x) \text{ (constant } x\text{-acceleration only).} \tag{2.10}$$

Equation (2.10) is *not* true if the x-acceleration varies during the time interval. We also know that with constant x-acceleration, the x-velocity υ_x at any time t is given by Eq. (2.8). Substituting that expression for υ_x into Eq. (2.10), we find

$$\upsilon_{\mathrm{av}-x} = \frac{1}{2}\left(\upsilon_{0x} + \upsilon_{0x} + \right) = \upsilon_{0x} + \frac{1}{2}a_x t \quad \text{(constant } x\text{-acceleration only).} \qquad (2.11)$$

Finally, we set Eqs. (2.9) and (2.11) equal to each other and simplify:

$$\upsilon_{0x} + \frac{1}{2}a_x t = \frac{x - x_0}{t} \quad \text{or}$$

Position of the particle at time 0

Time

Position at time t of a particle with **constant x-acceleration**

$$x = x_0 + \upsilon_{0x}t + \tfrac{1}{2}a_x t^2 \qquad (2.12)$$

x-velocity of the particle at time 0

Constant x-acceleration of the particle

Equation (2.12) tells us that the particle's position at time t is the sum of three terms: its initial position at t = 0, x_0, plus the displacement $\upsilon_{0x}t$ it would have if its x-velocity remained equal to its initial value, plus an additional displacement $a_x t^2/2$ caused by the change in x-velocity.

A graph of Eq. (2.12) - that is, an x-t graph for motion with constant x-acceleration **(Fig. 2.11a) -** is always a *parabola*. Figure 2.12b shows such a graph. The curve intercepts the vertical axis (x-axis) at x_0, the position at t = 0. The slope of the tangent at t = 0 equals υ_{0x}, the initial x-velocity, and the slope of the tangent at any time t equals the x-velocity υ_x at that time. The slope and x-velocity are continuously increasing, so the x-acceleration a_x is positive and the graph in Fig. 2.11b is concave up (it curves upward). If a_x is negative, the x-t graph is a parabola that is concave down (has a downward curvature). If there is zero x-acceleration, the x-t graph is a straight line; if there is a constant x-acceleration, the additional $a_x t^2/2$ term in Eq. (2.12) for x as a function of t curves the graph into a parabola **(Fig. 2.12a)**. Similarly, if there is zero x-acceleration, the υ_x-t graph is a horizontal line (the x-velocity is constant). Adding a constant x-acceleration in Eq. (2.8) gives a slope to the graph (Fig. 2.12b).

Here's another way to derive Eq. (2.12). Just as the change in x-velocity of the particle equals the area under the a_x-t graph, the displacement (change in position) equals the area under the υ_x-t graph. So the displacement $x - x_0$ of the particle between t = 0 and any later time t equals the area under the υ_x-t graph between those times. In Fig. 2.10 we divide the area under the graph into a dark-colored rectangle (vertical side υ_{0x}, horizontal side t, and area $\upsilon_{0x}t$) and a light-colored right triangle (vertical side $a_x t$, horizontal side t, and area $(a_x t)(t)/2 = a_x t^2/2$. The total area under the υ_x-t graph is $x - x_0 = \upsilon_{0x}t + a_x t^2/2$, in accord with Eq. (2.12).

It's often useful to have a relationship for position, x-velocity, and (constant) x- acceleration that does not involve time. To obtain this, we first solve Eq. (2.8) for t and then substitute the resulting expression into Eq. (2.12):

$$t = \frac{\upsilon_x - \upsilon_{0x}}{a_x},$$

$$x = x_0 + \upsilon_{0x}\left(\frac{\upsilon_x - \upsilon_{0x}}{a_x}\right) + \frac{1}{2}a_x\left(\frac{\upsilon_x - \upsilon_{0x}}{a_x}\right)^2.$$

We transfer the term x_0 to the left side, multiply through by $2a_x$, and simplify:

$$2a_x(x - x_0) = 2\upsilon_{0x}\upsilon_x - 2\upsilon_{0x}^2 + \upsilon_x^2 - 2\upsilon_{0x}\upsilon_x + \upsilon_{0x}^2.$$

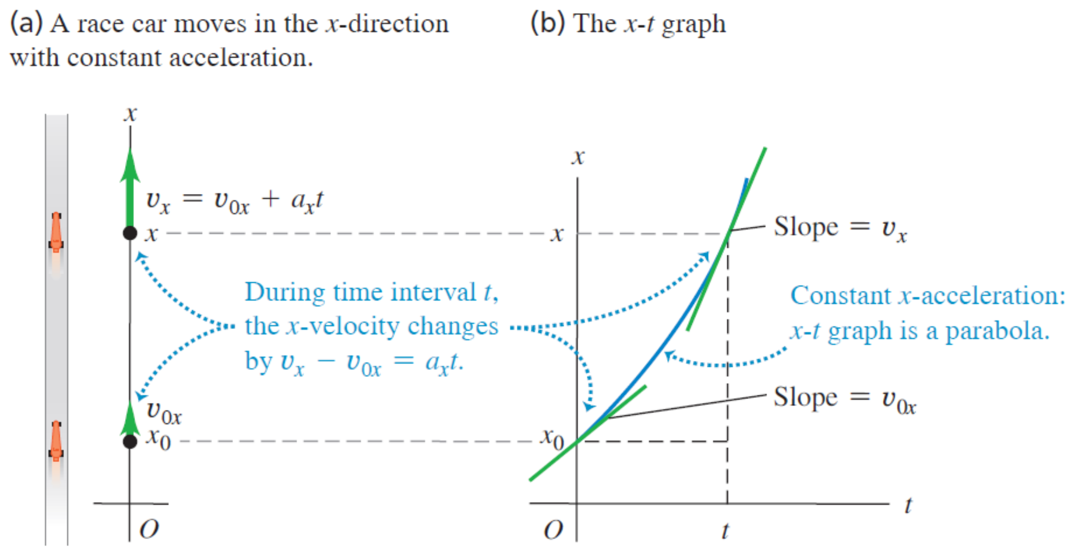


Figure 2.11 - (a) Straight-line motion with constant acceleration. (b) A position-time ($x-t$) graph for this motion (the same motion as is shown in Figs 2.9, and 2.10). For this motion the initial position x_0, the initial velocity υ_{0x}, and the acceleration *ax* are all positive

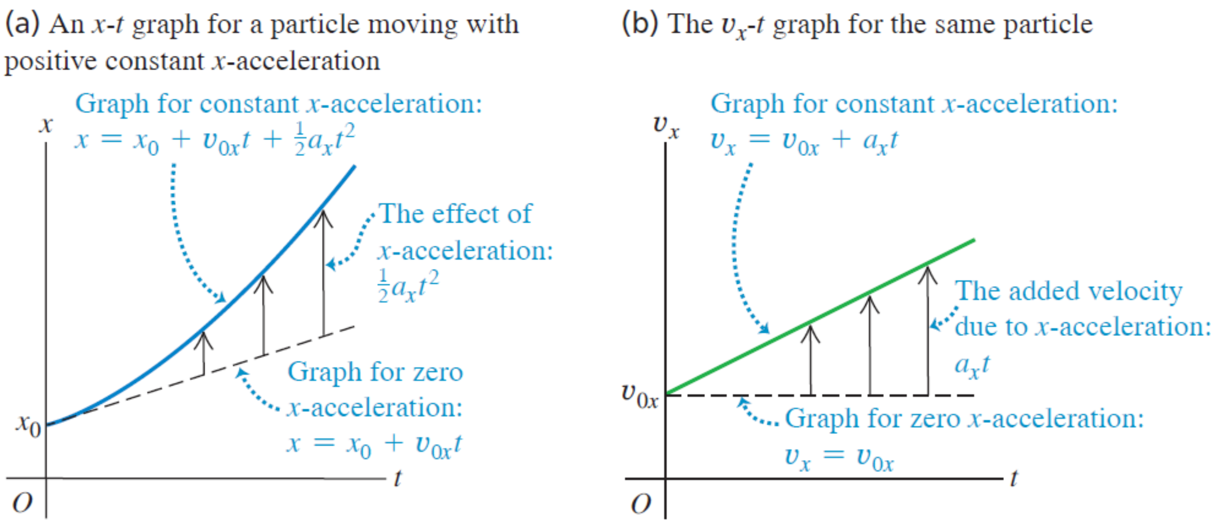


Figure 2.12 - (a) How a constant *x*-acceleration affects a particle's (a) *x-t* graph and (b) υ-*t* graph

Finally,

x-velocity of the particle at time 0

x-velocity at time *t* of a particle with **constant x-acceleration**

$$v_x^2 = v_{0x}^2 + 2a_x(x - x_0) \qquad (2.13)$$

Constant *x*-acceleration of the particle Position of the particle at time *t* Position of the particle at time 0

We can get one more useful relationship by equating the two expressions for υ_{av-x}, Eqs. (2.9) and (2.10), and multiplying through by *t*:

Position of the particle at time 0 Time

Position at time *t* of a particle with **constant x-acceleration**

$$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t \qquad (2.14)$$

x-velocity of the particle at time 0 *x*-velocity of the particle at time *t*

Note that Eq. (2.14) does not contain the *x*-acceleration a_x. This equation can be handy when a_x is constant but its value is unknown.

Equations (2.8), (2.12), (2.13), and (2.14) are the *equations of motion with constant acceleration* (**Table 2.1**). By using these equations, we can solve *any* problem involving straight-line motion of a particle with constant acceleration.

For the particular case of motion with constant *x*-acceleration graphed in Figs. 2.9, 2.10, and 2.11, the values of x_0, υ_{0x}, and a_x are all positive. We recommend that you redraw these figures for cases in which one, two, or all three of these quantities are negative.

Table 2.1 - Equations of Motion with Constant Acceleration

Equation		**Includes Quantities**			
$v_x = v_{0x} + a_x t$	(2.8)	t		v_x	a_x
$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$	(2.12)	t	x		a_x
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	(2.13)		x	v_x	a_x
$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$	(2.14)	t	x	v_x	

PROBLEM-SOLVING STRATEGY

IDENTIFY *the relevant concepts:* In most straight-line motion problems, you can use the constant-acceleration Equations (2.8), (2.12), (2.13), and (2.14). If you encounter a situation in which the acceleration *isn't* constant, you'll need a different approach (see Section 2.6).

SET UP *the problem* using the following steps:

1. Read the problem carefully. Make a motion diagram showing the location of the particle at the times of interest. Decide where to place the origin of coordinates and which axis direction is positive. It's often helpful to place the particle at the origin at time $t = 0$; then $x_0 = 0$. Your choice of the positive axis direction automatically determines the positive directions for *x*-velocity and *x*-acceleration. If x is positive to the right of the origin, then υ_x and a_x are also positive toward the right.

2. Identify the physical quantities (times, positions, velocities, and accelerations) that appear in Eqs. (2.8), (2.12), (2.13), and (2.14) and assign them appropriate symbols: t, x, x_0, υ_x, υ_{0x}, and a_x, or symbols related to those. Translate the prose into physics: "*When* does the particle arrive at its highest point" means "What is the value of t when x has its maximum value?" In Example 2.4, "Where is he when his speed is 25 m/s?" means "What is the value of x when υ_x= 25 m/s?" Be alert for implicit information. For example, "A car sits at a stop light" usually means $\upsilon_{0x} = 0$.

3. List the quantities such as x, x_0, υ_x, υ_{0x}, a_x, and t. Some of them will be known and some will be unknown. Write down the values of the known quantities, and decide which of the unknowns are the target variables. Make note of the *absence* of any of the quantities that appear in the four constant-acceleration equations.

4. Use Table 2.5 to identify the applicable equations. (These are often the equations that don't include any of the absent quantities that you identified in step 3). Usually you'll find a single equation that contains only one of the target variables. Sometimes you must find two equations, each containing the same two unknowns.

5. Sketch graphs corresponding to the applicable equations. The υ_x-t graph of Eq. (2.8) is a straight line with slope *ax*. The *x*-*t* graph of Eq. (2.12) is a parabola that's concave up if a_x is positive and concave down if *ax* is negative.

6. On the basis of your experience with such problems, and taking account of what your sketched graphs tell you, make any qualitative and quantitative predictions you can about the solution.

EXECUTE *the solution:* If a single equation applies, solve it for the target variable, *using symbols only*; then substitute the known values and calculate the value of the target variable. If you have two equations in two unknowns, solve them simultaneously for the target variables.

EVALUATE *your answer:* Take a hard look at your results to see whether they make sense. Are they within the general range of values that you expected?

2.5 Freely Falling Objects

The most familiar example of motion with (nearly) constant acceleration is an object falling under the influence of the earth's gravitational attraction. Such motion has held the attention of philosophers and scientists since ancient times. In the fourth century b.c., Aristotle thought (erroneously) that heavy objects fall faster than light objects, in proportion to their weight. Nineteen centuries later, Galileo (see Section 1.1) argued that an object should fall with a downward acceleration that is constant and independent of its weight.

Experiment shows that if the effects of the air can be ignored, Galileo is right; all objects at a particular location fall with the same downward acceleration, regardless of their size or weight. If in addition the distance of the fall is small compared with the radius of the earth, and if we ignore small effects due to the earth's rotation, the acceleration is constant. The idealized motion that results under all of these assumptions is called **free fall,** although it includes rising as well as falling motion. (In Chapter 3 we'll extend the discussion of free fall to include the motion of projectiles, which move both vertically and horizontally).

The constant acceleration of a freely falling object is called the **acceleration due to gravity,** and we denote its magnitude with the letter g. We'll frequently use the approximate value of g at or near the earth's surface:

$$g = 9.80 \text{ m/s}^2 \text{ (approximate value near the earth's surface).}$$

The exact value varies with location, so we'll often give the value of g at the earth's surface to only two significant figures as 9.8 m/s^2. On the moon's surface, the acceleration due to gravity is caused by the attractive force of the moon rather than the earth, and $g = 1.6$ m/s^2. Near the surface of the sun, $g = 270$ m/s^2.

CAUTION! ***g* is always a positive number.** Because g is the *magnitude* of a vector quantity, it is always a *positive* number. If you take the positive y-direction to be upward, as we do in most situations involving free fall, the y-component of the acceleration is negative and equal to $-g$. Be careful with the sign of g, or you'll have trouble with free-fall problems. ❚

2.6 Velocity and Position by Integration

This section is intended for students who have already learned a little integral calculus. In Section 2.4 we analyzed the special case of straight-line motion with constant acceleration. When a_x is not constant, as is frequently the case, the equations that we derived in that section are no longer valid. But even when *ax* varies with time, we can still use the relationship $\upsilon_x = dx/dt$ to find the x-velocity υ_x as a function of time if the position x is a known function of time. And we can still use $a_x = d\upsilon_x/dt$ to find the x-acceleration a_x as a function of time if the x-velocity υ_x is a known function of time.

In many situations, however, position and velocity are not known functions of time. How can we find the position and velocity in straightline motion from the acceleration function $a_x(t)$?

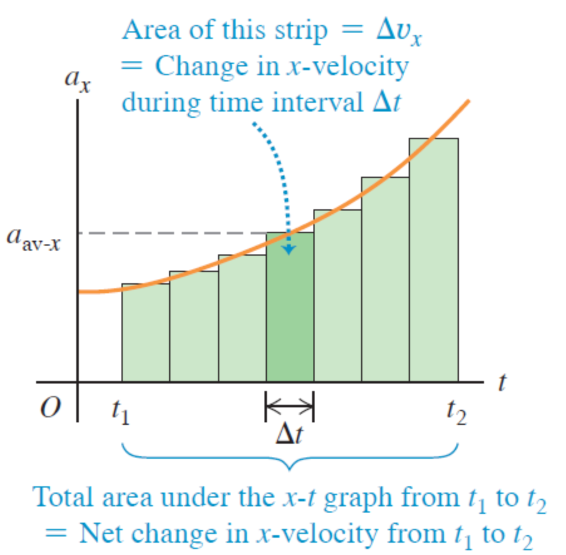


Figure 2.28 - An a_x–t graph for a particle whose x-acceleration is not constant

Figure 2.28 is a graph of x-acceleration versus time for a particle whose acceleration is not constant. We can divide the time interval between times t_1 and t_2 into many smaller subintervals, calling a typical one Δt. Let the average x-acceleration during Δt be $a_{\text{av-}x}$. From Eq. (2.4) the change in x-velocity $\Delta\upsilon_x$ during Δt is

$$\Delta\upsilon_x = a_{\text{av}-x}\cdot\Delta t.$$

Graphically, $\Delta\upsilon_x$ equals the area of the shaded strip with height $a_{\text{av-}x}$ and width Δt - that is, the area under the curve between the left and right sides of Δt. The total change in x-velocity from t_1 to t_2 is the sum of the x-velocity changes $\Delta\upsilon_x$ in the small subintervals. So the total x-velocity change is represented graphically by the *total* area under the $a_{\text{av-}x}$ curve between the vertical lines t_1 and t_2. (In Section 2.4 we showed this for the special case in which a_x is constant).

In the limit that all the Δt's become very small and they become very large in number, the value of $a_{\text{av-}x}$ for the interval from any time t to $t + \Delta t$ approaches the instantaneous x-acceleration a_x at time t. In this limit, the area under the a_x–t curve is the *integral* of a_x (which is in general a function of t) from t_1 to t_2. If υ_{1x} is the x-velocity of the particle at time t_1 and υ_{2x} is the velocity at time t_2, then

$$\upsilon_{2x} - \upsilon_{1x} = \int_{\upsilon_{1x}}^{\upsilon_{2x}} d\upsilon_x = \int_{t_1}^{t_2} a_x dt. \tag{2.15}$$

The change in the x-velocity υ_x is the time integral of the x-acceleration a_x.

We can carry out exactly the same procedure with the curve of x-velocity versus time. If x_1 is a particle's position at time t_1 and x_2 is its position at time t_2, from Eq. (2.2) the displacement Δx during a small time interval Δt is equal to $\upsilon_{\text{av-}x}\Delta t$, where $\upsilon_{\text{av-}x}$ is the average x-velocity during Δt. The total displacement x_2–x_1 during the interval t_2–t_1 is given by

$$x_2 - x_1 = \int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} \upsilon_x dt. \tag{2.16}$$

The change in position x - that is, the displacement - is the time integral of x-velocity υ_x. Graphically, the displacement between times t_1 and t_2 is the area under the υ_x-t curve between those two times. [This is the same result that we obtained in Section 2.4 for the special case in which vx is given by Eq. (2.8)].

If $t_1 = 0$ and t_2 is any later time t, and if x_0 and υ_{0x} are the position and velocity, respectively, at time $t = 0$, then we can rewrite Eqs. (2.15) and (2.16) as

x-velocity of a particle at time t — x-velocity of the particle at time 0

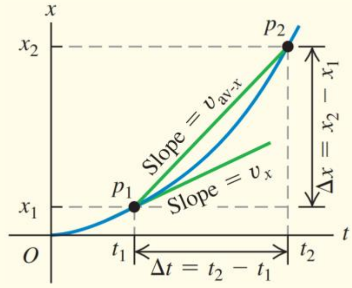
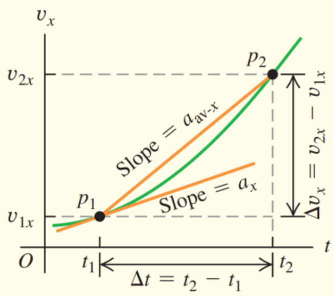
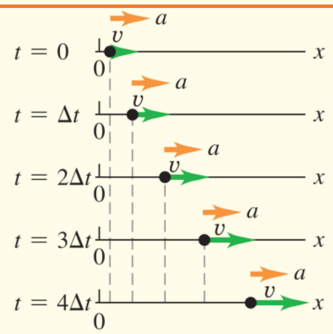
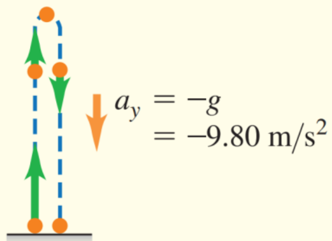
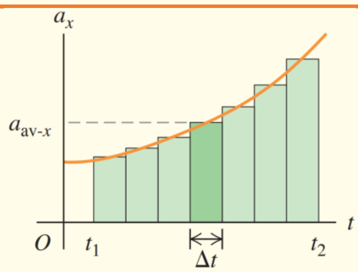
$$\upsilon_x = \upsilon_{0x} + \underbrace{\int_0^t a_x\, dt}_{\text{Integral of the } x\text{-acceleration of the particle from time 0 to time } t} \tag{2.17}$$

Position of a particle at time t — Position of the particle at time 0

$$x = x_0 + \underbrace{\int_0^t \upsilon_x\, dt}_{\text{Integral of the } x\text{-velocity of the particle from time 0 to time } t} \tag{2.18}$$

If we know the x-acceleration a_x as a function of time and we know the initial velocity υ_{0x}, we can use Eq. (2.17) to find the x-velocity υ_x at any time; that is, we can find υ_x as a function of time. Once we know this function, and given the initial position x_0, we can use Eq. (2.18) to find the position x at any time.

CHAPTER 2: SUMMARY

Straight-line motion, average and instantaneous x-velocity: When a particle moves along a straight line, we describe its position with respect to an origin O by means of a coordinate such as x. The particle's average x-velocity $\upsilon_{\text{av}-x}$ during a time interval $\Delta t = t_2 - t_1$ is equal to its displacement $\Delta x = x_2 - x_1$ divided by Δt. The instantaneous x-velocity υ_x at any time t is equal to the average x-velocity over the time interval from t to $t + \Delta t$ in the limit that Δt goes to zero. Equivalently, υ_x is the derivative of the position function with respect to time	$\upsilon_{\text{av}-x} = \dfrac{\Delta x}{\Delta t} = \dfrac{x_2 - x_1}{t_2 - t_1}$ $\upsilon_x = \lim\limits_{\Delta t \to 0} \dfrac{\Delta x}{\Delta t} = \dfrac{dx}{dt}$	
Average and instantaneous x-acceleration: The average xacceleration $a_{\text{av}-x}$ during a time interval Δt is equal to the change in velocity $\Delta\upsilon_x = \upsilon_{2x} - \upsilon_{1x}$ during that time interval divided by Δt. The instantaneous x-acceleration a_x is the limit of $a_{\text{av}-x}$ as Δt goes to zero, or the derivative of υ_x with respect to t	$a_{\text{av}-x} = \dfrac{\Delta\upsilon_x}{\Delta t} = \dfrac{\upsilon_{2x} - \upsilon_{1x}}{t_2 - t_1}$ $a_x = \lim\limits_{\Delta t \to 0} \dfrac{\Delta\upsilon_x}{\Delta t} = \dfrac{d\upsilon_x}{dt}$	
Straight-line motion with constant acceleration: When the x-acceleration is constant, four equations relate the position x and the x-velocity υ_x at any time t to the initial position x_0, the initial x-velocity υ_{0x} (both measured at time $t = 0$), and the x-acceleration a_x	Constant x-acceleration only: $\upsilon_x = \upsilon_{0x} + a_x t$ $x = x_0 + \upsilon_{0x} t + \frac{1}{2} a_x t^2$ $\upsilon_x^2 = \upsilon_{0x}^2 + 2a_x (x - x_0)$ $x - x_0 = \frac{1}{2}(\upsilon_{0x} + \upsilon_x) t$	
Freely falling objects: Free fall (vertical motion without air resistance, so only gravity affects the motion) is a case of motion with constant acceleration. The magnitude of the acceleration due to gravity is a positive quantity, g. The acceleration of an object in free fall is always downward. (See Examples 2.6–2.8)		
Straight-line motion with varying acceleration: When the acceleration is not constant but is a known function of time, we can find the velocity and position as functions of time by integrating the acceleration function	$\upsilon_x = \upsilon_{0x} + \int_0^t a_x dt$ $x = x_0 + \int_0^t \upsilon_x dt$	

3 MOTION IN TWO OR THREE DIMENSIONS

What determines where a batted baseball lands? How do you describe the motion of a roller coaster car along a curved track or the flight of a circling hawk? Which hits the ground first: a cricket ball that you simply drop or one that you throw horizontally?

We can't answer these kinds of questions by using the techniques of Chapter 2, in which particles moved only along a straight line. Instead, we need to extend our descriptions of motion to two- and three-dimensional situations. We'll still use the vector quantities displacement, velocity, and acceleration, but now these quantities will no longer lie along a single line. We'll find that several important kinds of motion take place in two dimensions only - that is, in a *plane.*

We also need to consider how the motion of a particle is described by different observers who are moving relative to each other. The concept of *relative velocity* will play an important role later in the book when we explore electromagnetic phenomena and when we introduce Einstein's special theory of relativity.

This chapter merges the vector mathematics of Chapter 1 with the kinematic language of Chapter 2. As before, we're concerned with describing motion, not with analyzing its causes. But the language you learn here will be an essential tool in later chapters when we study the relationship between force and motion.

3.1 Position and Velocity Vectors

Let's see how to describe a particle's motion in space. If the particle is at a point P at a certain instant, the **position vector** $\vec{r}$ of the particle at this instant is a vector that goes from the origin of the coordinate system to point P (**Fig. 3.1**). The Cartesian coordinates x, y, and z of point P are the x-, y-, and z-components of vector $\vec{r}$. Using the unit vectors we introduced in Section 1.9, we can write

Position vector of a particle at a given instant

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \qquad (3.1)$$

Unit vectors in x-, y-, and z-directions

Coordinates of particle's position

During a time interval Δt the particle moves from P_1, where its position vector is $\vec{r}_1$, to P_2, where its position vector is $\vec{r}_2$. The change in position (the displacement) during this interval is $\Delta\vec{r} = \vec{r}_2 - \vec{r}_2 = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$. We define the **average velocity** vS av during this interval in the same way we did in Chapter 2 for straight-line motion, as the displacement divided by the time interval (**Fig. 3.2**):

$$\vec{\upsilon}_{\text{av}} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}. \qquad (3.2)$$

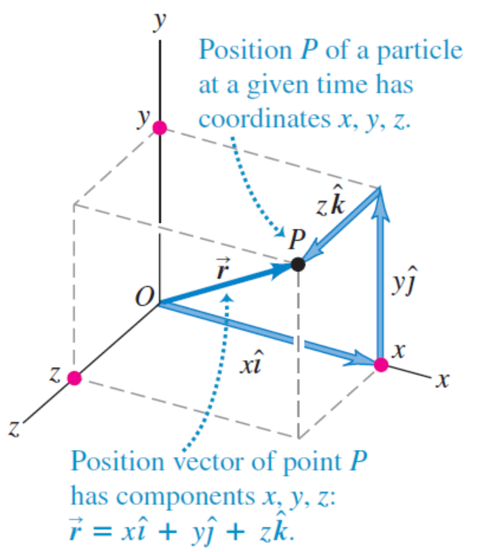


Figure 3.1 - The position vector $\vec{r}$ from origin O to point P has components x, y, and z

At any instant, the *magnitude* of $\vec{\upsilon}$ is the *speed* υ of the particle at that instant, and the *direction* of $\vec{\upsilon}$ is the direction in which the particle is moving at that instant.

As $\Delta t \to 0$, points P_1 and P_2 in Fig. 3.2 move closer and closer together. In this limit, the vector $\Delta\vec{r}$ becomes tangent to the path. The direction of $\Delta\vec{r}$ in this limit is also the direction of $\vec{\upsilon}$. So *at every point along the path, the instantaneous velocity vector is tangent to the path at that point*.

It's often easiest to calculate the instantaneous velocity vector by using components. During any displacement $\Delta\vec{r}$, the changes Δx, Δy, and Δz in the three coordinates of the particle are the *components* of $\Delta\vec{r}$. It follows that the components υ_x, υ_y, and υ_z of the instantaneous velocity $\vec{\upsilon} = \upsilon_x\hat{i} + \upsilon_y\hat{j} + \upsilon_z\hat{k}$ are simply the time derivatives of the coordinates x, y, and z:

Each **component of** a particle's **instantaneous velocity vector** ...

$$\upsilon_x = \frac{dx}{dt} \qquad \upsilon_y = \frac{dy}{dt} \qquad \upsilon_z = \frac{dz}{dt} \tag{3.4}$$

... equals the instantaneous rate of change of its corresponding coordinate.

The x-component of $\vec{\upsilon}$ is $\upsilon_x = dx/dt$, which is the same as Eq. (2.3) for straight-line motion (see Section 2.2). Hence Eq. (3.4) is a direct extension of instantaneous velocity to motion in three dimensions.

We can also get Eq. (3.4) by taking the derivative of Eq. (3.1). The unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ don't depend on time, so their derivatives are zero and we find

$$\vec{\upsilon} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}. \tag{3.5}$$

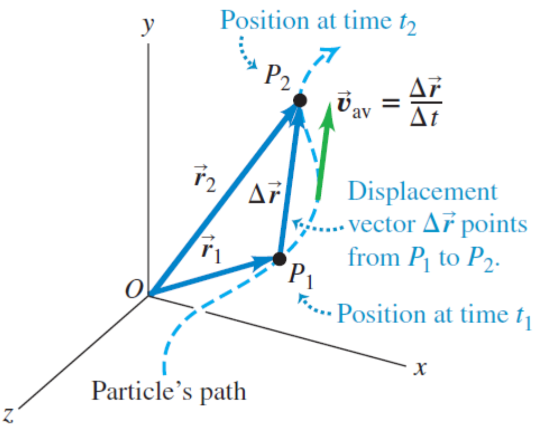


Figure 3.2 - The average velocity υ_{av} between points P_1 and P_2 has the same direction as the displacement $\Delta\vec{r}$

This shows again that the components of $\vec{\upsilon}$ are dx/dt, dy/dt, and dz/dt.

The magnitude of the instantaneous velocity vector $\vec{\upsilon}$ - that is, the speed - is given in terms of the components υ_x, υ_y, and υ_z by the Pythagorean relationship:

$$|\vec{\upsilon}| = \upsilon = \sqrt{\upsilon_x^2 + \upsilon_y^2 + \upsilon_z^2}. \tag{3.6}$$

From now on, when we use the word "velocity," we'll always mean the *instantaneous* velocity vector $\vec{\upsilon}$ (rather than the average velocity vector). Usually, we won't even bother to call $\vec{\upsilon}$ a vector; it's up to you to remember that velocity is a vector quantity with both magnitude and direction.

3.2 The Acceleration Vector

Now let's consider the *acceleration* of a particle moving in space. Just as for motion in a straight line, acceleration describes how the velocity of the particle changes. But since we now treat velocity as a vector, acceleration will describe changes in the velocity magnitude (that is, the speed) *and* changes in the direction of velocity (that is, the direction in which the particle is moving).

In **Fig. 3.3a**, a car (treated as a particle) is moving along a curved road. Vectors $\vec{\upsilon}_1$ and $\vec{\upsilon}_2$ represent the car's instantaneous velocities at time t_1, when the car is at pointP_1, and at time t_2, when the car is at point P_2 . During the time interval from t_1 to t_2, the *vector change in velocity* is $\vec{\upsilon}_2 - \vec{\upsilon}_1 = \Delta\vec{\upsilon}$, so $\vec{\upsilon}_2 = \vec{\upsilon}_1 + \Delta\vec{\upsilon}$ (Fig. 3.3b). The **average acceleration** $\vec{a}_{av}$ of the car during this time interval is the velocity change divided by the time interval $t_2 – t_1 = \Delta t$:

Change in the particle's velocity

Average acceleration vector of a particle during time interval from t_1 to t_2

$$\vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} \qquad (3.8)$$

Final velocity minus initial velocity

Time interval

Final time minus initial time

Average acceleration is a *vector* quantity in the same direction as $\Delta\vec{\upsilon}$ (Fig. 3.3c). The *x*-component of Eq. (3.8) is $a_{av-x} = (\upsilon_{2x} - \upsilon_{1x})/(t_2 - t_1) = \Delta\upsilon_x / \Delta t$, which is just Eq. (2.4) for average acceleration in straight-line motion.

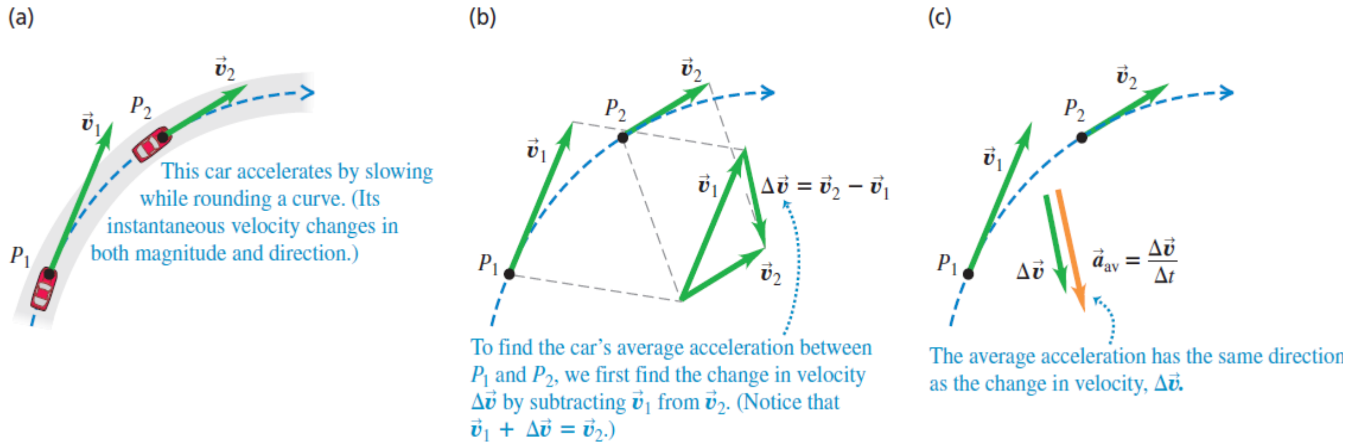


Figure 3.3 (a) A car moving along a curved road from P_1 to P_2. (b) How to obtain the change in velocity $\Delta\vec{\upsilon} = \vec{\upsilon}_2 - \vec{\upsilon}_1$ by vector subtraction. (c) The vector $\vec{a}_{av} = \Delta\vec{\upsilon}/dt$ represents the average acceleration between P_1 and P_2

As in Chapter 2, we define the **instantaneous acceleration** $\vec{a}$ (a *vector* quantity) at point P_1 as the limit of the average acceleration vector when point P_2 approaches point P_1, so both $\Delta\vec{\upsilon}$ and Δt approach zero:

The **instantaneous acceleration vector** of a particle ...

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \qquad (3.9)$$

... equals the limit of its average acceleration vector as the time interval approaches zero ...

... and equals the instantaneous rate of change of its velocity vector.

The velocity vector $\vec{\upsilon}$ is always tangent to the particle's path, but the instantaneous acceleration vector $\vec{a}$ does *not* have to be tangent to the path. If the path is curved, $\vec{a}$ points toward the concave side of the path - that is, toward the inside of any turn that the particle is making. The acceleration is tangent to the path only if the particle moves in a straight line.

CAUTION! **Any particle following a curved path is accelerating.** When a particle is moving in a curved path, it always has nonzero acceleration, even when it moves with constant speed. This conclusion is contrary to the everyday use of the word "acceleration" to mean that speed is increasing. The more precise definition given in Eq. (3.9) shows that there is a nonzero acceleration whenever the velocity vector changes in *any* way, whether there is a change of speed, direction, or both.

To convince yourself that a particle is accelerating as it moves on a curved path with constant speed, think of your sensations when you ride in a car. When the car accelerates, you tend to move inside the car in a direction *opposite* to the car's acceleration. (In Chapter 4 we'll learn why this is so). Thus you tend to slide toward the back of the car when it accelerates forward (speeds up) and toward the front of the car when it accelerates backward (slows down). If the car makes a turn on a level road, you tend to slide toward the outside of the turn; hence the car is accelerating toward the inside of the turn.

We'll usually be interested in instantaneous acceleration, not average acceleration. From now on, we'll use the term "acceleration" to mean the instantaneous acceleration vector $\vec{a}$.

Each component of the acceleration vector $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ is the derivative of the corresponding component of velocity:

Each **component of** a particle's **instantaneous acceleration vector** ...

$$a_x = \frac{dv_x}{dt} \qquad a_y = \frac{dv_y}{dt} \qquad a_z = \frac{dv_z}{dt} \tag{3.10}$$

... equals the instantaneous rate of change of its corresponding velocity component.

In terms of unit vectors,

$$\vec{a} = \frac{d\vec{\upsilon}}{dt} = \frac{d\upsilon_x}{dt}\hat{i} + \frac{d\upsilon_y}{dt}\hat{j} + \frac{d\upsilon_z}{dt}\hat{k}. \tag{3.11}$$

The x-component of Eqs. (3.10) and (3.11), $a_x = d\upsilon_x/dt$, is just Eq. (2.5) for instantaneous acceleration in one dimension.

Since each component of velocity is the derivative of the corresponding coordinate, we can express the components a_x, a_y, and a_z of the acceleration vector $\vec{a}$ as

$$a_x = \frac{d^2x}{dt^2},\ a_y = \frac{d^2y}{dt^2},\ a_z = \frac{d^2z}{dt^2}. \tag{3.12}$$

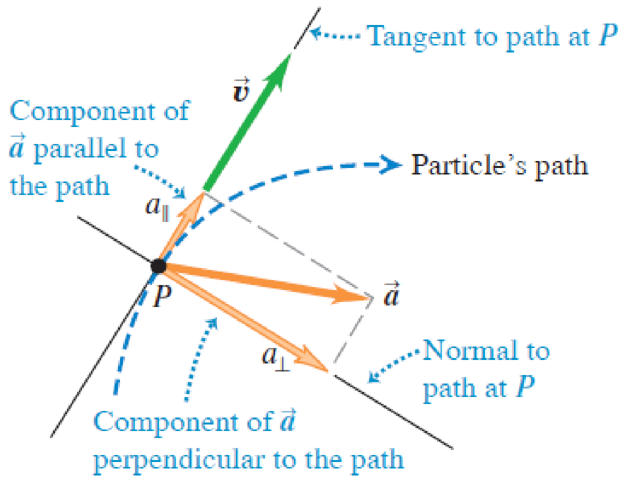


Figure 3.4 - The acceleration can be resolved into a component $a_\parallel$ parallel to the path (that is, along the tangent to the path) and a component $a_\perp$ perpendicular to the path (that is, along the normal to the path)

Parallel and Perpendicular Components of Acceleration

Equations (3.10) tell us about the components of a particle's instantaneous acceleration vector ***a***S along the x-, y-, and z-axes. Another useful way to think about $\vec{a}$ is in terms of one component *parallel* to the particle's path and to its velocity $\vec{\upsilon}$, and one component *perpendicular* to the path and to $\vec{\upsilon}$ (**Fig. 3.4**). That's because the parallel component $a_\parallel$ tells us about changes in the particle's *speed*, while the perpendicular component $a_\perp$ tells us about changes in the particle's *direction of motion*. To see why the parallel and perpendicular components of $\vec{a}$ have these properties, let's consider two special cases.

In **Fig. 3.5a** the acceleration vector is in the same direction as the velocity $\vec{\upsilon}_1$, so $\vec{a}$ has only a parallel component $a_\parallel$ (that is, $a_\perp = 0$). The velocity change $\Delta\vec{\upsilon}$ during a small time interval Δt is in the same direction as $\vec{a}$ and hence in the same direction as $\vec{\upsilon}_1$. The velocity $\vec{\upsilon}_2$ at the end of Δt is in the same direction as $\vec{\upsilon}_1$ but has greater magnitude. Hence during the time interval Δt the particle in Fig. 3.5a moved in a straight line with increasing speed.

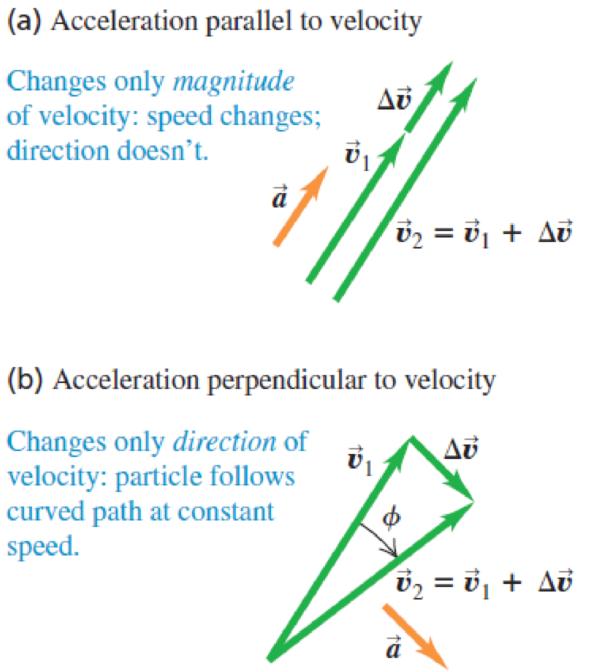


Figure 3.5 - The effect of acceleration directed (a) parallel to and (b) perpendicular to a particle's velocity

In **Fig. 3.5b** the acceleration is *perpendicular* to the velocity, so $\vec{a}$ has only a perpendicular component $a_\perp$ (that is, $a_\parallel = 0$). In a small time interval Δt, the velocity change $\Delta\vec{\upsilon}$ is very nearly perpendicular to $\vec{\upsilon}_1$, and so $\vec{\upsilon}_1$ and $\vec{\upsilon}_2$ have different directions. As the time interval Δt approaches zero, the angle f in the figure also approaches zero, $\Delta\vec{\upsilon}$ comes perpendicular to *both* $\vec{\upsilon}_1$ and $\vec{\upsilon}_2$, and $\vec{\upsilon}_1$ and $\vec{\upsilon}_2$ have the same magnitude. In other words, the speed of the particle stays the same, but the direction of motion changes and the path of the particle curves. In the most general case, the acceleration $\vec{a}$ has *both* components parallel and perpendicular to the velocity $\vec{\upsilon}$, as in Fig. 3.4. Then the particle's speed will change (described by the parallel component $a_\parallel$) *and* its direction of motion will change (described by the perpendicular component $a_\perp$).

Figure 3.6 shows a particle moving along a curved path for three situations: constant speed, increasing speed, and decreasing speed. If the speed is constant, $\vec{a}$ is perpendicular, or *normal*, to the path and to $\vec{\upsilon}$ and points toward the concave side of the path (Fig. 3.6a). If the speed is increasing, there is still a perpendicular component of $\vec{a}$, but there is also a parallel component with the same direction as $\vec{\upsilon}$ (Fig. 3.6b). Then $\vec{a}$ points ahead of the normal to the path. If the speed is decreasing, the parallel component has the direction opposite to *v*S, and $\vec{a}$ points behind the normal to the path (Fig. 3.6c). We'll use these ideas again in Section 3.4 when we study the special case of motion in a circle.

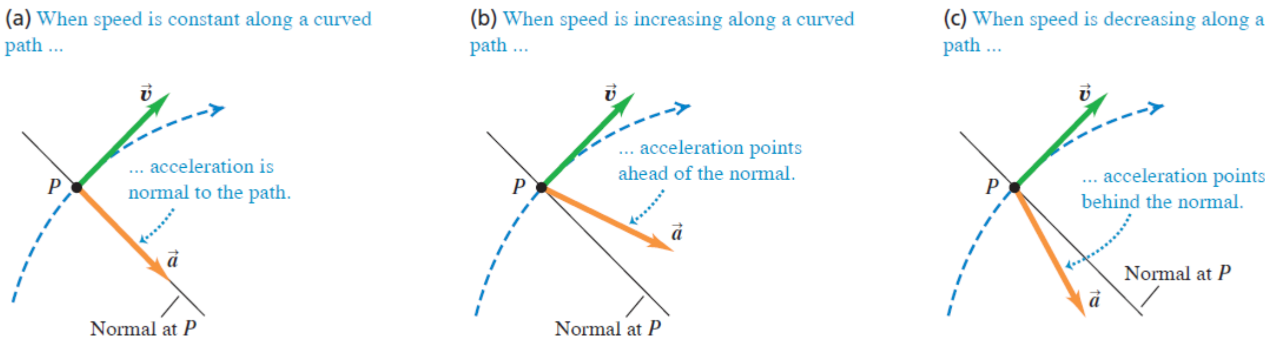


Figure 3.6 - Velocity and acceleration vectors for a particle moving through a point *P* on a curved path with (a) constant speed, (b) increasing speed, and (c) decreasing speed

3.3 Projectile Motion

A **projectile** is any object that is given an initial velocity and then follows a path determined entirely by the effects of gravitational acceleration and air resistance. A batted baseball, a thrown basketball, and a bullet shot from a rifle are all projectiles. The path followed by a projectile is called its **trajectory.**

To analyze the motion of a projectile, we'll use an idealized model. We'll represent the projectile as a particle with an acceleration (due to gravity) that is constant in both magnitude and direction. We'll ignore the effects of air resistance and the curvature and rotation of the earth. This model has limitations, however: We have to consider the earth's curvature when we study the flight of long-range missiles, and air resistance is of crucial importance to a sky diver. Nevertheless, we can learn a lot from analysis of this simple model. For the remainder of this chapter the phrase "projectile motion" will imply that we're ignoring air resistance. In Chapter 5 we'll see what happens when air resistance cannot be ignored.

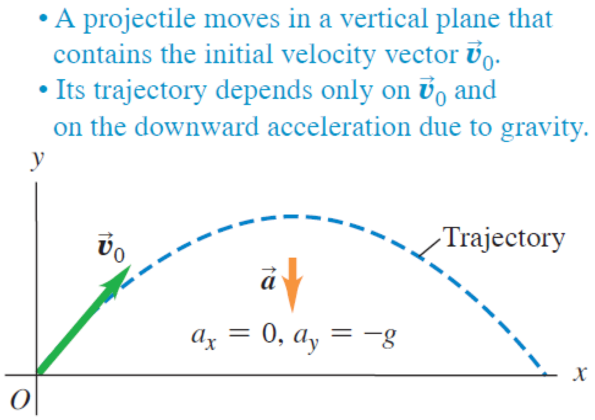


Figure 3.7 - The trajectory of an idealized projectile

Projectile motion is always confined to a vertical plane determined by the direction of the initial velocity (**Fig. 3.7**). This is because the acceleration due to gravity is purely vertical; gravity can't accelerate the projectile sideways. Thus projectile motion is *twodimensional*. We'll call the plane of motion the *xy*-coordinate plane, with the *x*-axis horizontal and the *y*-axis vertically upward.

The key to analyzing projectile motion is that we can treat the *x*- and *y*-coordinates separately. The horizontal motion of the projectile has *no* effect on its vertical motion. For projectiles, the *x*-component of acceleration is zero and the *y*-component is constant and equal to $-g$. So *we can analyze projectile motion as a combination of horizontal motion with constant velocity and vertical motion with constant acceleration.*

We can then express all the vector relationships for the projectile's position, velocity, and acceleration by separate equations for the horizontal and vertical components. The components of $\vec{a}$ are

$$a_x = 0 \quad a_y = -g \text{ (projectile motion, no air resistance).} \tag{3.13}$$

Since both the *x*-acceleration and *y*-acceleration are constant, we can use Eqs. (2.8), (2.12), (2.13), and (2.14) directly. Suppose that at time $t = 0$ our particle is at the point (x_0, y_0) and its initial velocity at this time has components υ_{0x} and υ_{0y}. The components of acceleration are $a_x = 0$, $a_y = -g$. Considering the *x*-motion first, we substitute 0 for a_x in Eqs. (2.8) and (2.12). We find

$$\upsilon_x = \upsilon_{0x}, \qquad x = x_0 + \upsilon_{0x}\, t. \tag{3.14–15}$$

For the *y*-motion we substitute *y* for *x*, υ_y for υ_x, υ_{0y} for υ_{0x}, and $a_y = -g$ for a_x:

$$\upsilon_y = \upsilon_{0y} - gt, \qquad y = y_0 + \upsilon_{0y}\, t - gt^2/2. \tag{3.16–17}$$

It's usually simplest to take the initial position (at $t = 0$) as the origin; then $x_0 = y_0 = 0$. This might be the position of a ball at the instant it leaves the hand of the person who throws it or the position of a bullet at the instant it leaves the gun barrel.

Figure 3.8 shows the trajectory of a projectile that starts at (or passes through) the origin at time $t = 0$, along with its position, velocity, and velocity components at equal time intervals. The *x*-velocity υ_x is constant; the *y*-velocity υ_y changes by equal amounts in equal times, just as if the projectile were launched vertically with the same initial *y*-velocity.

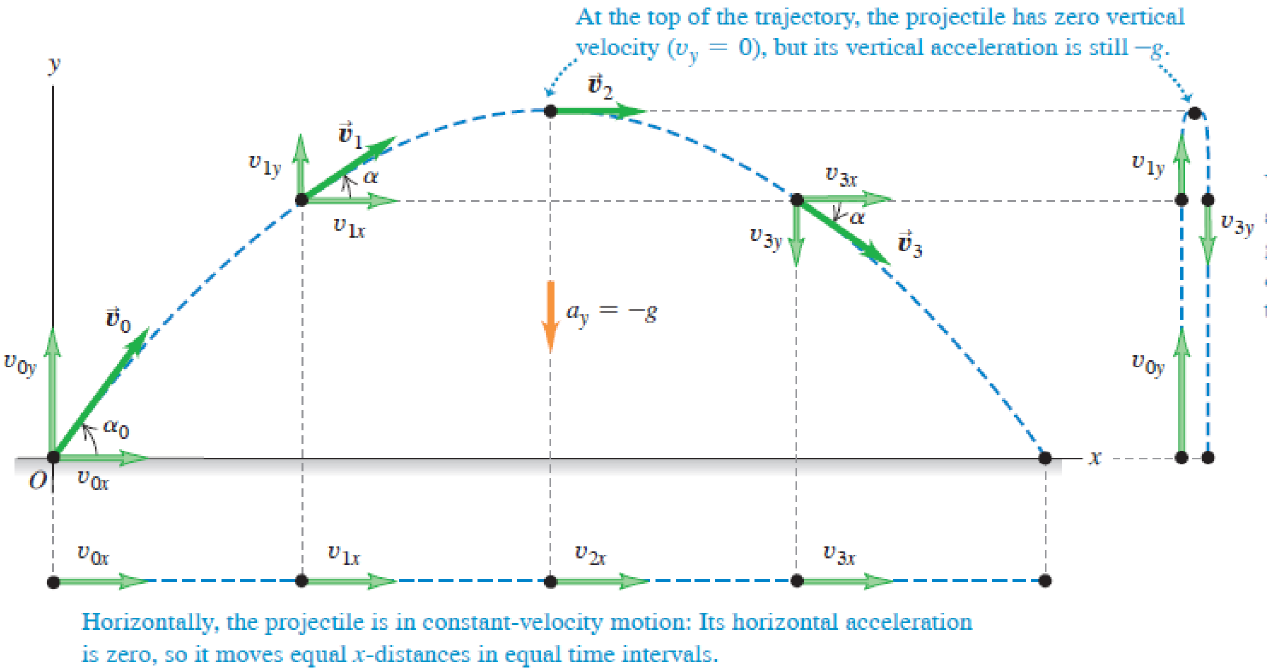


Figure 3.8 - If air resistance is negligible, the trajectory of a projectile is a combination of horizontal motion with constant velocity and vertical motion with constant acceleration

We can also represent the initial velocity $\vec{\upsilon}_0$ by its magnitude υ_0 (the initial speed) and its angle α_0 with the positive *x*-axis. In terms of these quantities, the components υ_{0x} and υ_{0y} of the initial velocity are

$$\upsilon 0x = \upsilon 0 \cos(\alpha_0) \upsilon 0y = \upsilon 0 \sin(\alpha_0). \tag{3.18}$$

If we substitute Eqs. (3.18) into Eqs. (3.14) through (3.17) and set $x_0 = y_0 = 0$, we get the following equations. They describe the position and velocity of the projectile in Fig. 3.8 at any time *t*:

Coordinates at time *t* of a **projectile** (positive *y*-direction is upward, and $x = y = 0$ at $t = 0$)

$$x = (v_0 \cos \alpha_0)t \tag{3.19}$$

Speed at $t = 0$ | Direction at $t = 0$ | Time

$$y = (v_0 \sin \alpha_0)t - \tfrac{1}{2}gt^2 \tag{3.20}$$

Acceleration due to gravity: Note $g > 0$.

Velocity components at time *t* of a **projectile** (positive *y*-direction is upward)

$$v_x = v_0 \cos \alpha_0 \tag{3.21}$$

Speed at $t = 0$ | Direction at $t = 0$ | Time

$$v_y = v_0 \sin \alpha_0 - gt \tag{3.22}$$

We can get a lot of information from Eqs. (3.19) through (3.22). For example, the distance *r* from the origin to the projectile at any time *t* is

$$r = \sqrt{x^2 + y^2}. \tag{3.23}$$

The projectile's speed (the magnitude of its velocity) at any time is

$$\upsilon = \sqrt{\upsilon_x^2 + \upsilon_y^2}. \tag{3.24}$$

The *direction* of the velocity, in terms of the angle a it makes with the positive *x*-direction (see Fig. 3.17), is

$$\tan \alpha = \frac{\upsilon_y}{\upsilon_x}. \tag{3.25}$$

The velocity vector $\vec{\upsilon}$ is tangent to the trajectory at each point.

We can derive an equation for the trajectory's shape in terms of *x* and *y* by eliminating *t*. From Eqs. (3.19) and (3.20), we find $t = x/(\upsilon_0 \cos\alpha_0)$ and

$$y = (\tan \alpha_0)x - \frac{g}{2\upsilon_0^2 \cos^2 \alpha_0} x^2. \tag{3.36}$$

Don't worry about the details of this equation; the important point is its general form. Since *v*0, tan a0, cos a0, and *g* are constants, Eq. (3.26) has the form

$$y = bx - cx^2,$$

where b and c are constants. This is the equation of a *parabola*. In our simple model of projectile motion, the trajectory is always a parabola.

When air resistance *isn't* negligible and has to be included, calculating the trajectory becomes a lot more complicated; the effects of air resistance depend on velocity, so the acceleration is no longer constant. Air resistance has a very large effect; the projectile does not travel as far or as high, and the trajectory is no longer a parabola.

3.4 Motion in a Circle

When a particle moves along a curved path, the direction of its velocity changes. As we saw in Section 3.2, this means that the particle *must* have a component of acceleration perpendicular to the path, even if its speed is constant (see Fig. 3.5b). In this section we'll calculate the acceleration for the important special case of motion in a circle.

Uniform Circular Motion

When a particle moves in a circle with *constant speed*, the motion is called **uniform circular motion.** A car rounding a curve with constant radius at constant speed, a satellite moving in a circular orbit, and an ice skater skating in a circle with constant speed are all examples of uniform circular motion. There is no component of acceleration parallel (tangent) to the path; otherwise, the speed would change. The acceleration vector is perpendicular (normal) to the path and hence directed inward (never outward!) toward the center of the circular path. This causes the direction of the velocity to change without changing the speed.

We can find a simple expression for the magnitude of the acceleration in uniform circular motion. We begin with **Fig. 3.9a**, which shows a particle moving with constant speed in a circular path of radius R with center at O. The particle moves a distance Δs from P_1 to P_2 in a time interval Δt. Figure 3.28b shows the vector change in velocity $\Delta\vec{\upsilon}$ during this interval.

The angles labeled $\Delta\phi$ in Figs. 3.9a and 3.9b are the same because $\vec{\upsilon}_1$ is perpendicular to the line OP_1 and $\vec{\upsilon}_2$ is perpendicular to the line OP_2 . Hence the triangles in Figs. 3.9a and 3.9b are *similar*. The ratios of corresponding sides of similar triangles are equal, so

$$\frac{|\Delta\vec{\upsilon}|}{\upsilon_1} = \frac{\Delta s}{R} \quad \text{or} \quad |\Delta\vec{\upsilon}| = \frac{\upsilon_1}{R}\Delta.$$

The magnitude a_{av} of the average acceleration during Δt is therefore

$$a_{av} = \frac{|\Delta\vec{\upsilon}|}{\Delta t} = \frac{\upsilon_1}{R}\frac{\Delta s}{\Delta t}.$$

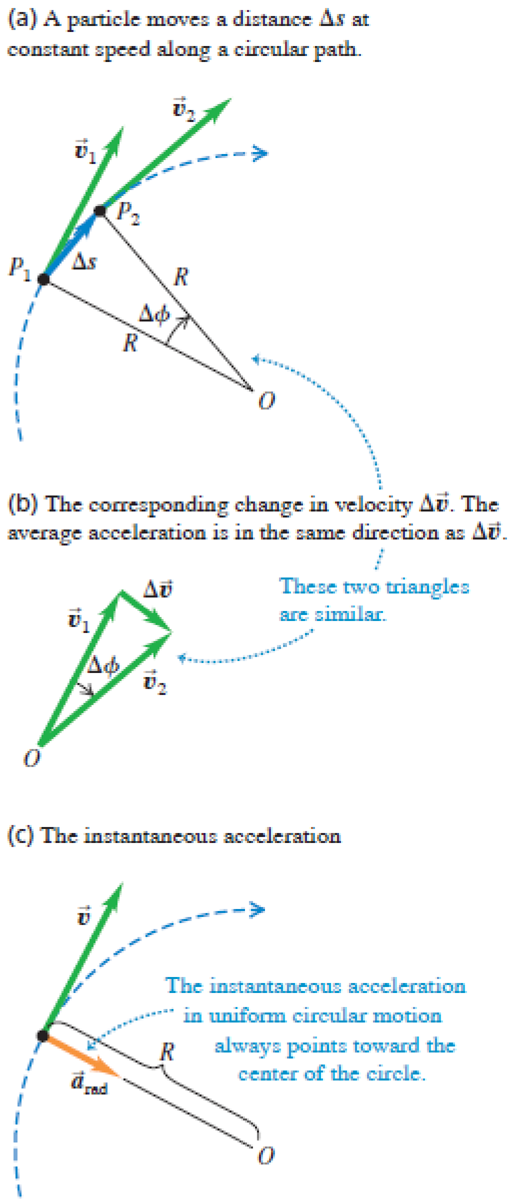


Figure 3.9 - Finding the velocity change $\Delta\vec{\upsilon}$, average acceleration a_{av}, and instantaneous acceleration $\vec{a}_{rad}$ for a particle

The magnitude a of the *instantaneous* acceleration $\vec{a}$ at point P_1 is the limit of this expression as we take point P_2 closer and closer to point P_1:

$$a = \lim_{\Delta t \to 0}\left(\frac{\upsilon_1}{R}\frac{\Delta s}{\Delta t}\right) = \frac{\upsilon_1}{R}\lim_{\Delta t \to 0}\left(\frac{\Delta s}{\Delta t}\right).$$

If the time interval Δt is short, Δs is the distance the particle moves along its curved path. So the limit of $\Delta s/\Delta t$ is the speed υ_1 at point P_1. Also, P_1 can be any point on the path, so we can drop the subscript and let υ represent the speed at any point. Then

Magnitude of acceleration of an object in **uniform circular motion**

$$a_{\text{rad}} = \frac{\upsilon^2}{R} \tag{3.27}$$

Speed of object

Radius of object's circular path

The subscript "rad" is a reminder that the direction of the instantaneous acceleration at each point is always along a radius of the circle (toward the center of the circle; see Fig. 3.9c). So *in uniform circular motion, the magnitude* a_{rad} *of the instantaneous acceleration is equal to the square of the speed* υ *divided by the radius R of the circle. Its direction is perpendicular to* $\vec{\upsilon}$ *and inward along the radius*. Because the acceleration in uniform circular motion is along the radius, we often call it **radial acceleration.**

Because the acceleration in uniform circular motion is always directed toward the center of the circle, it is sometimes called **centripetal acceleration.** The word "centripetal" is derived from two Greek words meaning "seeking the center."

We can also express the magnitude of the acceleration in uniform circular motion in terms of the **period** T of the motion, the time for one revolution (one complete trip around the circle). In a time T the particle travels a distance equal to the circumference $2\pi R$ of the circle, so its speed is

$$\upsilon = \frac{2\pi R}{T}. \tag{3.28}$$

When we substitute this into Eq. (3.27), we obtain the alternative expression

Magnitude of acceleration of an object in **uniform circular motion**

$$a_{\text{rad}} = \frac{4\pi^2 R}{T^2} \tag{3.29}$$

Radius of object's circular path

Period of motion

Nonuniform Circular Motion

We have assumed throughout this section that the particle's speed is constant as it goes around the circle. If the speed varies, we call the motion **nonuniform circular motion.** In nonuniform circular motion, Eq. (3.27) still gives the *radial* component of acceleration $a_{\text{rad}} = \upsilon^2/R$, which is always *perpendicular* to the instantaneous velocity and directed toward the center of the circle. But since the speed υ has different values at different points in the motion, the value of a_{rad} is not constant. The radial (centripetal) acceleration is greatest at the point in the circle where the speed is greatest.

In nonuniform circular motion there is also a component of acceleration that is *parallel* to the instantaneous velocity (see Figs. 3.8b and 3.8c). This is the component $a_{\parallel}$ that we discussed in Section 3.2; here we call this component a_{tan} to emphasize that it is *tangent* to the circle. This component, called the **tangential acceleration** a_{tan}, is equal to the rate of change of *speed*. Thus

$$a_{\text{rad}} = \frac{\upsilon^2}{R} \quad \text{and} \quad a_{\text{tan}} = \frac{d|\vec{\upsilon}|}{dt} \quad \text{(nonuniform circular motion).} \tag{3.30}$$

The tangential component is in the same direction as the velocity if the particle is speeding up, and in the opposite direction if the particle is slowing down. If the particle's speed is constant, $a_{\tan} = 0$.

CAUTION! **Uniform vs. nonuniform circular motion.** The two quantities $\frac{d|\vec{\upsilon}|}{dt}$ and $\left|\frac{d\vec{\upsilon}}{dt}\right|$ are *not* the same. The first, equal to the tangential acceleration, is the rate of change of speed; it is zero whenever a particle moves with constant speed, even when its direction of motion changes (such as in *uniform* circular motion). The second is the magnitude of the vector acceleration; it is zero only when the particle's acceleration *vector* is zero—that is, when the particle moves in a straight line with constant speed. In *uniform* circular motion $|d\vec{\upsilon}/dt| = a_{\text{rad}} = \upsilon^2/r$; in *nonuniform* circular motion there is also a tangential component of acceleration, so $|d\vec{\upsilon}/dt| = \sqrt{a_{\text{rad}}^2 + a_{\tan}^2}$.

3.5 Relative Velocity

If you stand next to a one-way highway, all the cars appear to be moving forward. But if you're driving in the fast lane on that highway, slower cars appear to be moving backward. In general, when two observers measure the velocity of the same object, they get different results if one observer is moving relative to the other. The velocity seen by a particular observer is called the velocity *relative* to that observer, or simply **relative velocity.** In many situations relative velocity is extremely important

We'll first consider relative velocity along a straight line and then generalize to relative velocity in a plane.

Relative Velocity in One Dimension

A passenger walks with a velocity of +1.0 m/s along the aisle of a train that is moving with a velocity of +3.0 m/s (**Fig. 3.10a)**. What is the passenger's velocity? It's a simple enough question, but it has no single answer. As seen by a second passenger sitting in the train, she is moving at +1.0 m/s. A person on a bicycle standing beside the train sees the walking passenger moving at +1.0 m/s + 3.0 m/s = +4.0 m/s. An observer in another train going in the opposite direction would give still another answer. We have to specify which observer we mean, and we speak of the velocity *relative* to a particular observer. The walking passenger's velocity relative to the train is +1.0 m/s, her velocity relative to the cyclist is +4.0 m/s, and so on. Each observer, equipped in principle with a meter stick and a stopwatch, forms what we call a **frame of reference**. Thus a frame of reference is a coordinate system plus a time scale.

Let's use the symbol *A* for the cyclist's frame of reference (at rest with respect to the ground) and the symbol *B* for the frame of reference of the moving train. In straight-line motion the position of a point *P* relative to frame *A* is given by $x_{P/A}$(the position of *P* with respect to *A*), and the position of *P* relative to frame *B* is given by $x_{P/B}$(Fig. 3.10b). The position of the origin of *B* with respect to the origin of *A* is $x_{B/A}$. Figure 3.10b shows that

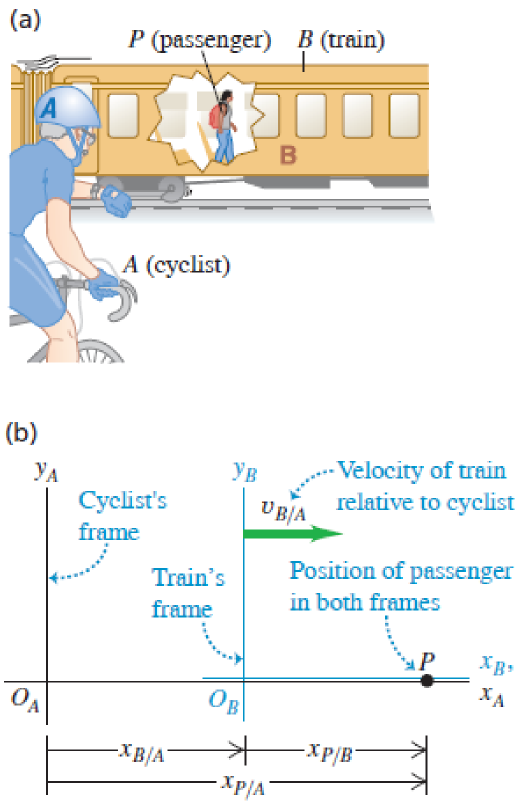


Figure 3.10 - (a) A passenger walking in a train. (b) The position of the passenger relative to the cyclist's frame of reference and the train's frame of reference

$$x_{P/A} = x_{P/B} + x_{B/A} \tag{3.31}$$

In words, the coordinate of *P* relative to *A* equals the coordinate of *P* relative to *B* plus the coordinate of *B* relative to *A*.

The *x*-velocity of *P* relative to frame *A*, denoted by $\upsilon_{P/A\text{-}x}$, is the derivative of $x_{P/A}$ with respect to time. We can find the other velocities in the same way. So the time derivative of Eq. (3.31) gives us a relationship among the various velocities:

$$\frac{dx_{P/A}}{dt} = \frac{dx_{P/B}}{dt} + \frac{dx_{B/A}}{dt} \quad \text{or}$$

Relative velocity along a line:

$$\underset{\substack{x\text{-velocity of}\\ P \text{ relative to } A}}{v_{P/A\text{-}x}} = \underset{\substack{x\text{-velocity of}\\ P \text{ relative to } B}}{v_{P/B\text{-}x}} + \underset{\substack{x\text{-velocity of}\\ B \text{ relative to } A}}{v_{B/A\text{-}x}} \tag{3.32}$$

Getting back to the passenger on the train in Fig. 3.10a, we see that *A* is the cyclist's frame of reference, *B* is the frame of reference of the train, and point *P* represents the passenger. Using the above notation, we have

$$\upsilon_{P/B\text{-}x} = +1.0 \text{ m/s} \quad \upsilon_{B/A\text{-}x} = +3.0 \text{ m/s}.$$

From Eq. (3.32) the passenger's velocity $\upsilon_{P/B\text{-}x}$ relative to the cyclist is

$$\upsilon_{P/A\text{-}x} = +1.0 \text{ m/s} + 3.0 \text{ m/s} = +4.0 \text{ m/s},$$

as we already knew.

In this example, both velocities are toward the right, and we have taken this as the positive *x*-direction. If the passenger walks toward the *left* relative to the train, then $\upsilon_{P/B\text{-}x} = -1.0$ m/s, and her *x*-velocity relative to the cyclist is $\upsilon_{P/A\text{-}x} = -1.0$ m/s + 3.0 m/s = +2.0 m/s. The sum in Eq. (3.32) is always an algebraic sum, and any or all of the *x*-velocities may be negative.

When the passenger looks out the window, the stationary cyclist on the ground appears to her to be moving backward; we call the cyclist's velocity relative to her $\upsilon_{A/P\text{-}x}$. This is just the negative of the *passenger's* velocity relative to the *cyclist*, $\upsilon_{P/A\text{-}x}$. In general, if *A* and *B* are any two points or frames of reference,

$$\upsilon_{A/B\text{-}x} = -\upsilon_{B/A\text{-}x}. \tag{3.33}$$

Relative Velocity in Two or Three Dimensions

Let's extend the concept of relative velocity to include motion in a plane or in space. Suppose that the passenger in Fig. 3.10a is walking not down the aisle of the railroad car but from one side of the car to the other, with a speed of 1.0 m/s (**Fig. 3.11a)**. We can again describe the passenger's position *P* in two frames of reference: *A* for the stationary ground observer and *B* for the moving train. But instead of coordinates *x*, we use position vectors ***r***S because the problem is now two-dimensional. Then, as Fig. 3.11b shows,

$$\vec{r}_{P/A} = \vec{r}_{P/B} + \vec{r}_{B/A}. \tag{3.34}$$

Just as we did before, we take the time derivative of this equation to get a relationship among the various velocities; the velocity of *P* relative to *A* is $\vec{\upsilon}_{P/A} = \vec{r}_{P/A}/dt$ and so on for the other velocities.

We get

Relative velocity in space:

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A} \qquad (3.35)$$

Velocity of P relative to A — Velocity of P relative to B — Velocity of B relative to A

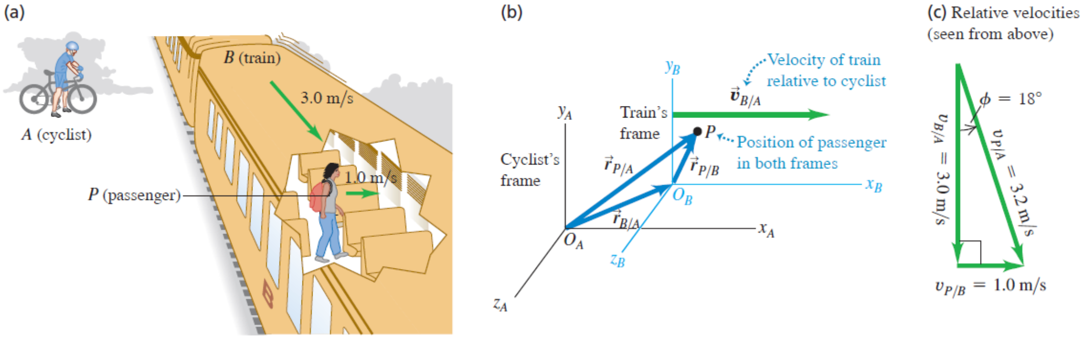


Figure 3.34 - (a) A passenger walking across a railroad car. (b) Position of the passenger relative to the cyclist's frame and the train's frame. (c) Vector diagram for the velocity of the passenger relative to the ground (the cyclist's frame), $\vec{\upsilon}_{P/A}$

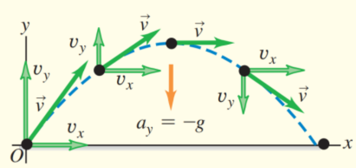
Equation (3.35) is known as the *Galilean velocity transformation*. It relates the velocity of an object *P* with respect to frame *A* and its velocity with respect to frame *B* ($\vec{\upsilon}_{P/A}$ and $\vec{\upsilon}_{P/B}$, respectively) to the velocity of frame *B* with respect to frame *A* ($\vec{\upsilon}_{B/A}$). If all three of these velocities lie along the same line, then Eq. (3.35) reduces to Eq. (3.32) for the components of the velocities along that line.

As in the case of motion along a straight line, we have the general rule that if *A* and *B* are *any* two points or frames of reference,

$$\vec{\upsilon}_{A/B} = -\vec{\upsilon}_{B/A}. \qquad (3.36)$$

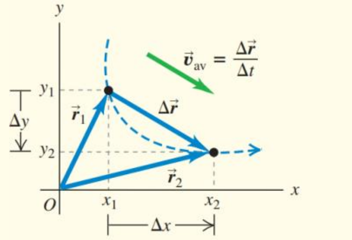
The velocity of the passenger relative to the train is the negative of the velocity of the train relative to the passenger, and so on.

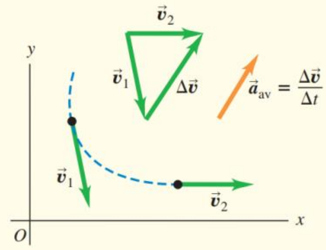
In the early 20th century Albert Einstein showed that Eq. (3.35) has to be modified when speeds approach the speed of light, denoted by *c*. It turns out that if the passenger in Fig. 3.10a could walk down the aisle at 0.30*c* and the train could move at 0.90*c*, then her speed relative to the ground would be not 1.20*c* but 0.94*c*; nothing can travel faster than light! We'll return to Einstein and his *special theory of relativity*.

CHAPTER 3: SUMMARY		
Projectile motion: In projectile motion with no air resistance, $a_x = 0$ and $a_y = -g$. The coordinates and velocity components are simple functions of time, and the shape of the path is always a parabola. We usually choose the origin to be at the initial position of the projectile	$x = (\upsilon_0 \cos\alpha_0)t$ $y = (\upsilon_0 \sin\alpha_0)t - \frac{1}{2}gt^2$ $\upsilon_x = \upsilon_0 \cos\alpha_0$ $\upsilon_y = \upsilon_0 \sin\alpha_0 - gt$	

<table>
<tr>
<td>Position, velocity, and acceleration vectors: The position vector $\vec{r}$ of a point P in space is the vector from the origin to P. Its components are the coordinates x, y, and z.

The average velocity vector $\vec{\upsilon}_{av}$ during the time interval Δt is the displacement $\Delta\vec{r}$ (the change in position vector $\vec{r}$) divided by Δt. The instantaneous velocity vector $\vec{\upsilon}$ is the time derivative of $\vec{r}$, and its components are the time derivatives of x, y, and z. The instantaneous speed is the magnitude of $\vec{\upsilon}$ The velocity $\vec{\upsilon}$ of a particle is always tangent to the particle's path.
The average acceleration vector $\vec{a}_{av}$ during the time interval Δt equals $\Delta\vec{\upsilon}$ (the change in velocity vector $\vec{\upsilon}$) divided by Δt. The instantaneous acceleration vector $\vec{a}$ is the time derivative of $\vec{\upsilon}$, and its components are the time derivatives of υ_x, υ_y, and υ_z.

The component of acceleration parallel to the direction of the instantaneous velocity affects the speed, while the component of $\vec{a}$ perpendicular to $\vec{\upsilon}$ affects the direction of motion</td>
<td>$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
$\vec{\upsilon}_{av} = \dfrac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \dfrac{\Delta\vec{r}}{\Delta t}$
$\vec{\upsilon} = \lim\limits_{\Delta t\to 0}\dfrac{\Delta\vec{r}}{\Delta t} = \dfrac{d\vec{r}}{dt}$
$\upsilon_x = \dfrac{dx}{dt} \quad \upsilon_y = \dfrac{dy}{dy} \quad \upsilon_z = \dfrac{dz}{dt}$
$\vec{a}_{av} = \dfrac{\vec{\upsilon}_2 - \vec{\upsilon}_1}{t_2 - t_1} = \dfrac{\Delta\vec{\upsilon}}{\Delta t}$
$\vec{a} = \lim\limits_{\Delta t\to 0}\dfrac{\Delta\vec{\upsilon}}{\Delta t} = \dfrac{d\vec{\upsilon}}{dt}$
$a_x = \dfrac{d\upsilon_x}{dt}$
$a_y = \dfrac{d\upsilon_y}{dt}$
$a_z = \dfrac{d\upsilon_z}{dt}$</td>
<td>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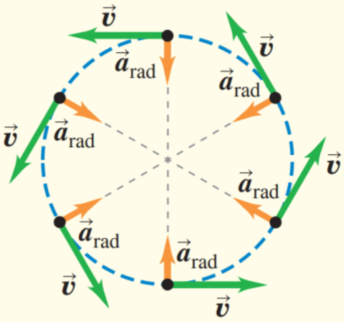
<tr>
<td>Uniform and nonuniform circular motion: When a particle moves in a circular path of radius R with constant speed υ (uniform circular motion), its acceleration $\vec{a}$ is directed toward the center of the circle and perpendicular to $\vec{\upsilon}$. The magnitude a_{rad} of this radial acceleration can be expressed in terms of υ and R or in terms of R and the period T (the time for one revolution), where $\upsilon = 2\pi R/T$.
If the speed is not constant in circular motion (nonuniform circular motion), there is still a radial component of $\vec{a}$ given by Eq. (3.27) or (3.29), but there is also a component of $\vec{a}$ parallel (tangential) to the path. This tangential component is equal to the rate of change of speed, $d\upsilon/dt$</td>
<td>$a_{rad} = \dfrac{\upsilon^2}{R}$
$a_{rad} = \dfrac{4\pi^2 R}{T^2}$</td>
<td>
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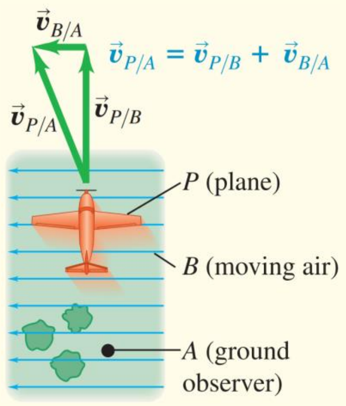
Relative velocity: *When an object P moves relative to an object (or reference frame) B, and B moves relative to an object (or reference frame) A, we denote the velocity of P relative to B by* $\vec{\upsilon}_{P/B}$*, the velocity of P relative to A by* $\vec{\upsilon}_{P/A}$*, and the velocity of B relative to A by* $\vec{\upsilon}_{B/A}$*. If these velocities are all along the same line, their components along that line are related by Eq. (3.32). More generally, these velocities are related by Eq. (3.35)*	$\upsilon_{P/A-x} = \upsilon_{P/B-x} + \upsilon_{B/A-x}$ (relative velocity along a line) $\vec{\upsilon}_{P/A} = \vec{\upsilon}_{P/B} + \vec{\upsilon}_{B/A}$ (relative velocity in space)	

4 NEWTON’S LAWS OF MOTION

We’ve seen in the last two chapters how to use *kinematics* to describe motion inone, two, or three dimensions. But what *causes* objects to move the way that theydo? For example, why does a dropped feather fall more slowly than a droppedbowling ball? Why do you feel pushed backward in a car that accelerates forward? Theanswers to such questions take us into the subject of **dynamics,** the relationship of motionto the forces that cause it.

The principles of dynamics were clearly stated for the first time by Sir Isaac Newton(1642–1727); today we call them **Newton’s laws of motion.** Newton did not *derive* thelaws of motion, but rather *deduced* them from a multitude of experiments performed byother scientists, especially Galileo Galilei (who died the year Newton was born). Newton’slaws are the foundation of **classical mechanics** (also called **Newtonian mechanics**); usingthem, we can understand most familiar kinds of motion. Newton’s laws need modificationonly for situations involving extremely high speeds (near the speed of light) or very smallsizes (such as within the atom).

4.1 Force and Interactions

A **force** is a push or a pull. More precisely, a force is an *interaction* between two objects or between an object and its environment (**Fig. 4.1**). That’s why we always refer to the force that one object *exerts* on a second object. When you push on a car that is stuck in the snow, you exert a force on the car; a steel cable exerts a force on the beam it is hoisting at a construction site; and so on. As Fig. 4.1 shows, force is a *vector* quantity; you can push or pull an object in different directions.

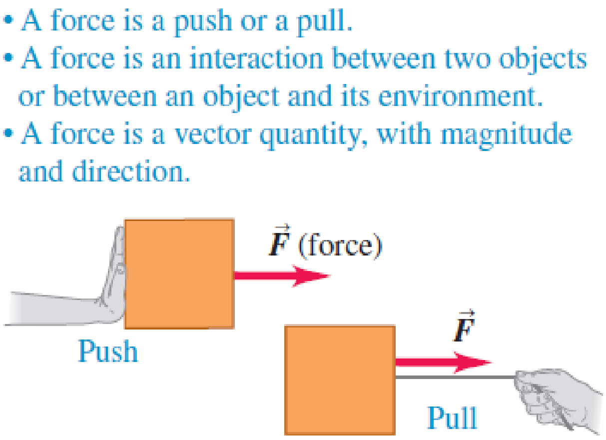


Figure 4.1 - Some properties of forces

When a force involves direct contact between two objects, such as a push or pull that you exert on an object with your hand, we call it a **contact force. Figures 4.2a,** 4.2b, and 4.2c show three common types of contact forces. The **normal force** (Fig. 4.2a) is exerted on an object by any surface with which it is in contact. The adjective “normal” means that the force always acts *perpendicular* to the surface of contact, no matter what the angle of that surface. By contrast, the **friction force** (Fig. 4.2b) exerted on an object by a surface acts *parallel* to the surface, in the direction that opposes sliding. The pulling force exerted by a stretched rope or cord on an object to which it’s attached is called a **tension force** (Fig. 4.2c). When you tug on your dog’s leash, the force that pulls on her collar is a tension force.

In addition to contact forces, there are **long-range forces** that act even when the objects are separated by empty space. The force between two magnets is an example of a longrange force, as is the force of gravity (Fig. 4.2d); the earth pulls a dropped object toward it even though there is no direct contact between the object and the earth. The gravitational force that the earth exerts on your body is called your **weight.**

To describe a force vector $\vec{F}$, we need to describe the *direction* in which it acts as well as its *magnitude,* the quantity that describes “how much” or “how hard” the force pushes or pulls. The SI unit of the magnitude of force is the *newton,* abbreviated N. (We’ll give a precise definition of the newton in Section 4.3).

A common instrument for measuring force magnitudes is the *spring balance.* It consists of a coil spring enclosed in a case with a pointer attached to one end. When forces are applied to the ends of the spring, it stretches by an amount that depends on the force. We can make a scale for the pointer by using a number of identical objects with weights of exactly 1 N each. When one, two, or more of these are suspended simultaneously from the balance, the total force stretching the spring is 1 N, 2 N, and so on, and we can label the corresponding positions of the pointer 1 N, 2 N, and so on. Then we can use this instrument to measure the magnitude of an unknown force. We can also make a similar instrument that measures pushes instead of pulls.

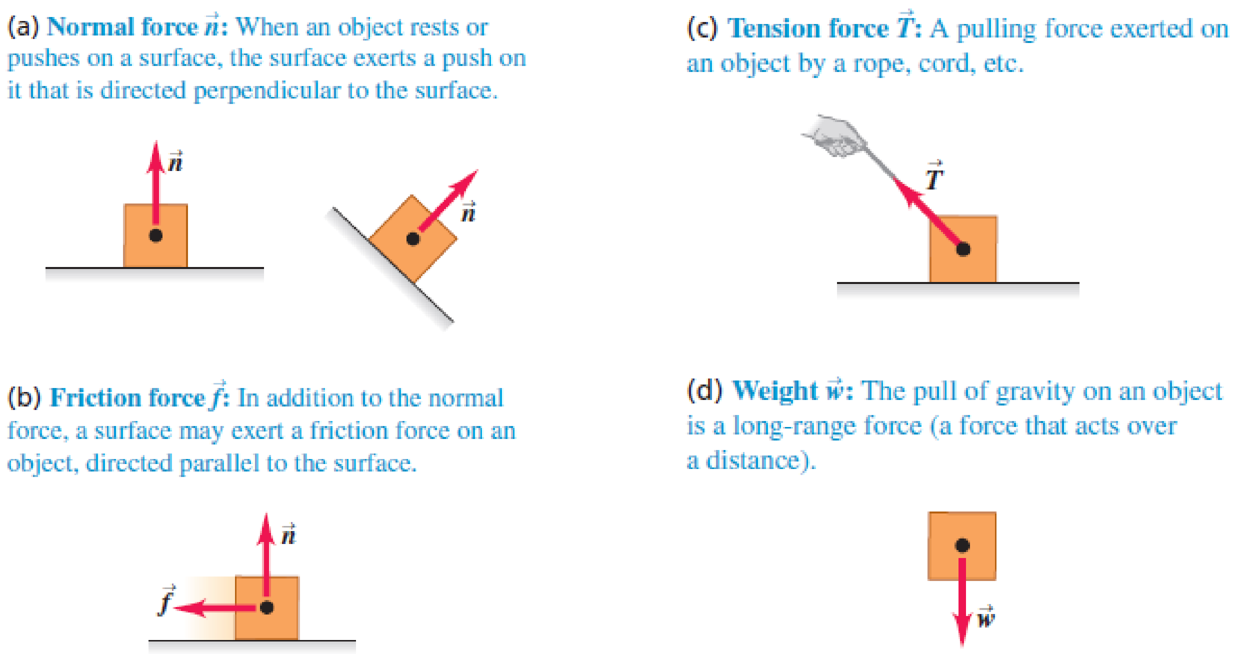


Figure 4.2 - Four common types of forces

Superposition of Forces

When you hold a ball in your hand to throw it, at least two forces act on it: the push of your hand and the downward pull of gravity. Experiment shows that when two forces $\vec{F}_1$ and $\vec{F}_2$ act at the same time at the same point on an object (**Fig. 4.3**), the effect on the object's motion is the same as if a single force $\vec{R}$ were acting equal to the vector sum, or resultant, of the original forces: $\vec{R} = \vec{F}_1 + \vec{F}_2$. More generally, *any number offorces applied at a point on an object have the same effect as a single force equal to the vector sum of the forces.* This important principle is called **superposition of forces.**

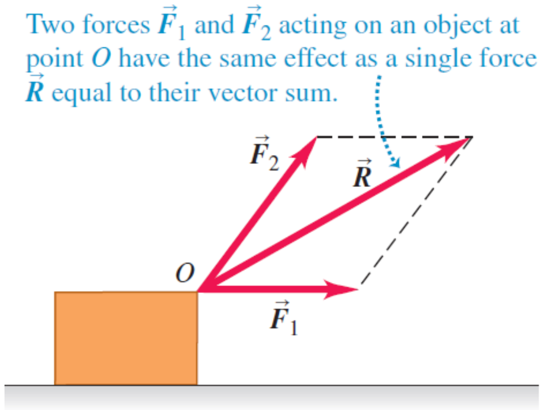


Figure 4.3 - Superposition of forces

Since forces are vector quantities and add like vectors, we can use all of the rules of vector mathematics that we learned in Chapter 1 to solve problems that involve vectors. This would be a good time to review the rules for vector addition presented in Sections 1.7 and 1.8.

We learned in Section 1.8 that it's easiest to add vectors by using components. That's why we often describe a force $\vec{F}$ in terms of its x- and y-components $F_x$and F_y. Note that the x- and y-coordinate axes do *not* have to be horizontal and vertical, respectively. As an example, **Fig. 4.4** shows a crate being pulled up a ramp by a force $\vec{F}$. In this situation it's most convenient to choose one axis to be parallel to the ramp and the other to be perpendicular to the ramp. For the case shown in Fig. 4.4, both $F_x$and $F_y$are positive; in other situations, depending on your choice of axes and the orientation of the force $\vec{F}$, either F_x or F_y may be negative or zero.

CAUTION! **Using a wiggly line in force diagramsю** In Fig. 4.4 we draw a wiggly line through the force vector $\vec{F}$ to show that we have replaced it by its x- and y-components. Otherwise, the diagram would include the same force twice. We'll draw such a wiggly line in any force diagram where a force is replaced by its components. We encourage you to do the same in your own diagrams!

We'll often need to find the vector sum (resultant) of *all* forces acting on an object. We call this the **net force** acting on the object. We'll use the Greek letter Σ (capital sigma, equivalent to the Roman *S*) as a shorthand notation for a sum. If the forces are labeled $\vec{F}_1, \vec{F}_2, \vec{F}_3$, and so on, we can write

$$\vec{R} = \sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots . \qquad (4.1)$$

We read $\Sigma\vec{F}$ as "the vector sum of the forces" or "the net force." The *x*-component of the net force is the sum of the *x*-components of the individual forces, and likewise for the *y*-component (**Fig. 4.5**):

$$R_x = \sum F_x \quad R_y = \sum F_y . \qquad (4.2)$$

Each component may be positive or negative, so be careful with signs when you evaluate these sums.

Once we have R_x and R_y we can find the magnitude and direction of the net force $\vec{R} = \Sigma\vec{F}$ acting on the object. The magnitude is

$$R = \sqrt{R_x^2 + R_y^2},$$

and the angle θ between $\vec{R}$ and the +*x*–axis can be found from the relationship $\tan\theta = R_y / R_x$. The components R_x and R_y may be positive, negative, or zero, and the angle u may be in any of the four quadrants.

In three-dimensional problems, forces may also have *z*-components; then we add the equation $R_z = \Sigma F_z$ to Eqs. (4.2). The magnitude of the net force is then

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}.$$

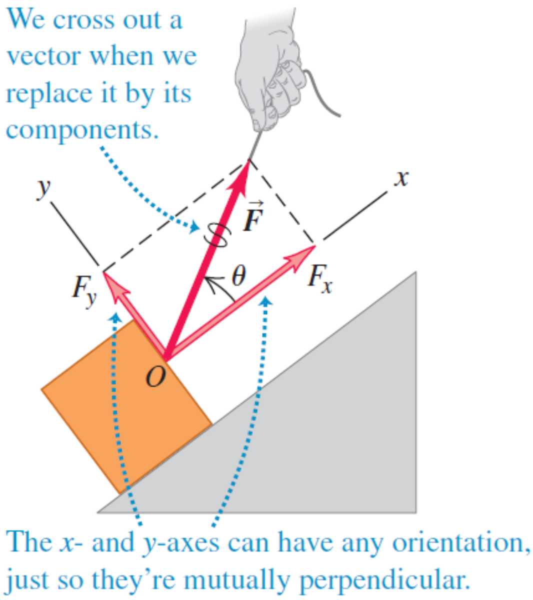


Figure 4.4 - F_x and F_y are the components of $\vec{F}$ parallel and perpendicular to the sloping surface of the inclined plane

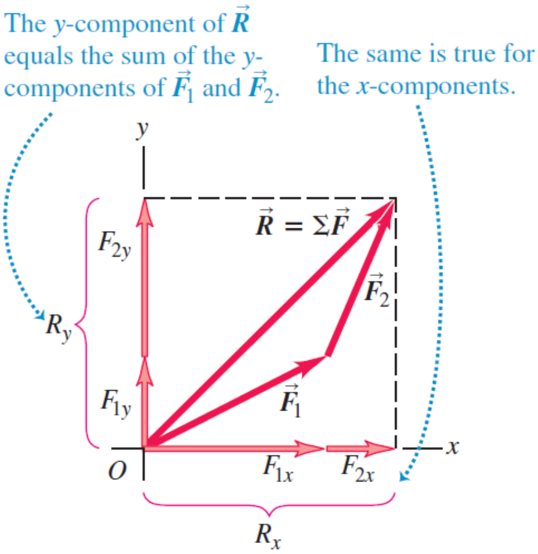


Figure 4.5 - Finding the components of the vector sum (resultant) $\vec{R}$ of two forces $\vec{F}_1$ and $\vec{F}_2$

4.2 Newton's First Law

How do the forces acting on an object affect that object's motion? Let's first note that it's impossible for an object to affect its own motion by exerting a force on itself. If that were possible, you could lift yourself to the ceiling by pulling up on your belt! The forces that affect an object's motion are **external forces**, those forces exerted on the object by other objects in its environment. So the question we must answer is this: How do the external forces that act on an object affect its motion?

To begin to answer this question, let's first consider what happens when the net external force on an object is *zero*. You would almost certainly agree that if an object is at rest, and if no net external force acts on it (that is, no net push or pull from other objects), that object will remain at rest. But what if there is zero net external force acting on an object in *motion*?

To see what happens in this case, suppose you slide a hockey puck along a horizontal tabletop, applying a horizontal force to it with your hand. After you stop pushing, the puck does *not* continue to move indefinitely; it slows down and stops. To keep it moving, you have to keep pushing (that is, applying a force). You might come to the "common sense" conclusion that objects in motion naturally come to rest and that a force is required to sustain motion.

But now imagine pushing the puck across a smooth surface of ice. After you quit pushing, the puck will slide a lot farther before it stops. Put it on an air-hockey table, where it floats on a thin cushion of air, and it moves still farther. In each case, what slows the puck down is *friction*, an interaction between the lower surface of the puck and the surface on which it slides. Each surface exerts a friction force on the puck that resists the puck's motion; the difference in the three cases is the magnitude of the friction force. The ice exerts less friction than the tabletop, so the puck travels farther. The gas molecules of the air-hockey table exert the least friction of all. If we could eliminate friction completely, the puck would never slow down, and we would need no force at all to keep the puck moving once it had been started. Thus the "common sense" idea that a force is required to sustain motion is *incorrect.*

Experiments like the ones we've just described show that when *no* net external force acts on an object, the object either remains at rest *or* moves with constant velocity in a straight line. Once an object has been set in motion, no net external force is needed to keep it moving. We call this observation *Newton's first law of motion:*

NEWTON'S FIRST LAW OF MOTION: An object acted on by no net external force has a constant velocity (which may be zero) and zero acceleration.

The tendency of an object to keep moving once it is set in motion is called **inertia.** You use inertia when you try to get ketchup out of a bottle by shaking it. First you start the bottle (and the ketchup inside) moving forward; when you jerk the bottle back, the ketchup tends to keep moving forward and, you hope, ends up on your burger. Inertia is also the tendency of an object at rest to remain at rest. You may have seen a tablecloth yanked out from under a table setting without breaking anything. The force on the table setting isn't great enough to make it move appreciably during the short time it takes to pull the tablecloth away.

It's important to note that the *net* external force is what matters in Newton's first law. For example, a physics book at rest on a horizontal tabletop has two forces acting on it: an upward supporting force, or normal force, exerted by the tabletop (see Fig. 4.2a) and the downward force of the earth's gravity (see Fig. 4.2d). The upward push of the surface is just as great as the downward pull of gravity, so the *net* external force acting on the book (that is, the vector sum of the two forces) is zero. In agreement with Newton's first law, if the book is at rest on the tabletop, it remains at rest. The same principle applies to a hockey puck sliding on a horizontal, frictionless surface: The vector sum of the upward push of the surface and the downward pull of gravity is zero. Once the puck is in motion, it continues to move with constant velocity because the *net* external force acting on it is zero.

We find that if the object is at rest at the start, it remains at rest; if it is initially moving, it continues to move in the same direction with constant speed. These results show that in Newton's first law, *zero net external force is equivalent to no external force at all.* This is just the principle of superposition of forces that we saw in Section 4.1. When an object is either at rest or moving with constant velocity (in a straight line with constant speed), we say that the object is in **equilibrium.**

For an object to be in equilibrium, it must be acted on by no forces, or by several forces such that their vector sum - that is, the net external force - is zero:

$$\underbrace{\sum \vec{F}}_{\text{Newton's first law: Net external force on an object ...}} = \underbrace{\mathbf{0}}_{\text{... must be zero if the object is in equilibrium.}} \qquad (4.3)$$

We're assuming that the object can be represented adequately as a point particle. When the object has finite size, we also have to consider *where* on the object the forces are applied. We'll return to this point in Chapter 11.

Inertial Frames of Reference

In discussing relative velocity in Section 3.5, we introduced the concept of *frame of reference.* This concept is central to Newton's laws of motion. Suppose you are in a bus that is traveling on a straight road and speeding up. If you could stand in the aisle on roller skates, you would start moving *backward* relative to the bus as the bus gains speed. If instead the bus was slowing to a stop, you would start moving *forward* down the aisle. In either case, it looks as though Newton's first law is not obeyed; there is no net external force acting on you, yet your velocity changes. What's wrong?

The point is that the bus is accelerating with respect to the earth and is *not* a suitable frame of reference for Newton's first law. This law is valid in some frames of reference and not valid in others. A frame of reference in which Newton's first law *is* valid is called an **inertial frame of reference.** The earth is at least approximately an inertial frame of reference, but the bus is not. (The earth is not a completely inertial frame, owing to the acceleration associated with its rotation and its motion around the sun. These effects are quite small, however). Because Newton's first law is used to define what we mean by an inertial frame of reference, it is sometimes called the *law of inertia.*

Figure 4.6 helps us understand what you experience when riding in a vehicle that's accelerating. In Fig. 4.6a, a vehicle is initially at rest and then begins to accelerate to the right. A passenger standing on roller skates (which nearly eliminate the effects of friction) has virtually no net external force acting on her, so she tends to remain at rest relative to the inertial frame of the earth. As the vehicle accelerates around her, she moves backward relative to the vehicle. In the same way, a passenger in a vehicle that is slowing down tends to continue moving with constant velocity relative to the earth, and so moves forward relative to the vehicle (Fig. 4.6b). A vehicle is also accelerating if it moves at a constant speed but is turning (Fig. 4.6c). In this case a passenger tends to continue moving relative to the earth at constant speed in a straight line; relative to the vehicle, the passenger moves to the side of the vehicle on the outside of the turn.

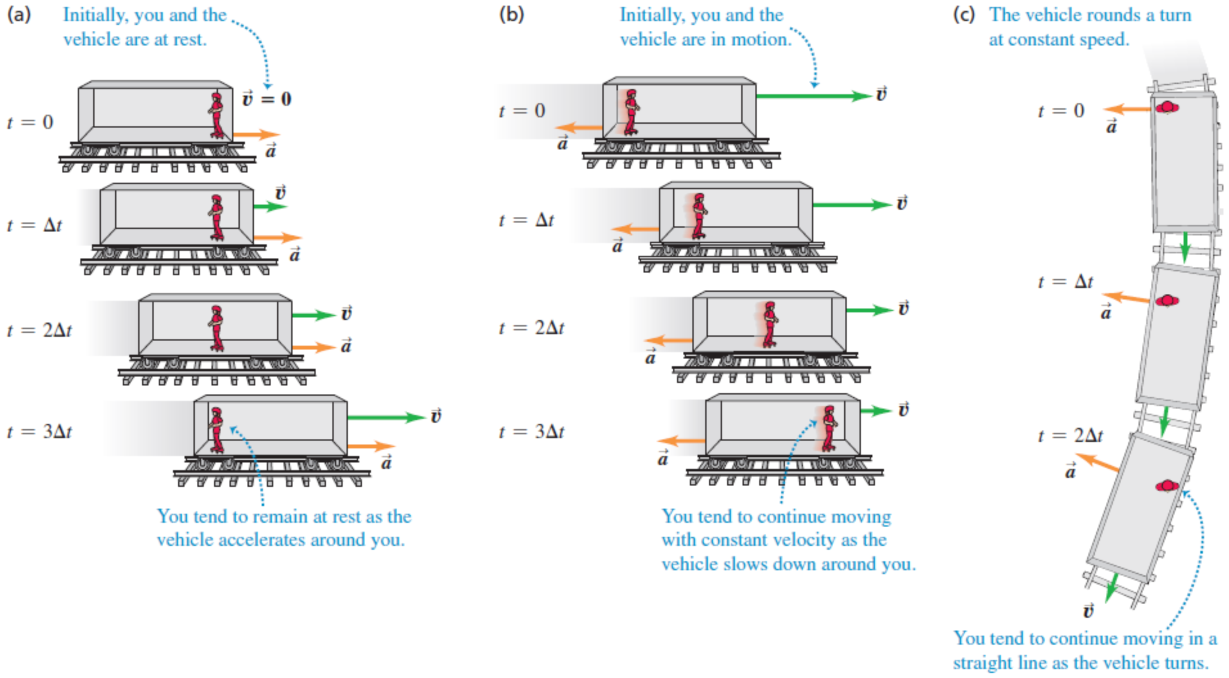


Figure 4.6 - Riding in an accelerating vehicle

In each case shown in Fig. 4.6, an observer in the vehicle's frame of reference might be tempted to conclude that there *is* a net external force acting on the passenger, since the passenger's velocity *relative to the vehicle* changes in each case. This conclusion is simplywrong; the net external force on the passenger is indeed zero. The vehicle observer's mistake is in trying to apply Newton's first law in the vehicle's frame of reference, which is *not* an inertial frame and in which Newton's first law isn't valid. In this book we'll use *only* inertial frames of reference.

We've mentioned only one (approximately) inertial frame of reference: the earth's surface. But there are many inertial frames. If we have an inertial frame of reference *A*, in which Newton's first law is obeyed, then *any* second frame of reference *B* will also be inertial if it moves relative to *A* with constant velocity $\vec{\upsilon}_{B/A}$. We can prove this by using the relative-velocity relationship Eq. (3.35) from Section 3.5:

$$\vec{\upsilon}_{P/A} = \vec{\upsilon}_{P/B} + \vec{\upsilon}_{B/A}.$$

Suppose that P is an object that moves with constant velocity $\vec{\upsilon}_{P/A}$ with respect to an inertial frame A. By Newton's first law the net external force on this object is zero. The velocity of P relative to another frame B has a different value, $\vec{\upsilon}_{P/B} = \vec{\upsilon}_{P/A} - \vec{\upsilon}_{B/A}$. But if the relative velocity $\vec{\upsilon}_{B/A}$ of the two frames is constant, then $\vec{\upsilon}_{P/B}$ is constant as well. Thus B is also an inertial frame; the velocity of P in this frame is constant, and the net external force on P is zero, so Newton's first law is obeyed in B. Observers in frames A and B will disagree about the velocity of P, but they will agree that P has a constant velocity (zero acceleration) and has zero net external force acting on it.

There is no single inertial frame of reference that is preferred over all others for formulating Newton's laws. If one frame is inertial, then every other frame moving relative to it with constant velocity is also inertial. Viewed in this light, the state of rest and the state of motion with constant velocity are not very different; both occur when the vector sum of forces acting on the object is zero.

4.3 Newton's Second Law

Newton's first law tells us that when an object is acted on by zero net external force, the object moves with constant velocity and zero acceleration. In **Fig. 4.7a** (next page), a hockey puck is sliding to the right on wet ice. There is negligible friction, so there are no horizontal forces acting on the puck; the downward force of gravity and the upward normal force exerted by the ice surface sum to zero. So the net external force $\Sigma\vec{F}$ acting on the puck is zero, the puck has zero acceleration, and its velocity is constant.

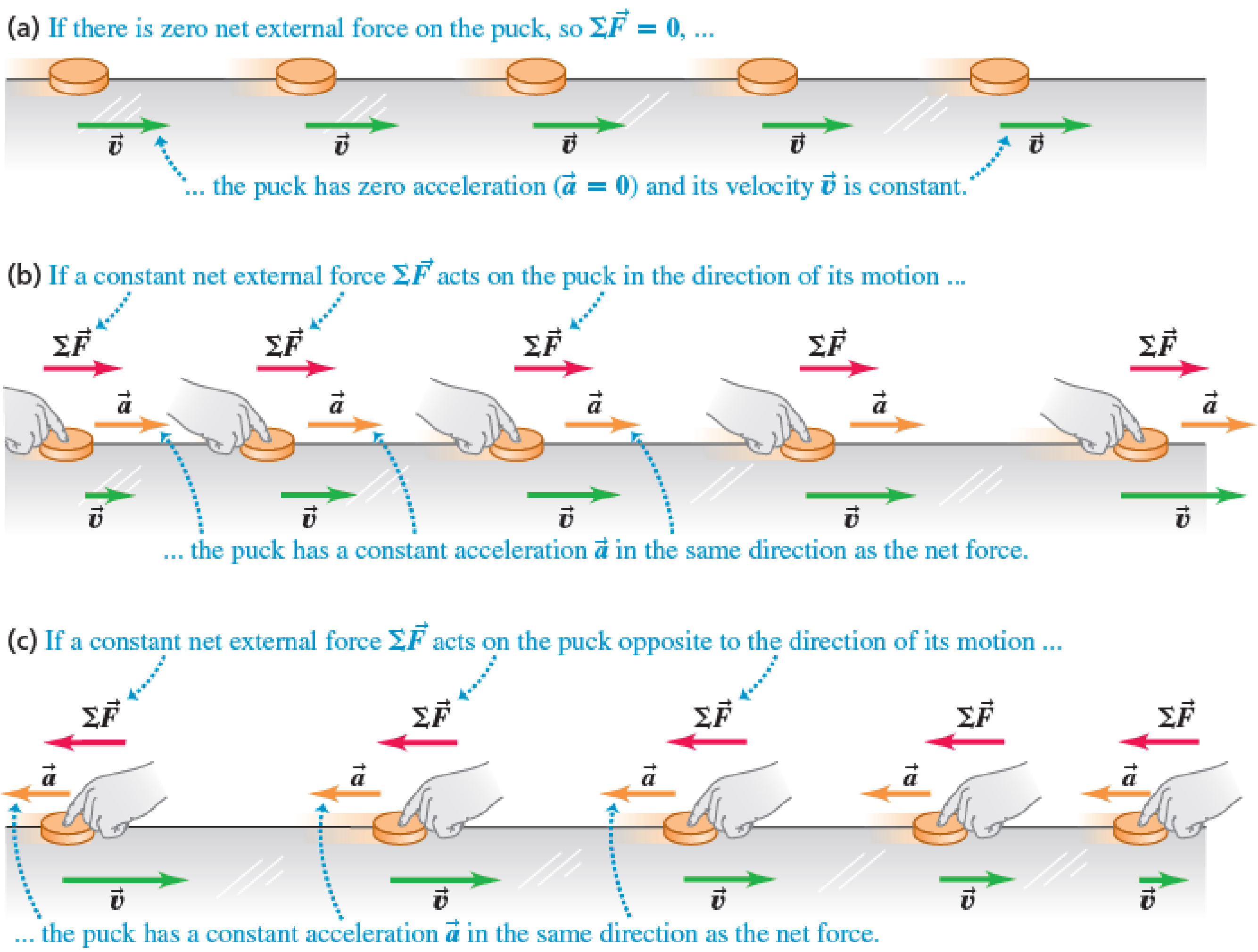


Figure 4.7 - Using a hockey puck on a frictionless surface to explore the relationship between the net external force $\Sigma\vec{F}$ on an object and the resulting acceleration ***a***S of the object

But what happens when the net external force is *not* zero? In Fig. 4.7b we apply a constant horizontal force to a sliding puck in the same direction that the puck is moving. Then $\Sigma\vec{F}$ is constant and in the same horizontal direction as $\vec{\upsilon}$. We find that during the time the force is acting, the velocity of the puck changes at a constant rate; that is, the puck moves with constant acceleration. The speed of the puck increases, so the acceleration $\vec{a}$ is in the same direction as $\vec{\upsilon}$ and $\Sigma\vec{F}$.

In Fig. 4.7c we reverse the direction of the force on the puck so that $\Sigma\vec{F}$ acts opposite to *v*S. In this case as well, the puck has an acceleration; the puck moves more and more slowly to the right. The acceleration $\vec{a}$ in this case is to the left, in the same direction as $\Sigma\vec{F}$. As in the previous case, experiment shows that the acceleration is constant if $\Sigma\vec{F}$ is constant. We conclude that *a net external force acting on an object causes the object to accelerate in the same direction as the net external force.* If the magnitude of the net external force is constant, as in Fig. 4.7b and Fig. 4.7c, then so is the magnitude of the acceleration.

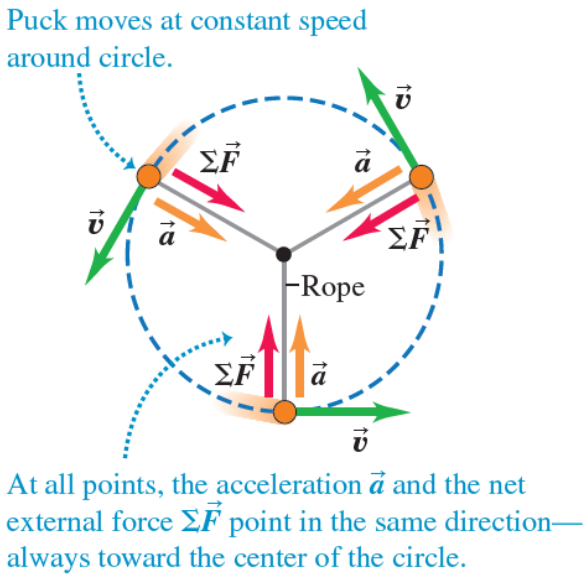


Figure 4.8 - A top view of a hockeypuck in uniform circular motion on africtionless horizontal surface

These conclusions about net external force and acceleration also apply to an object moving along a curved path. For example, **Fig. 4.8** shows a hockey puck moving in a horizontal circle on an ice surface of negligible friction. A rope is attached to the puck and to a stick in the ice, and this rope exerts an inward tension force of constant magnitude on the puck. The net external force and acceleration are both constant in magnitude and directed toward the center of the circle. The speed of the puck is constant, so this is uniform circular motion (see Section 3.4).

Figure 4.9a shows another experiment involving acceleration and net external force. We apply a constant horizontal force to a puck on a frictionless horizontal surface, using the spring balance described in Section 4.1 with the spring stretched a constant amount. As in Figs. 4.7b and Figs. 4.7c, this horizontal force equals the net external force on the puck. If we change the magnitude of the net external force, the acceleration changes in the same proportion.

Doubling the net external force doubles the acceleration (Fig. 4.9b), halving the net external force halves the acceleration (Fig. 4.9c), and so on.

Many such experiments show that *for any given object, the magnitude of the acceleration is directly proportional to the magnitude of the net external force acting on the object.*

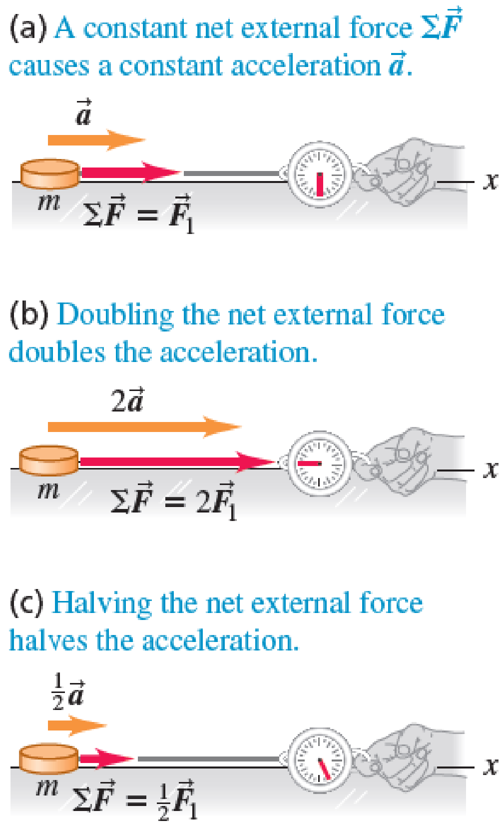


Figure 4.9 - The magnitude of an object's acceleration $\vec{a}$ is directly proportional to the magnitude of the net external force $\vec{F}$ acting on the object of mass *m*

Mass and Force

Our results mean that for a given object, the *ratio* of the magnitude $|\Sigma\vec{F}|$ of the net external force to the magnitude $a = |\vec{a}|$ of the acceleration is constant, regardless of the magnitude of the net external force. We call this ratio the *inertial mass,* or simply the **mass,** of the object and denote it by *m*. That is,

$$m = \frac{|\Sigma\vec{F}|}{a} \quad \text{or} \quad |\Sigma\vec{F}| = ma \quad \text{or} \quad a = \frac{|\Sigma\vec{F}|}{m}. \tag{4.4}$$

Mass is a quantitative measure of inertia, which we discussed in Section 4.2. The last of the equations in Eqs. (4.4) says that the greater an object's mass, the more the object "resists" being accelerated. When you hold a piece of fruit in your hand at the supermarket and move it slightly up and down to estimate its heft, you're applying a force and seeing how much the fruit accelerates up and down in response. If a force causes a large acceleration, the fruit has a small mass; if the same force causes only a small acceleration, the fruit has a large mass. In the same way, if you hit a table-tennis ball and then a basketball with the same force, the basketball has much smaller acceleration because it has much greater mass.

The SI unit of mass is the **kilogram.** We mentioned in Section 1.3 that the kilogram is officially defined in terms of the definitions of the second and the meter, as well as the value of a fundamental quantity called Planck's constant. We can use this definition:

One newton is the amount of net external force that gives an acceleration of 1 meter per second squared to an object with a mass of 1 kilogram.

This definition allows us to calibrate the spring balances and other instruments used to measure forces. Because of the way we have defined the newton, it is related to the units of mass, length, and time. For Eqs. (4.4) to be dimensionally consistent, it must be true that

1 newton = (1 kilogram)(1 meter per second squared) or 1 N = 1 kg · m/s^2.

We'll use this relationship many times in the next few chapters, so keep it in mind.

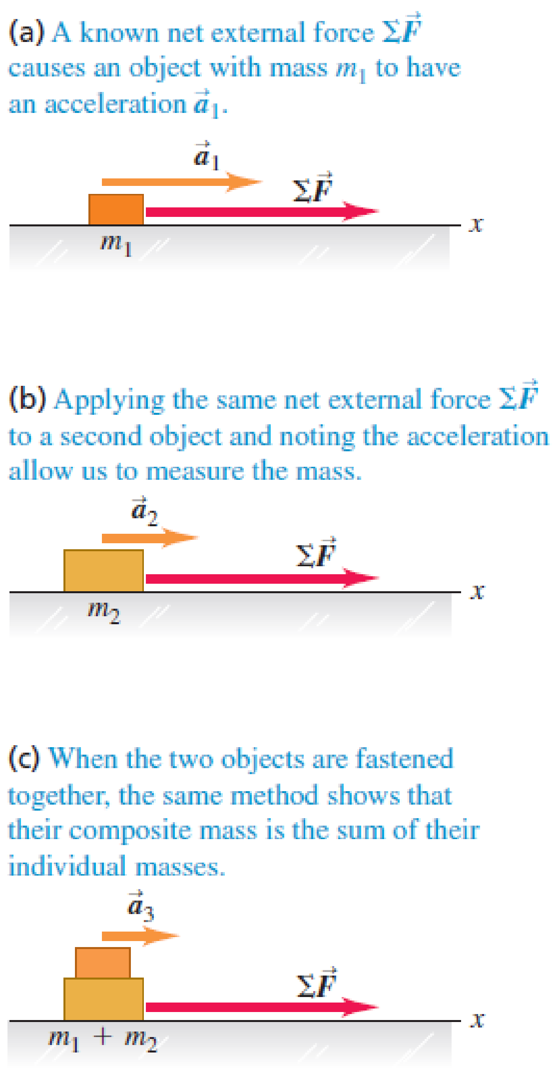


Figure 4.10 - For a given net external force $\Sigma\vec{F}$ acting an object, the acceleration is inversely proportional to the mass of the object. Masses add like ordinary scalars

Here's an application of Eqs. (4.4). Suppose we apply a constant net external force $\Sigma\vec{F}$ to an object of known mass m_1 and we find an acceleration of magnitude a_1 (**Fig. 4.10a**). We then apply the same force to another object of unknown mass m_2 , and we find an acceleration of magnitude a_2 (Fig. 4.10b). Then, according to Eqs. (4.4),

$$m_1 a_1 = m_2 a_2 \quad \frac{m_2}{m1} = \frac{a_1}{a_2} \quad \text{(same net external force).} \tag{4.5}$$

For the same net external force, the ratio of the masses of two objects is the inverse of the ratio of their accelerations. In principle we could use Eq. (4.5) to measure an unknown mass m_2, but it is usually easier to determine mass indirectly by measuring the object's *weight.* We'll return to this point in Section 4.4.

When two objects with masses m_1 and m_2 are fastened together, we find that the mass of the composite object is always $m_1 + m_2$ (Fig. 4.10c). This additive property of mass may seem obvious, but it has to be verified experimentally. Ultimately, the mass of an object is related to the number of protons, electrons, and neutrons it contains. This wouldn't be a good way to *define* mass because there is no practical way to count these particles. But the concept of mass is the most fundamental way to characterize the quantity of matter in an object.

Stating Newton's Second Law

Experiment shows that the *net* external force on an object is what causes that object to accelerate. If a combination of forces $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$, and so on is applied to an object, the object will have the same acceleration vector $\vec{a}$ as when only a single force is applied, if that single force is equal to the vector sum $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots$. In other words, the principle of superposition of forces (see Fig. 4.3) also holds true when the net external force is not zero and the object is accelerating.

Equations (4.4) relate the magnitude of the net external force on an object to the magnitude of the acceleration that it produces. We have also seen that the direction of the net external force is the same as the direction of the acceleration, whether the object's path is straight or curved. Finally, we've seen that the forces that affect an object's motion are *external* forces, those exerted on the object by other objects in its environment. Newton wrapped up all these results into a single concise statement that we now call *Newton's second law of motion:*

NEWTON'S SECOND LAW OF MOTION: If a net external force acts on an object, the object accelerates. The direction of acceleration is the same as the direction of the net external force. The mass of the object times the acceleration vector of the object equals the net external force vector.

In symbols,

Newton's second law: If there is a net external force on an object ...

$$\sum \vec{F} = m\vec{a} \qquad (4.6)$$

Mass of object

... the object accelerates in the same direction as the net external force.

An alternative statement is that the acceleration of an object is equal to the net external force acting on the object divided by the object's mass:

$$\vec{a} = \frac{\sum \vec{F}}{m}.$$

Newton's second law is a fundamental law of nature, the basic relationship between force and motion. Most of the remainder of this chapter and all of the next are devoted to learning how to apply this principle in various situations.

Equation (4.6) has many practical applications. You've actually been using it all your life to measure your body's acceleration. In your inner ear, microscopic hair cells are attached to a gelatinous substance that holds tiny crystals of calcium carbonate called *otoliths*. When your body accelerates, the hair cells pull the otoliths along with the rest of your body and sense the magnitude and direction of the force that they exert. By Newton's second law, the acceleration of the otoliths - and hence that of your body as a whole - is proportional to this force and has the same direction. In this way, you can sense the magnitude and direction of your acceleration even with your eyes closed!

Using Newton's Second Law

At least four aspects of Newton's second law deserve special attention. First, Eq. (4.6) is avector equation. Usually we'll use it in component form, with a separate equation for eachcomponent of force and the corresponding component of acceleration:

Newton's second law: Each component of the net external force on an object ...

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \sum F_z = ma_z \tag{4.7}$$

... equals the object's mass times the corresponding acceleration component.

This set of component equations is equivalent to the single vector Eq. (4.6).

Second, the statement of Newton's second law refers to *external* forces. As an example, how a kicked football moves isn't affected by the *internal* forces that hold the pieces of the ball together. That's why only external forces are included in the sum $\Sigma\vec{F}$ in Eqs. (4.6) and (4.7).

Third, Eqs. (4.6) and (4.7) are valid only when the mass m is *constant.* It's easy to think of systems whose masses change, such as a leaking tank truck or a moving railroad car being loaded with coal. Such systems are better handled by using the concept of momentum; we'll get to that in Chapter 8.

Finally, Newton's second law is valid in inertial frames of reference *only,* just like the first law. It's not valid in the reference frame of any of the accelerating vehicles in Fig. 4.6; relative to any of these frames, the passenger accelerates even though the net external force on the passenger is zero. We'll usually treat the earth as an adequate approximation to an inertial frame, although because of its rotation and orbital motion it is not precisely inertial.

CAUTION! **$m\vec{a}$ is not a force.** Even though the vector $m\vec{a}$ is equal to the vector sum $\Sigma\vec{F}$ of all the forces acting on the object, the vector $m\vec{a}$ is *not* a force. Acceleration is the *result* of the net external force; it is not a force itself. It's "common sense" to think that a "force of acceleration" pushes you back into your seat when your car accelerates forward from rest. But *there is no such force;* instead, your inertia causes you to tend to stay at rest relative to the earth, and the car accelerates around you (see Fig. 4.6a). The "common sense" confusion arises from trying to apply Newton's second law where it isn't valid - in the noninertial reference frame of an accelerating car. We'll always examine motion relative to *inertial* frames of reference only, and we strongly recommend that you do the same in solving problems.

In learning how to use Newton's second law, we'll begin in this chapter with examples of straight-line motion. Then in Chapter 5 we'll consider more general kinds of motion and develop more detailed problem-solving strategies.

4.4 Mass and Weight

The *weight* of an object is the gravitational force that the earth exerts on the object. (If you are on another planet, your weight is the gravitational force that planet exerts on you). Unfortunately, the terms "mass" and "weight" are often misused and interchanged in everyday conversation. It's absolutely essential for you to understand clearly the distinctions between these two physical quantities.

Mass characterizes the *inertial* properties of an object. Mass is what keeps the table setting on the table when you yank the tablecloth out from under it. The greater the mass, the greater the force needed to cause a given acceleration; this is reflected in Newton's second law, $\Sigma\vec{F} = m\vec{a}$.

Weight, on the other hand, is a *force* exerted on an object by the pull of the earth. Mass and weight are related: Objects that have large mass also have large weight. A large stone is hard to throw because of its large *mass,* and hard to lift off the ground because of its large *weight.*

To understand the relationship between mass and weight, note that a freely falling object has an acceleration of magnitude g (see Section 2.5). Newton's second law tells us that a force must act to produce this acceleration. If a 1 kg object falls with an acceleration of 9.8 m/s^2, the required force has magnitude

$$F = ma = (1\ \text{kg})(9.8\ \text{m/s}^2) = 9.8\ \text{kg}\cdot\text{m/s}^2 = 9.8\ \text{N}.$$

The force that makes the object accelerate downward is its weight. Any object near the surface of the earth that has a mass of 1 kg *must* have a weight of 9.8 N to give it the acceleration we observe when it is in free fall. More generally,

$$\underset{\text{Magnitude of weight of an object}}{w} = \overset{\text{Mass of object}}{m}\underset{\text{Magnitude of acceleration due to gravity}}{g} \tag{4.8}$$

Hence the magnitude *w* of an object’s weight is directly proportional to its mass *m*. The weight of an object is a force, a vector quantity, and we can write Eq. (4.8) as a vector equation:

$$\vec{w} = m\vec{g}. \tag{4.9}$$

Remember that g is the *magnitude* of $\vec{g}$, the acceleration due to gravity, so g is always a positive number, by definition. Thus w, given by Eq. (4.8), is the *magnitude* of the weight and is also always positive.

CAUTION! **An object’s weight acts at all times.** When keeping track of the external forces on an object, remember that the weight is present *all the time*, whether the object is in free fall or not. If we suspend an object from a rope, it is in equilibrium and its acceleration is zero. But its weight, given by Eq. (4.9), is still pulling down on it. In this case the rope pulls up on the object, applying an upward force. The *vector sum* of the external forces is zero, but the weight still acts.

Variation of *g* with Location

We’ll use $g = 9.80\ \text{m/s}^2$ for problems set on the earth (or, if the other data in the problem are given to only two significant figures, $g = 9.8\ \text{m/s}^2$). In fact, the value of g varies somewhat from point to point on the earth’s surface - from about 9.78 to 9.82 m/s^2 - because the earth is not perfectly spherical and because of effects due to its rotation. At a point where $g = 9.80\ \text{m/s}^2$, the weight of a standard kilogram is $w = 9.80$ N.At a different point, where $g = 9.78\ \text{m/s}^2$, the weight is $w = 9.78$ N but the mass is still 1 kg. The weight of an object varies from one location to another; the mass does not.

Measuring Mass and Weight

In Section 4.3 we described a way to compare masses by comparing their accelerations when they are subjected to the same net external force. Usually, however, the easiest way to measure the mass of an object is to measure its weight, often by comparing with a standard. Equation (4.8) says that two objects that have the same weight at a particular location also have the same mass. We can compare weights very precisely; the familiar equal-arm balance can determine with great precision (up to 1 part in 10^6) when the weights of two objects are equal and hence when their masses are equal.

The concept of mass plays two rather different roles in mechanics. The weight of an object (the gravitational force acting on it) is proportional to its mass as stated in the equation $w = mg$; we call the property related to gravitational interactions *gravitational mass.* On the other hand, we call the inertial property that appears in Newton’s second law ($\Sigma\vec{F} = m\vec{a}$) the *inertial mass.* If these two quantities were different, the acceleration due to gravity might well be different for different objects. However, extraordinarily precise experiments have established that in fact the two *are* the same to a precision of better than one part in 10^{12}.

CAUTION! **Don’t confuse mass and weight!** The SI units for mass and weight are often misused in everyday life. For example, it’s incorrect to say “This box weighs 6 kg.” What this really means is that the *mass* of the box, probably determined indirectly by *weighing*, is 6 kg. Avoid this sloppy usage in your own work! In SI units, weight (a force) is measured in newtons, while mass is measured in kilograms. ▮

4.5 Newton's Third Law

A force acting on an object is always the result of its interaction with another object, so forces always come in pairs. You can't pull on a doorknob without the doorknob pulling back on you. When you kick a football, your foot exerts a forward force on the ball, but you also feel the force the ball exerts back on your foot.

In each of these cases, the force that you exert on the other object is in the opposite direction to the force that object exerts on you. Experiments show that whenever two objects interact, the two forces that they exert on each other are always *equal in magnitude* and *opposite in direction.* This fact is called *Newton's third law of motion:*

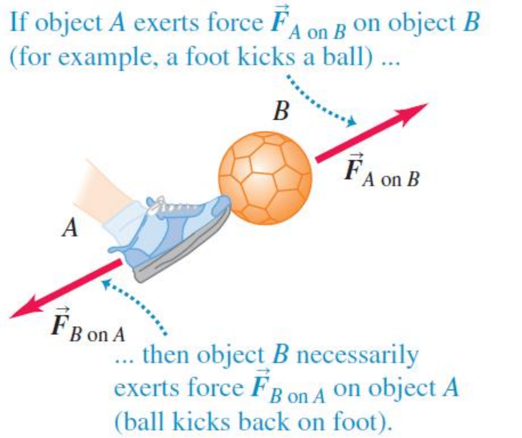


Figure 4.11 - Newton's third law of motion

NEWTON'S THIRD LAW OF MOTION: If object *A* exerts a force on object *B* (an "action"), then object *B* exerts a force on object *A* (a "reaction"). These two forces have the same magnitude but are opposite in direction. These two forces act on *different* objects.

For example, in **Fig. 4.11** $\vec{F}_{A\ on\ B}$ is the force applied *by* object *A* (first subscript) *on* object *B* (second subscript), and $\vec{F}_{B\ on\ A}$ is the force applied *by* object *B* (first subscript) *on* object *A* (second subscript). In equation form,

Newton's third law: When two objects *A* and *B* exert forces on each other ...

$$\vec{F}_{A\text{ on }B} = -\vec{F}_{B\text{ on }A} \qquad (4.10)$$

... the two forces have the same magnitude but opposite directions.

Note: The two forces act on *different* objects.

It doesn't matter whether one object is inanimate (like the football in Fig. 4.11) and the other is not (like the kicker's foot): They necessarily exert forces on each other that obey Eq. (4.10).

In the statement of Newton's third law, "action" and "reaction" are the two opposite forces (in Fig. 4.11, $\vec{F}_{A\ on\ B}$ and $\vec{F}_{B\ on\ A}$); we sometimes refer to them as an **action-reaction pair.** This is *not* meant to imply any cause-and-effect relationship; we can consider either force as the "action" and the other as the "reaction". We often say simply that the forces are "equal and opposite," meaning that they have equal magnitudes and opposite directions.

CAUTION! **The two forces in an action-reaction pair act on different objects.** We stress that the two forces described in Newton's third law act on *different* objects. This is important when you solve problems involving Newton's first or second law, which involve the forces that act *on* an object. For instance, the net external force on the football in Fig. 4.11 is the vector sum of the weight of the ball and the force $\vec{F}_{A\ on\ B}$ exerted by kicker *A* on the ball *B*. You wouldn't include the force $\vec{F}_{B\ on\ A}$ because this force acts on the kicker *A*, *not* on the ball.

In Fig. 4.23 the action and reaction forces are *contact* forces that are present only when the two objects are touching. But Newton's third law also applies to *long-range* forces that do not require physical contact, such as the force of gravitational attraction. A table-tennis ball exerts an upward gravitational force on the earth that's equal in magnitude to the downward gravitational force the earth exerts on the ball. When you drop the ball, both the ball and the earth accelerate toward each other. The net force on each object has the same magnitude, but the earth's acceleration is microscopically small because its mass is so great. Nevertheless, it does move!

CAUTION! **Contact forces need contact.** If your fingers push on an object, the force you exert acts only when your fingers and the object are in contact. Once contact is broken, the force is no longer present even if the object is still moving.

4.6 Free-Body Diagrams

Newton’s three laws of motion contain all the basic principles we need to solve a wide variety of problems in mechanics. These laws are very simple in form, but the process of applying them to specific situations can pose real challenges. In this brief section we’ll point out three key ideas and techniques to use in any problems involving Newton’s laws. You’ll learn others in Chapter 5, which also extends the use of Newton’s laws to cover more complex situations.

1. *Newton’s first and second laws apply to a specific object.* Whenever you use Newton’s first law, $\Sigma\vec{F} = 0$, for an equilibrium situation or Newton’s second law, $\Sigma\vec{F} = m\vec{a}$, for a nonequilibrium situation, you must decide at the beginning to which object you are referring. This decision may sound trivial, but it isn’t.

2. *Only forces acting on the object matter*. The sum $\Sigma\vec{F}$ includes all the forces that act *on* the object in question. Hence, once you’ve chosen the object to analyze, you have to identify all the forces acting on it. Don’t confuse the forces acting on a object with the forces exerted by that object on some other object. For example, to analyze a person walking, you would include in $\Sigma\vec{F}$ the force that the ground exerts on the person as he walks, but *not* the force that the person exerts on the ground. These forces form an action–reaction pair and are related by Newton’s third law, but only the member of the pair that acts on the object you’re working with goes into $\Sigma\vec{F}$.

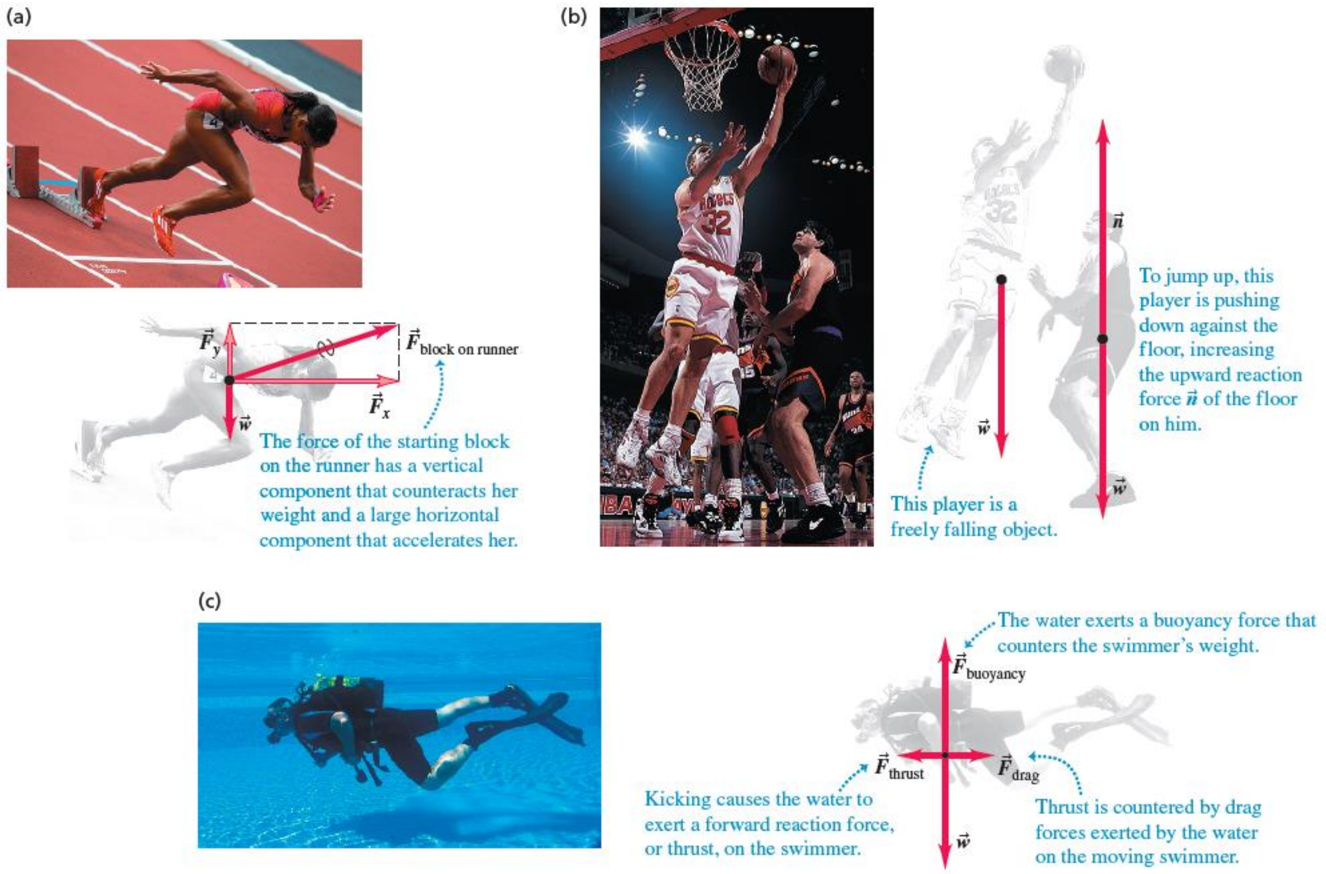
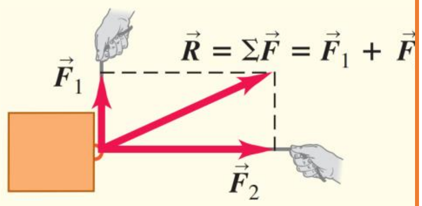
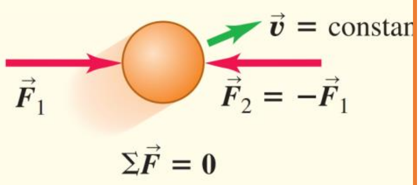
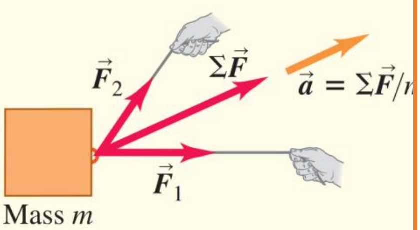
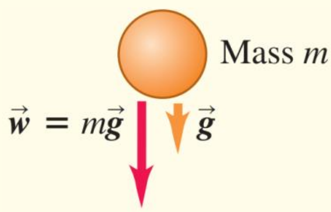


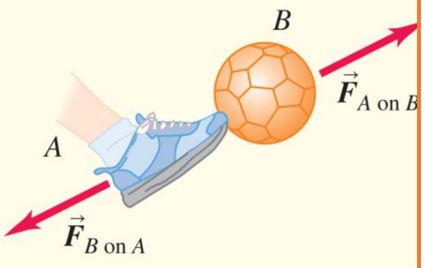
Figure 4.12 - Examples of free-body diagrams. Each free-body diagram shows all of the external forces that act on the object in question

3. *Free-body diagrams are essential to help identify the relevant forces.* A **free-body diagram** shows the chosen object by itself, “free” of its surroundings, with vectors drawn to show the

magnitudes and directions of all the forces that act on the object. (Here “body” is another word for “object”). Be careful to include all the forces acting *on* the object, but be equally careful *not* to include any forces that the object exerts on any other object. In particular, the two forces in an action–reaction pair must *never* appear in the same free-body diagram because they never act on the same object. Furthermore, never include forces that a object exerts on itself, since these can’t affect the object’s motion.

When a problem involves more than one object, you have to take the problem apart and draw a separate free-body diagram for each object. In **Fig. 4.12** we present three real-life situations and the corresponding complete free-body diagrams. Note that in each situation a person exerts a force on something in his or her surroundings, but the force that shows up in the person’s free-body diagram is the surroundings pushing back *on* the person.CAUTION! **Forces in free-body diagrams.** For a free-body diagram to be complete, you *must* be able to answer this question for each force: What other object is applying this force? If you can’t answer that question, you may be dealing with a nonexistent force. Avoid nonexistent forces such as “the force of acceleration” or “the $m\vec{a}$ force”, discussed in Section 4.3.

CHAPTER 4: SUMMARY		
Force as a vector: Force is a quantitative measure of the interaction between two objects. It is a vector quantity. When several external forces act on an object, the effect on its motion is the same as if a single force, equal to the vector sum (resultant) of the forces, acts on the object	$\vec{R} = \sum\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots$	
The net external force on an object and Newton’s first law: Newton’s first law states that when the vector sum of all external forces acting on a object (the *net external force*) is zero, the object is in equilibrium and has zero acceleration. If the object is initially at rest, it remains at rest; if it is initially in motion, it continues to move with constant velocity. This law is valid in inertial frames of reference only	$\sum\vec{F} = 0$.	
Mass, acceleration, and Newton’s second law: The inertial properties of an object are characterized by its *mass*. Newton’s second law states that the acceleration of an object under the action of a given set of external forces is directly proportional to the vector sum of the forces (the *net force*) and inversely proportional to the mass of the object. Like Newton’s first law, this law is valid in inertial frames of reference only. In SI units, the unit of force is the newton (N), equal to $1\ \mathrm{kg \cdot m/s^2}$	$\sum\vec{F} = m\vec{a}$ $\sum F_x = ma_x$ $\sum F_y = ma_y$ $\sum F_z = ma_z$	
Weight: The weight $\vec{w}$ of an object is the gravitational force exerted on it by the earth. Weight is a vector quantity. The magnitude of the weight of an object at any specific location is equal to the product of its mass m and the magnitude of the acceleration due to gravity g at that location. The weight of an object depends on its	$w = mg$.	

location; its mass does not		
Newton's third law and action–reaction pairs: Newton's third law states that when two objects interact, they exert forces on each other that are equal in magnitude and opposite in direction. These forces are called action and reaction forces. Each of these two forces acts on only one of the two objects; they never act on the same object	$\vec{F}_{A\text{ on }B} = -\vec{F}_{B\text{ on A}}$.	

5 APPLYING NEWTON'S LAWS

We saw in Chapter 4 that Newton's three laws of motion, the foundation of classicalmechanics, can be stated very simply. But applying these laws to situationssuch as an iceboat skating across a frozen lake, a toboggan sliding down a hill, oran airplane making a steep turn requires analytical skills and problem-solving technique.In this chapter we'll help you extend the problem-solving skills you began to develop inChapter 4.

We'll begin with equilibrium problems, in which we analyze the forces that act on anobject that is at rest or moving with constant velocity. We'll then consider objects that arenot in equilibrium. For these we'll have to take account of the relationship between forcesand acceleration. We'll learn how to describe and analyze the contact force that acts on anobject when it rests on or slides over a surface. We'll also analyze the forces that act on anobject that moves in a circle with constant speed. We close the chapter with a brief look atthe fundamental nature of force and the classes of forces found in our physical universe.

5.1 Using Newton's First Law: Particles in Equilibrium

We learned in Chapter 4 that an object is in equilibrium when it is at rest or moving with constant velocity in an inertial frame of reference. A hanging lamp, a kitchen table, an airplane flying straight and level at a constant speed – all are examples of objects in equilibrium. In this section we consider only the equilibrium of an object that can be modelled as a particle. (In Chapter 11 we'll see how to analyze an object in equilibrium that can't be represented adequately as a particle, such as a bridge that's supported at various points along its span). The essential physical principle is Newton's first law:

Newton's first law: *Net* force on an object ... $\sum \vec{F} = \mathbf{0}$ ... must be *zero* for an **object in equilibrium.**

Sum of x-components of force on object must be zero. $\sum F_x = 0$ Sum of y-components of force on object must be zero. $\sum F_y = 0$ (5.1)

This section is about using Newton's first law to solve problems dealing with objects in equilibrium. Some of these problems may seem complicated, but remember that all problems involving particles in equilibrium are done in the same way. Problem-Solving Strategy 5.1 details the steps you need to follow for any and all such problems. Study this strategy carefully, look at how it's applied in the worked-out examples, and try to apply it when you solve assigned problems.

PROBLEM-SOLVING STRATEGY
5.1 Newton's First Law: Equilibrium of a Particle

IDENTIFY *the relevant concepts:*

- Use Newton's first law,Eqs. (5.1), for any problem that involves forces acting on an object inequilibrium—that is, either at rest or moving with constant velocity.A car is in equilibrium when it's parked, but also when it's travelingdown a straight road at a steady speed.
- If the problem involves more than one object and the objectsinteract with each other, you'll also need to use Newton's *third law*.This law allows you to relate the force that one object exerts on asecond object to the force that the second object exerts on the first one.
- Identify the **target variable(s)**. Common target variables inequilibrium problems include the magnitude and direction (angle) ofone of the forces, or the components of a force.

SET UP *the problem*:

- Draw a very simple sketch of the physical situation, showingdimensions and angles. You don't have to be an artist!
- Draw a free-body diagram for each object that is in equilibrium.For now, we consider the object as a particle, so you can representit as a large dot. In your free-body diagram, do not include theother objects that interact with it, such as a surface it may be restingon or a rope pulling on it.
- Ask yourself what is interacting with the object by contact or inany other way. On your free-body diagram, draw a force vectorfor each interaction. Label each force with a symbol for themagnitude of the force. If you know the angle at which a forceis directed, draw the angle accurately and label it. Include theobject's weight, unless the object has negligible mass. If themass is given, use $w = mg$ to find the weight. A surface in contactwith the object exerts a normal force perpendicular to the surface and possibly a friction force parallel to the surface. A rope or chain exerts a pull (never a push) in a direction along itslength.
- Do not show in the free-body diagram any forces exerted by the object on any other object. The sums in Eqs. (5.1) include only forces that act on the object. For each force on the object, ask yourself "What other object causes that force?" If you can't answer thatquestion, you may be imagining a force that isn't there.
- Choose a set of coordinate axes and include them in your freebody diagram. (If there is more than one object in the problem, choose axes for each object separately). Label the positive direction for each axis. If an object rests or slides on a plane surface, for simplicity choose axes that are parallel and perpendicular to this surface, even when the plane is tilted.

EXECUTE *the solution*:

- Find the components of each force along each of the object's coordinate axes. Draw a wiggly line through each force vector that has been replaced by its components, so you don't count it twice. The magnitude of a force is always positive, but its componentsmay be positive or negative.
- Set the sum of all x-components of force equal to zero. In a separate equation, set the sum of all y-components equal to zero.(Never add x- and y-components in a single equation).
- If there are two or more objects, repeat all of the above steps for each object. If the objects interact with each other, use Newton'sthird law to relate the forces they exert on each other.
- Make sure that you have as many independent equations as the number of unknown quantities. Then solve these equations to obtain thetarget variables.

EVALUATE *your answer:*

- Look at your results and ask whether they make sense. When the result is a symbolic expression or formula, check to see that your formula works for any special cases (particular values or extreme cases for the various quantities) for which you can guess what the results ought to be.

EXAMPLE 5.1

A gymnast with mass m_G = 50.0 kg suspends herself from the lower end of a hanging rope of negligible mass. The upper end of the rope is attached to the gymnasium ceiling. (a) What is the gymnast's weight? (b) What force (magnitude and direction) does the rope exert on her? (c) What is the tension at the top of the rope?

IDENTIFY and SET UP

The gymnast and the rope are in equilibrium,so we can apply Newton's first law to both objects. We'll use Newton'sthird law to relate the forces that they exert on each other. The targetvariables are the gymnast's weight, w_G; the force that the bottom of the rope exerts on the gymnast (call it$T_{\text{R on G}}$); and the force that theceiling exerts on the top of the rope (call it $T_{\text{C on R}}$). **Figure 5.1** shows our sketch of the situation and free-body diagrams for the gymnastand for the rope. We take the positive y-axis to be upward in each diagram. Each force acts in the vertical direction and so has only a y-component.

The forces $T_{\text{R on G}}$ (the upward force of the rope on the gymnast, Fig. 5.1b) and $T_{\text{G on R}}$ (the downward force of the gymnast on the rope, Fig. 5.1c) form an action–reaction pair. By Newton's third law, theymust have the same magnitude.

Note that Fig. 5.1c includes only the forces that act *on* the rope. Inparticular, it doesn't include the force that the *rope* exerts on the *ceiling*.

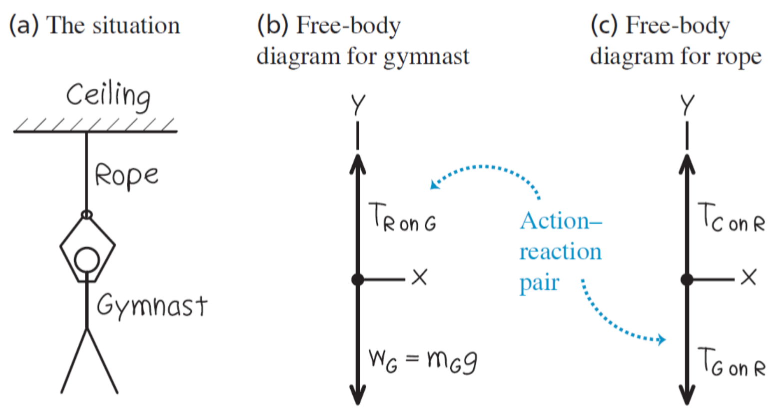


Figure 5.1 - Free-body diagrams for the gymnastand for the rope

EXECUTE

(a) The magnitude of the gymnast's weight is the product of her mass and the acceleration due to gravity, g:

$$w_G = m_G g = (50.0\ \text{kg})(9.80\ \text{m/s}^2) = 490\ \text{N}\,.$$

(b) The gravitational force on the gymnast (her weight) points in the negative y-direction, so its y-component is $-w_{\text{G}}$. The upward force of the rope on the gymnast has unknown magnitude $T_{\text{R on G}}$ and positive y-component $+T_{\text{R on G}}$. We find this by using Newton's first law from Eqs. (5.1):

$$\text{Gymnast:}\quad \sum F_y = T_{\text{R on G}} + (-w_{\text{G}}) = 0 \ \text{ so}$$
$$T_{\text{R on G}} = w_{\text{G}} = 490\ \text{N}$$

The rope pulls *up* on the gymnast with a force $T_{\text{R on G}}$ of magnitude 490 N. (By Newton's third law, the gymnast pulls *down* on the rope with a force of the same magnitude, $T_{\text{G on R}}$ = 490 N). (c) We have assumed that the rope is weightless, so the only forces on it are those exerted by the ceiling (upward force of unknown magnitude $T_{\text{C on R}}$) and by the gymnast (downward force of magnitude $T_{\text{G on R}}$ = 490 N). From Newton's first law, the net vertical force on the rope in equilibrium must be zero:

$$\text{Rope:}\quad \sum F_y = T_{\text{C on G}} + (-T_{\text{G on R}}) = 0 \ \text{ so}$$
$$T_{\text{C on G}} = T_{\text{G on R}} = 490\ \text{N}$$

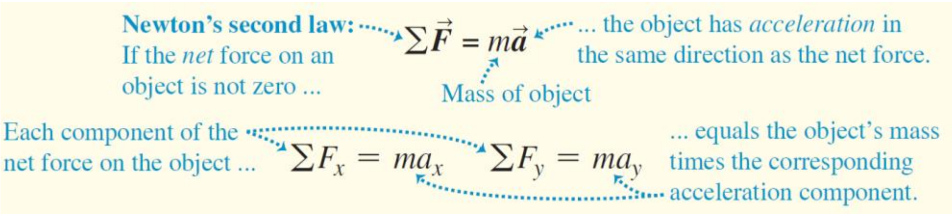
EVALUATE

The *tension* at any point in the rope is the magnitude of theforce that acts at that point. For this weightless rope, the tension $T_{\text{G on R}}$at the lower end has the same value as the tension $T_{\text{C on R}}$ at the upperend. For such an ideal weightless rope, the tension has the same valueat any point along the rope's length.

KEYCONCEPT. The sum of all the external forces on an object inequilibrium is zero. The tension has the same value at either end of arope or string of negligible mass.

5.2 Using Newton's Second Law: Dynamics of Particles

We are now ready to discuss *dynamics* problems. In these problems, we apply Newton's second law to objects on which the net force is *not* zero. These objects are *not* in equilibrium and hence are accelerating:



(5.2)

CAUTION! $m\vec{a}$ **doesn't belong in free-body diagrams.** Remember that the quantity $m\vec{a}$ is the *result* of forces acting on an object, *not* a force itself. When you draw the free-body diagram for an accelerating object (like the fruit in Fig. 5.2a), *never* include the "$m\vec{a}$ force" because *there is no such force* (Fig. 5.2c). Sometimes we draw the acceleration vector $\vec{a}$ *alongside* a free-body diagram, as in Fig. 5.2b. But we *never* draw the acceleration vector with its tail touching the object (a position reserved exclusively for forces that act on the object).

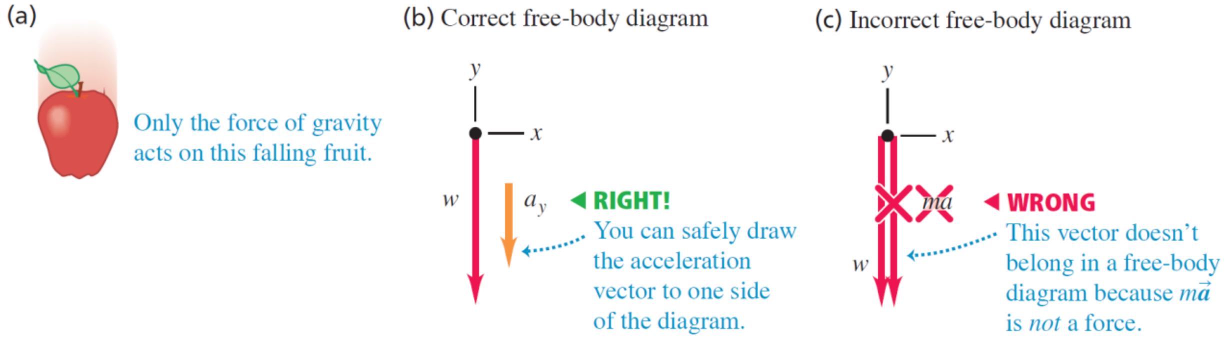


Figure 5.2 - Correct and incorrect free-body diagrams for a falling object

The following problem-solving strategy is very similar to Problem-Solving Strategy 5.1 for equilibrium problems in Section 5.1. Study it carefully, watch how we apply it in our examples, and use it when you tackle the end-of-chapter problems. You can use this strategy to solve any dynamics problem.

PROBLEM-SOLVING STRATEGY
5.2 Newton's Second Law: Dynamics of Particles

IDENTIFY *the relevant concepts:*

- Use Newton's second law, Eqs. (5.2), for any problem that involves forces acting on an acceleratingobject.
- Identify the **target variable** – usually an acceleration or a force. If the target variable is something else, you'll need to select another concept to use. For example, suppose the target variable is how fast a sled is moving when it reaches the bottom of a hill. Newton's second law will let you find the sled's acceleration; you'll then use the constant-acceleration relationshipsto find velocity from acceleration.

SET UP *the problem*:

- Draw a simple sketch of the situation that shows each moving object. For each object, draw a free-body diagram that shows all the forces acting *on* the object. [The sums in Eqs. (5.2) include the forces that act on the object, *not* the forces that it exerts on anything else.] Make sure you can answer the question “What other object is applying this force?” for each force in your diagram. Never include the quantity $m\vec{a}$ in your free-body diagram;it’s not a force!

- Label each force with an algebraic symbol for the force’s *magnitude*. Usually, one of the forces will be the object’s weight; it’susually best to label this as $w = mg$.
- Choose your x- and y-coordinate axes for each object, and show them in its free-body diagram. Indicate the positive direction for each axis. If you know the direction of the acceleration, it usually simplifies things to take one positive axis along that direction. If your problem involves two or more objects that accelerate in different directions, you can use a different set of axes for eachobject.
- In addition to Newton’s second law, $\sum \vec{F} = m\vec{a}$, identify any other equations you might need. For example, you might need one or more of the equations for motion with constant acceleration. If more than one object is involved, there may be relationships among their motions; for example, they may be connected by a rope. Express any such relationships as equations relating the accelerations of the various objects.

EXECUTE *the solution*:

- For each object, determine the components of the forces along each of the object’s coordinate axes. When you represent a force in terms of its components, draw a wiggly line through the originalforce vector to remind you not to include it twice.
- List all of the known and unknown quantities. In your list, identifythe target variable or variables.
- For each object, write a separate equation for each component of Newton’s second law, as in Eqs. (5.2). Write any additional equations that you identified in step 4 of “Set Up.” (You need as manyequations as there are target variables).
- Do the easy part—the math! Solve the equations to find the targetvariable(s).

EVALUATE*your answer:*

Does your answer have the correct units? (When appropriate, use the conversion $1\ \text{N} = 1\ \text{kg} \cdot \text{m/s}^2$). Does it have the correct algebraic sign? When possible, consider particular values or extreme cases of quantities and compare the results with your intuitive expectations. Ask, “Does this result make sense?”

EXAMPLE 5.2

An iceboat is at rest on a frictionless horizontal surface. Due to the blowing wind, 4.0 s after the iceboat is released, it is moving to the right at 6.0 m/s (about 22 km/h). What constant horizontal force F_W does the wind exert on the iceboat? The combined mass of iceboat and rider is 200 kg.

IDENTIFY and SET UP

Our target variable is one of the forces (F_W) acting on the accelerating iceboat, so we need to use Newton’s second law (Fig.5.3). The forces acting on the iceboat and rider (considered as a unit) are the weight w, the normal force n exerted by the surface, and the horizontal force F_W. Figure 5.3b shows the free-body diagram. The net force and hence the acceleration are to the right, so we chose the positive x-axis in this direction. The acceleration isn’t given; we’ll need to find it. Since the wind is assumed to exert a constant force,

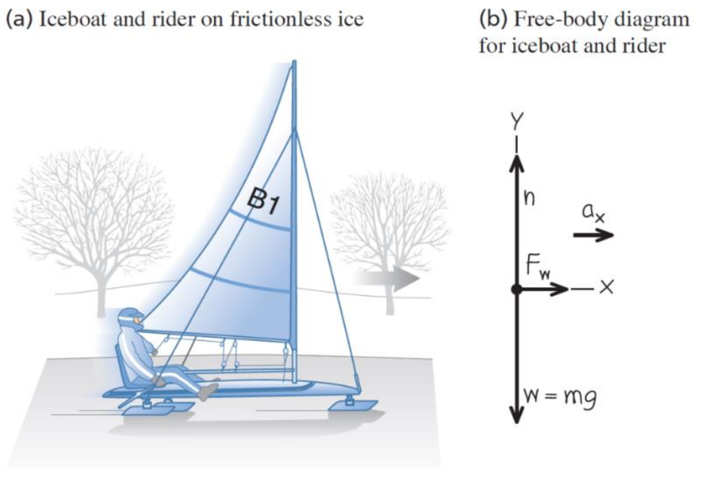


Figure 5.3

the resulting acceleration is constant and we can use one of the constant-acceleration formulas from Section 2.4.

The iceboat starts at rest (its initial x-velocity is $\upsilon_{0x} = 0$) and it attains an x-velocity $\upsilon_x = 6.0$ m/s after an elapsed time $t = 4.0$ s. To relate the x-acceleration a_x to these quantities we use $\upsilon_x = \upsilon_{0x} + a_x t$. There is no vertical acceleration, so we expect that the normal force on the iceboat is equal in magnitude to the iceboat's weight.

EXECUTE

The known quantities are the mass $m = 200$ kg, the initial and final x-velocities $\upsilon_{0x} = 0$ and $\upsilon_x = 6.0$ m/s, and the elapsed time $t = 4.0$ s. There are three unknown quantities: the acceleration a_x, the normal force n, and the horizontal force F_W. Hence we need three equations.

The first two equations are the x- and y-equations for Newton's second law, Eqs. (5.2). The force F_W is in the positive x-direction, while the forces n and $w = mg$ are in the positive and negative y-directions, respectively. Hence we have

$$\sum F_x = F_W = ma_x$$
$$\sum F_y = n + (-mg) = 0 \quad \text{so} \quad n = mg\,.$$

The third equation is equation for constant acceleration:

$$\upsilon_x = \upsilon_{0x} + a_x t\,.$$

To find F_W, we first solve this third equation for a_x and then substitute the result into the $\sum F_x$ equation:

$$a_x = \frac{\upsilon_x - \upsilon_{0x}}{t} = \frac{6.0\ \text{m/s} - 0}{4.0\ \text{s}} = 1.5\ \text{m/s}^2$$
$$F_W = ma_x = (200\ \text{kg})(1.5\ \text{m/s}^2) = 300\ \text{kg}\cdot\text{m/s}^2\,.$$

Since $1\ \text{kg}\cdot\text{m/s}^2 = 1\ \text{N}$, the final answer is

$$F_W = 300\ \text{N}\,.$$

EVALUATE

Our answers for F_W and n have the correct units for a force, and (as expected) the magnitude n of the normal force is equal to mg. Does it seem reasonable that the force F_W is substantially *less* than the weight of the boat, mg?

KEYCONCEPT. For problems in which an object is accelerating, it's usually best to choose one positive axis to be in the direction of the acceleration.

5.3 Friction Forces

We've seen several problems in which an object rests or slides on a surface that exerts forces on the object. Whenever two objects interact by direct contact (touching) of their surfaces, we describe the interaction in terms of *contact forces*. The normal force is one example of a contact force; in this section we'll look in detail at another contact force, the force of friction.

Friction is important in many aspects of everyday life. The oil in a car engine minimizes friction between moving parts, but without friction between the tires and the road we couldn't drive or turn the car. Air drag—the friction force exerted by the air on an object moving through it—decreases automotive fuel economy but makes parachutes work. Without friction, nails would pull out and most forms of animal locomotion would be impossible (**Fig. 5.4**).

Figure 5.4 - There is friction between the feet of this caterpillar (the larval stage of a butterfly of the family Papilionidae) and the surfaces over which it walks. Without friction, the caterpillar could not move forward or climb over obstacles

Kinetic and Static Friction

When you try to slide a heavy box of books across the floor, the box doesn't move at all unless you push with a certain minimum force. Once the box starts moving, you can usually keep it moving with less force than you needed to get it started. If you take some of the books out, you need less force to get it started or keep it moving. What can we say in general about this behavior?

First, when an object rests or slides on a surface, we can think of the surface as exerting a single contact force on the object, with force components perpendicular and parallel to the surface (**Fig. 5.5**). The perpendicular component vector is the normal force, denoted by $\vec{n}$. The component vector parallel to the surface (and perpendicular to $\vec{n}$) is the **friction force**, denoted by $\vec{f}$. If the surface is frictionless, then $\vec{f}$ is zero but there is still a normal force. (Frictionless surfaces are an unattainable idealization, like a massless rope. But we can approximate a surface as frictionless if the effects of friction are negligibly small). The direction of the friction force is always such as to oppose relative motion of the two surfaces.

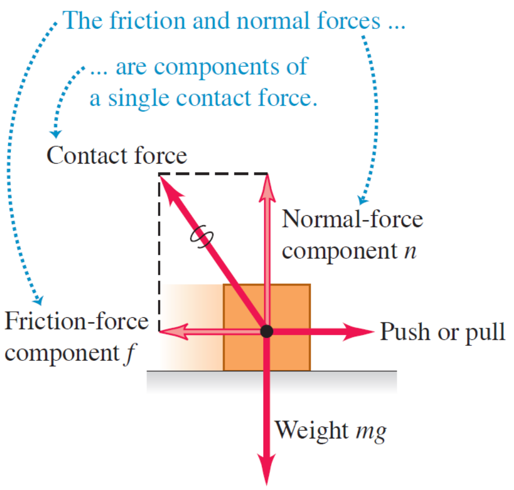


Figure 5.5 - When a block is pushed or pulled over a surface, the surface exerts a contact force on it

The kind of friction that acts when an object slides over a surface is called a **kinetic friction force** $\vec{f}_k$. The adjective "kinetic" and the subscript "k" remind us that the two surfaces are moving relative to each other. The *magnitude* of the kinetic friction force usually increases when the normal force increases. This is why it takes more force to slide a full box of books across the floor than an empty one. Automotive brakes use the same principle: The harder the brake pads are squeezed against the rotating brake discs, the greater the braking effect. In many cases the magnitude of the kinetic friction force f_k is found experimentally to be approximately *proportional* to the magnitude n of the normal force:

Magnitude of kinetic friction force · Coefficient of kinetic friction · Magnitude of normal force

$$f_k = \mu_k n \qquad (5.3)$$

Here μ_k (pronounced "mu-sub-k") is a constant called the **coefficient of kinetic friction**. The more slippery the surface, the smaller this coefficient. Because it is a quotient of two force magnitudes, μ_k is a pure number without units.

CAUTION! **Friction and normal forces are always perpendicular.** Remember that Eq. (5.3) is *not* a vector equation because $\vec{f}_k$ and $\vec{n}$ are always perpendicular. Rather, it is a scalar relationship between the magnitudes of the two forces.

Equation (5.3) is only an approximate representation of a complex phenomenon. On a microscopic level, friction and normal forces result from the intermolecular forces (electrical in nature) between two rough surfaces at points where they come into contact (**Fig. 5.6**). As a box slides over the floor, bonds between the two surfaces form and break, and the total number of such bonds varies. Hence the kinetic friction force is not perfectly constant. Smoothing the surfaces can actually increase friction, since more molecules can interact and bond; bringing two smooth surfaces of the same metal together can cause a "cold weld." Lubricating oils work because an oil film between two surfaces (such as the pistons and cylinder walls in a car engine) prevents them from coming into actual contact.

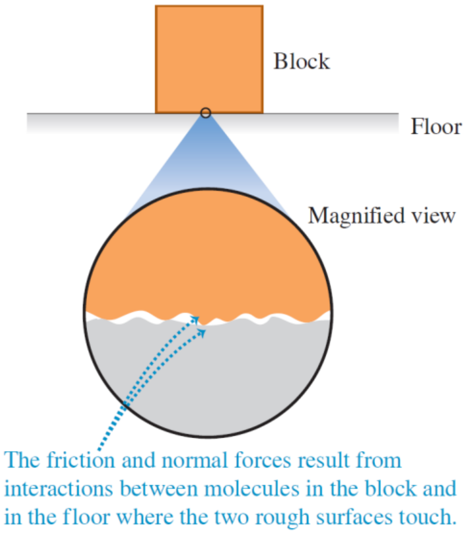


Figure 5.6 - A microscopic view of the friction and normal forces

Table 5.1 - Approximate Coefficients of Friction

Materials	**Coefficient of Static Friction,** μ_s	**Coefficient of Kinetic Friction,** μ_k
Steel on steel	0.74	0.57
Aluminum on steel	0.61	0.47
Copper on steel	0.53	0.36
Brass on steel	0.51	0.44
Zinc on cast iron	0.85	0.21
Copper on cast iron	1.05	0.29
Glass on glass	0.94	0.40
Copper on glass	0.68	0.53
Teflon on Teflon	0.04	0.04
Teflon on steel	0.04	0.04
Rubber on concrete (dry)	1.0	0.8
Rubber on concrete (wet)	0.30	0.25

Table 5.1 lists some representative values of μ_k. Although these values are givenwith two significant figures, they are only approximate, since friction forces can alsodepend on the speed of the object relative to the surface. For now we'll ignore this effectand assume that μ_k and f_k are independent of speed, in order to concentrate onthe simplest cases. Table 5.1 also lists coefficients of static friction; we'll define theseshortly.

Friction forces may also act when there is *no* relative motion. If you try to slide a boxacross the floor, the box may not move at all because the floor exerts an equal and oppositefriction force on the box. This is called a **static friction force** $\vec{f}_s$.

In **Fig. 5.7a**, the box is at rest, in equilibrium, under the action of its weight $\vec{w}$ and the upward normal force $\vec{n}$. The normal force is equal in magnitude to the weight ($n = w$) and is exerted on the box by the floor. Now we tie a rope to the box (Fig. 5.7b) and gradually increase the tension T in the rope. At first the box remains at rest because the force of static friction f_s fs also increases and stays equal in magnitude to T.

At some point T becomes greater than the maximum static friction force f_s the surface can exert. Then the box "breaks loose" and starts to slide. Figure 5.7c shows the forces when T is at this critical value. For a given pair of surfaces the maximum value of f_s depends on the normal force. Experiment shows that in many cases this maximum value, called $(f_s)_{max}$, is approximately *proportional* to n; we call the proportionality factor μ_s the **coefficient of static friction**. Table 5.1 lists some representative values of μ_s. In a particular situation, the actual force of static friction can have any magnitude between zero (when there is no other force parallel to the surface) and a maximum value given by $\mu_s n$:

$$\underbrace{f_s}_{\text{Magnitude of static friction force}} \le \underbrace{(f_s)_{max}}_{\text{Maximum static friction force}} = \underbrace{\mu_s}_{\text{Coefficient of static friction}}\underbrace{n}_{\text{Magnitude of normal force}} \qquad (5.4)$$

Like Eq. (5.3), this is a relationship between magnitudes, *not* a vector relationship. The equality sign holds only when the applied force T has reached the critical value at which motion is about to start (Fig. 5.7c). When T is less than this value (Fig. 5.6b), the inequality sign holds. In that case we have to use the equilibrium conditions ($\sum \vec{F} = 0$) to find f_s. If there is no applied force ($T = 0$) as in Fig. 5.7a, then there is no static friction force either ($f_s = 0$).

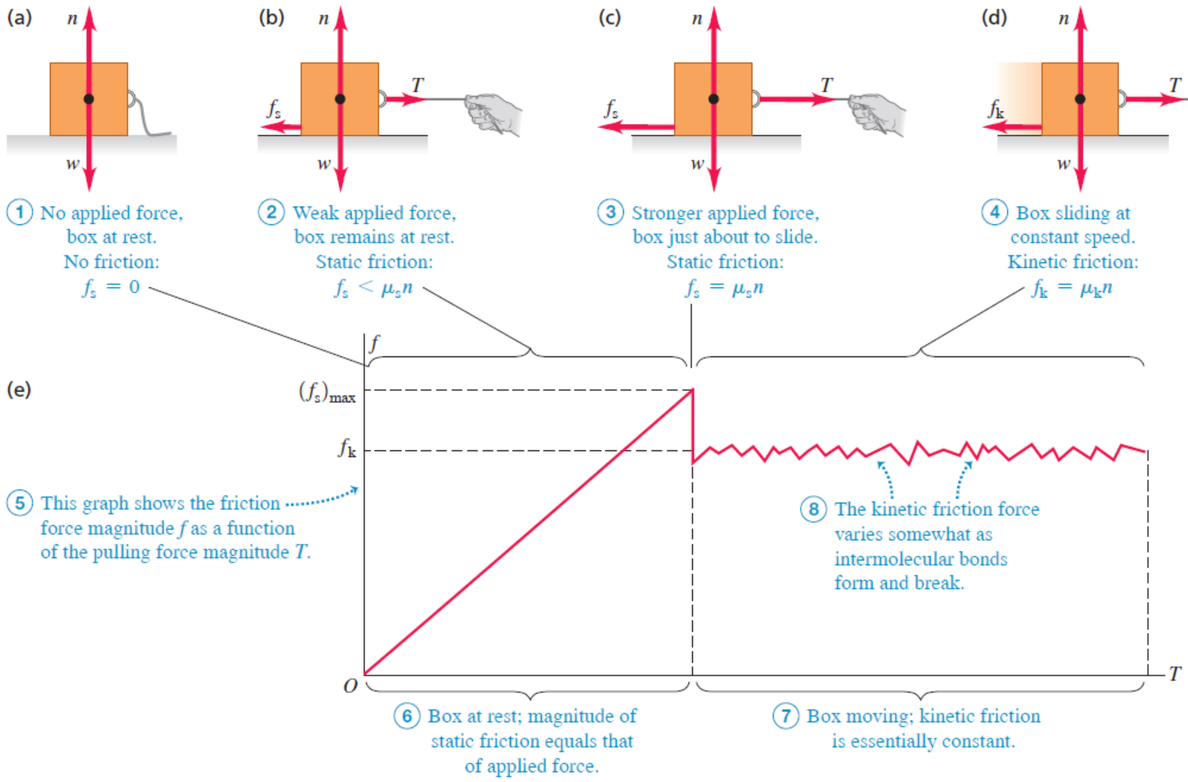


Figure 5.7 - When there is no relative motion, the magnitude of the static friction force f_s is less than or equal to $\mu_s n$. When there is relative motion, the magnitude of the kinetic friction force f_k equals $\mu_k n$

As soon as the box starts to slide (Fig. 5.6d), the friction force usually *decreases* (Fig. 5.7e); it's easier to keep the box moving than to start it moving. Hence the coefficient of kinetic friction is usually *less* than the coefficient of static friction for any given pair of surfaces, as Table 5.1 shows.

In some situations the surfaces will alternately stick (static friction) and slip (kinetic friction). This is what causes the horrible sound made by chalk held at the wrong angle on a blackboard and the shriek of tires sliding on asphalt pavement. A more positive example is the motion of a violin bow against the string.

In the linear air tracks used in physics laboratories, gliders move with very little friction because they are supported on a layer of air. The friction force is velocity dependent, but at typical speeds the effective coefficient of friction is of the order of 0.001.

APPLICATION. Static Friction and Windshield Wipers
The squeak of windshield wipers on dry glass is a stick-slip phenomenon. The moving wiper blade sticks to the glass momentarily, then slides when the force applied to the blade by the wiper motor overcomes the maximum force of static friction. When the glass is wet from rain or windshield cleaning solution, friction is reduced and the wiper blade doesn't stick.

Rolling Friction

It's a lot easier to move a loaded filing cabinet across a horizontal floor by using a cartwith wheels than by sliding it.

How much easier? We can define a **coefficient of rollingfriction** μ_r, which is the horizontal force needed for constant speed on a flat surface dividedby the upward normal force exerted by the surface. Transportation engineers call μ_r the *tractive resistance*. Typical values of μ_r are 0.002 to 0.003 for steel wheels on steelrails and 0.01 to 0.02 for rubber tires on concrete. These values show one reason trains aregenerally much more fuel efficient than trucks.

Fluid Resistance and Terminal Speed

Sticking your hand out the window of a fast-moving car will convince you of the existence of **fluid resistance**, the force that a fluid (a gas or liquid) exerts on an object moving through it. The moving object exerts a force on the fluid to push it out of the way. By Newton's third law, the fluid pushes back on the object with an equal and opposite force.

The *direction* of the fluid resistance force acting on an object is always opposite the direction of the object's velocity relative to the fluid. The *magnitude* of the fluid resistance force usually increases with the speed of the object through the fluid. This is very different from the kinetic friction force between two surfaces in contact, which we can usually regard as independent of speed.

For small objects moving at very low speeds, the magnitude f of the fluid resistance force is approximately proportional to the object's speed υ:

$$f = k\upsilon \quad \text{(fluid resistance at low speed)}, \qquad (5.5)$$

where k is a proportionality constant that depends on the shape and size of the object and the properties of the fluid. Equation (5.5) is appropriate for dust particles falling in air or a ball bearing falling in oil. For larger objects moving through air at the speed of a tossed tennis ball or faster, the resisting force is approximately proportional to υ^2 rather than to υ.

It is then called air drag or simply drag. Airplanes, falling raindrops, and bicyclists all experience air drag. In this case we replace Eq. (5.5) by

$$f = D\upsilon^2 \quad \text{(fluid resistance at high speed)}. \qquad (5.6)$$

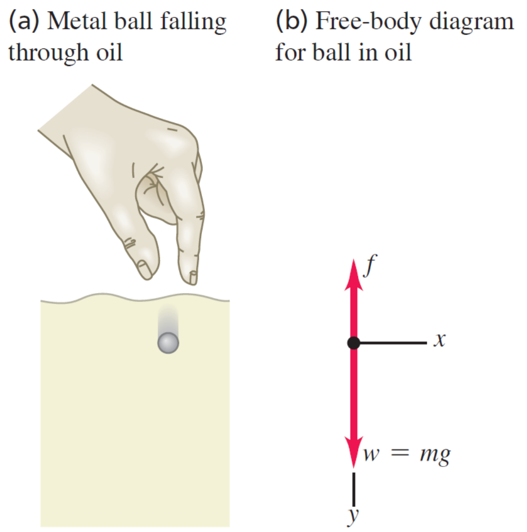


Figure 5.8 - Motion with fluid resistance

Because of the υ^2 dependence, air drag increases rapidly with increasing speed. The air drag on a typical car is negligible at low speeds but comparable to or greater than rolling resistance at highway speeds. The value of D depends on the shape and size of the object and on the density of the air. You should verify that the units of the constant k in Eq. (5.5) are $\mathrm{N \cdot s/m}$ or kg/s, and that the units of the constant D in Eq. (5.6) are $\mathrm{N \cdot s^2/m^2}$ or kg/m.

Because of the effects of fluid resistance, an object falling in a fluid does *not* have a constant acceleration. To describe its motion, we can't use the constant-acceleration relationships; instead, we have to start over with Newton's second law. As an example, suppose you drop a metal ball at the surface of a bucket of oil and let it fall to the bottom (**Fig. 5.8**). The fluid resistance force in this situation is given by Eq. (5.5). What are the acceleration, velocity, and position of the metal ball as functions of time?

Figure 5.8b shows the free-body diagram. We take the positive *y*-direction to be downward and neglect any force associated with buoyancy in the oil. Since the ball is moving downward, its speed υ is equal to its *y*-velocity υ_y and the fluid resistance force is in the –*y*-direction. There are no *x*-components, so Newton's second law gives

$$\sum F_y = mg + (-k\upsilon_y) = ma_y \,. \tag{5.7}$$

When the ball first starts to move, $\upsilon_y = 0$, the resisting force is zero and the initial acceleration is $a_y = g$. As the speed increases, the resisting force also increases, until finally it is equal in magnitude to the weight. At this time $mg - k\upsilon_y = 0$, the acceleration is zero, and there is no further increase in speed. The final speed υ_t, called the **terminal speed**, is given by $mg - k\upsilon_t = 0$, or

$$\upsilon_t = \frac{mg}{k} \quad \text{(terminal speed, fluid resistance } f = k\upsilon) \,. \tag{5.8}$$

Figure 5.9 shows how the acceleration, velocity, and position vary with time. As time goes by, the acceleration approaches zero and the velocity approaches υ_t (remember that we chose the positive *y*-direction to be down). The slope of the graph of *y* versus *t* becomes constant as the velocity becomes constant.

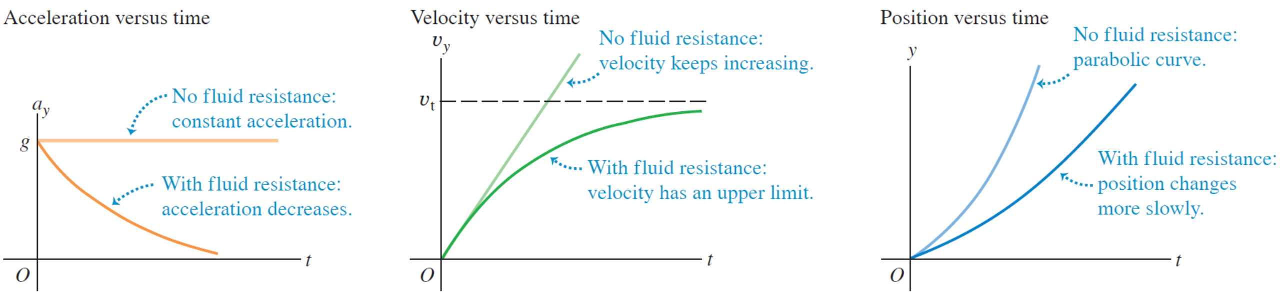


Figure 5.9 - Graphs of the motion of an object falling without fluid resistance and with fluid resistance proportional to the speed

To see how the graphs in Fig. 5.9 are derived, we must find the relationship between velocity and time during the interval before the terminal speed is reached. We go back to Newton's second law for the falling ball, Eq. (5.7), which we rewrite with $a_y = d\upsilon_y/dt$:

$$m\frac{d\upsilon_y}{dt} = mg - k\upsilon_y \,.$$

After rearranging terms and replacing mg/k by υ_t, we integrate both sides, noting that $\upsilon_y = 0$ when $t = 0$:

$$\int_0^{\upsilon} \frac{d\upsilon_y}{\upsilon_y - \upsilon_t} = -\frac{k}{m}\int_0^t dt \,,$$

which integrates to

$$\ln\frac{\upsilon_t - \upsilon_y}{\upsilon_t} = -\frac{k}{m}t \quad \text{or} \quad 1 - \frac{\upsilon_y}{\upsilon_t} = e^{-(k/m)t} \,,$$

and finally

$$\upsilon_y = \upsilon_t\left[1 - e^{-(k/m)t}\right]. \tag{5.9}$$

Note that υ_y becomes equal to the terminal speed υ_t only in the limit that $t \to \infty$; the ball cannot attain terminal speed in any finite length of time.

The derivative of υ_y in Eq. (5.9) gives a_y as a function of time, and the integral of υ_y gives y as a function of time. We leave the derivations for you to complete; the results are

$$a_y = ge^{-(k/m)t}. \qquad (5.10)$$

$$y = \upsilon_t\left[t - \frac{m}{k}\left(1 - e^{-(k/m)t}\right)\right]. \qquad (5.11)$$

Now look again at **Fig. 5.9**, which shows graphs of these three relationships.

In deriving the terminal speed in Eq. (5.8), we assumed that the fluid resistance force is proportional to the speed. For an object falling through the air at high speeds, so that the fluid resistance is equal to $D\upsilon^2$ as in Eq. (5.6), the terminal speed is reached when $D\upsilon^2$ equals the weight *mg* (Fig. 5.10a).
You can show that the terminal speed υ_t is given by

$$\upsilon_t = \sqrt{\frac{mg}{D}} \quad \text{(terminal speed, fluid resistance } f = D\upsilon^2\text{)}. \qquad (5.12)$$

This expression for terminal speed explains why heavy objects in air tend to fall fasterthan light objects. Two objects that have the same physical size but different mass (say,a table-tennis ball and a lead ball with the same radius) have the same value of *D* butdifferent values of *m*. The more massive object has a higher terminal speed and fallsfaster. The same idea explains why a sheet of paper falls faster if you first crumple itinto a ball; the mass *m* is the same, but the smaller size makes *D* smaller (less air dragfor a given speed) and υ_t larger. Skydivers use the same principle to control their descent(Fig. 5.10b).

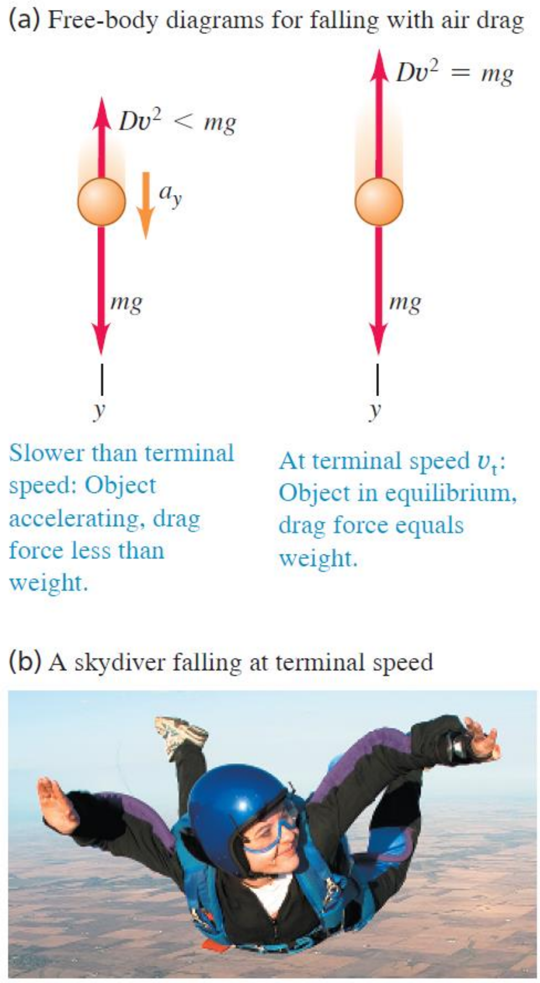


Figure 5.10 - (a) Air drag and terminalspeed. (b) By changing the positions of theirarms and legs while falling, skydivers canchange the value of the constant D in Eq.(5.6) and hence adjust the terminal speed oftheir fall [Eq. (5.12)]

Figure 5.11 shows the trajectories of a baseball with and without air drag, assuming a coefficient $D = 1.3\times10^{-3}$ kg/m (appropriate for a batted ball at sea level). Both the range of the baseball and the maximum height reached are substantially smaller than the zero-drag calculation would lead you to believe. Air drag is an important part of the game of baseball!

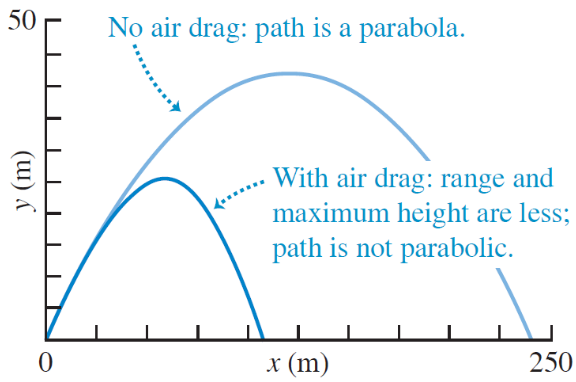


Figure 5.11 - Computer-generated trajectoriesof a baseball launched at 50 m/s at35º above the horizontal. Note that thescales are different on the horizontaland vertical axes

EXAMPLE 5.3 Terminal speed of a skydiver

For a human body falling through air in a spread-eagle position (Fig. 5.10b), the numerical value of the constant *D* in Eq. (5.6) is about 0.25 kg/m. Find the terminal speed for a 50 kg skydiver.

IDENTIFY and SET UP

This example uses the relationship among terminal speed, mass, and drag coefficient. We use Eq. (5.12) to find the target variable υ_t.

EXECUTE

We find for $m = 50$ kg :

$$\upsilon_t = \sqrt{\frac{mg}{D}} = \sqrt{\frac{(50 \text{ kg})(9.8 \text{ m/s}^2)}{0.25 \text{ kg/m}}} = 44 \text{ m/s (about 160 km/h)}.$$

EVALUATE

The terminal speed is proportional to the square root of the skydiver's mass. A skydiver with the same drag coefficient D but twice the mass would have a terminal speed $\sqrt{2} = 1.41$ times greater, or 63 m/s. (A more massive skydiver would also have more frontal area and hence a larger drag coefficient, so his terminal speed would be a bit less than 63 m/s). Even the 50 kg skydiver's terminal speed is quite high, so skydives don't last very long. A drop from 2800 m to the surface at the terminal speed takes only $(2800 \text{ m})/(44 \text{ m/s}) = 64$ s .

When the skydiver deploys the parachute, the value of D increases greatly. Hence the terminal speed of the skydiver with parachute decreases dramatically to a much lower value.

KEYCONCEPT. A falling object reaches its terminal speed when the upward force of fluid resistance equals the downward force of gravity. Depending on the object's speed, use either Eq. (5.8) or Eq. (5.12) to find the terminal speed.

5.4 Dynamics of Circular Motion

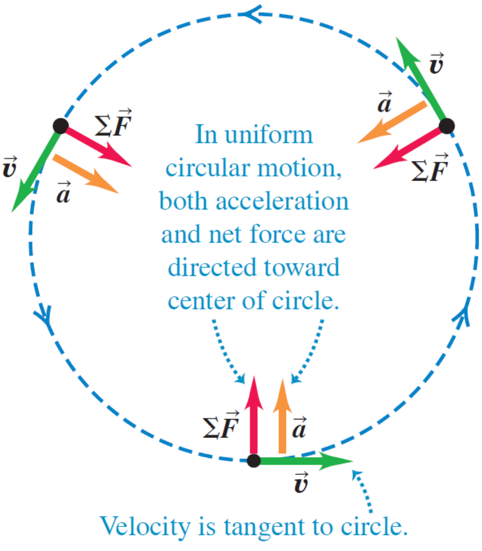


Figure 5.12 - Net force, acceleration, and velocity in uniform circular motion

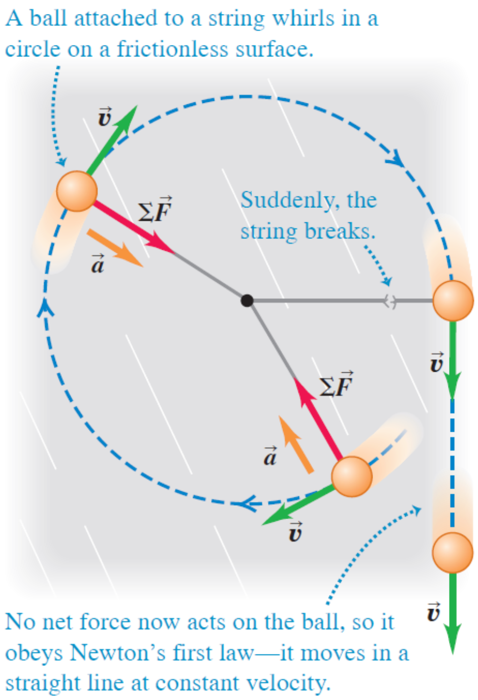


Figure 5.13 - What happens if the inward radial force suddenly ceases to act on an object in circular motion?

In case of uniform circular motion, when a particle moves in a circular path with constant speed, the particle's acceleration has a constant magnitude a_{rad} given by

Magnitude of acceleration of an object in **uniform circular motion**

$$a_{\text{rad}} = \frac{v^2}{R} \tag{5.13}$$

v: Speed of object; R: Radius of object's circular path

The subscript "rad" is a reminder that at each point the acceleration points radially inward toward the center of the circle, perpendicular to the instantaneous velocity. This acceleration is often called centripetal acceleration or radial acceleration. We can also express the centripetal acceleration a_{rad} in terms of the period T, the time for one revolution:

$$T = \frac{2\pi R}{v}. \tag{5.14}$$

In terms of the period, a_{rad} is

Magnitude of acceleration of an object in **uniform circular motion**

$$a_{\text{rad}} = \frac{4\pi^2 R}{T^2} \tag{5.15}$$

R: Radius of object's circular path; T: Period of motion

Uniform circular motion, like all other motion of a particle, is governed by Newton's second law. To make the particle accelerate toward the center of the circle, the net force $\sum \vec{F}$ on the particle must always be directed toward the center (Fig. 5.12). The magnitude of the acceleration is constant, so the magnitude F_{net} of the net force must also be constant. If the inward net force stops acting, the particle flies off in a straight line tangent to the circle (Fig. 5.13).

The magnitude of the radial acceleration is given by $a_{\text{rad}} = v^2/R$, so the magnitude F_{net} of the net force on a particle with mass m in uniform circular motion must be

$$F_{\text{net}} = ma_{\text{rad}} = m\frac{v^2}{R} \quad \text{(uniform circular motion)}. \tag{5.16}$$

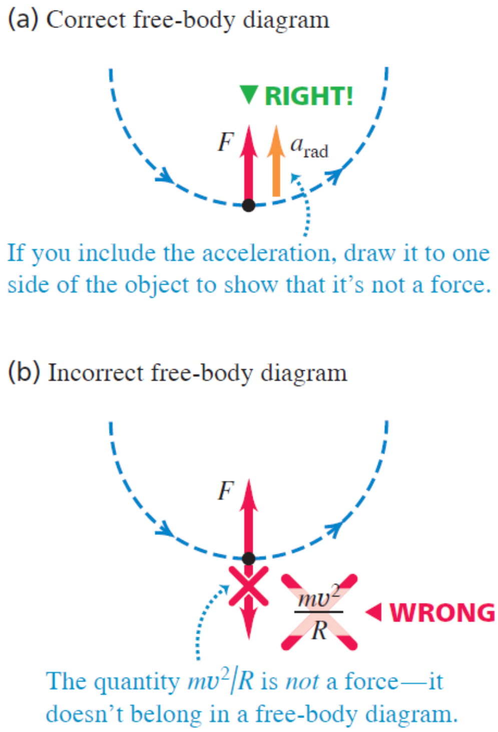


Figure 5.14 - Right and wrong ways to depictuniform circular motion

Uniform circular motion can result from any combination of forces, just so the net force $\sum \vec{F}$ is always directed toward the center of the circle and has a constant magnitude. Notethat the object need not move around a complete circle: Equation (5.16) is valid for *any*path that can be regarded as part of a circular arc.

CAUTION! **Avoid using "centrifugal force". Figure 5.14** shows a correct free-body diagram foruniform circular motion (Fig. 5.14a) and an *incorrect* diagram (Fig. 5.14b). Figure 5.14b is incorrectbecause it includes an extra outward force of magnitude $m(v^2/R)$ to "keep the object out there"or to "keep it in equilibrium". There are three reasons not to include such an outward force, called*centrifugal force* ("centrifugal" means "fleeing from the center"). First, the object does *not* "stayout there": It is in constant motion around its circular path. Because its velocity is constantly changingin direction, the object accelerates and is *not* in equilibrium. Second, if there *were* an outwardforce that balanced the inward force, the net force would be zero and the object would move in astraight line, not a circle (Fig. 5.13). Third, the quantity $m\left(\upsilon^2/R\right)$ is not a force; it corresponds tothe $m\vec{a}$ side of $\sum\vec{F}=m\vec{a}$ and does not appear in $\sum\vec{F}$ (Fig. 5.14a). It's true that when you ride in acar that goes around a circular path, you tend to slide to the outside of the turn as though there was a "centrifugal force". But what happens is that you tend to keep movingin a straight line, and the outer side of the car "runs into" you as the car turns. In an inertialframe of reference there is no such thing as "centrifugal force". We won't mention this termagain, and we strongly advise you to avoid it.

Banked Curves and the Flight of Airplanes

When an airplane is flying in a straight line at a constant speed and at a steady altitude, the airplane's weight is exactly balanced by the lift force $\vec{L}$ exerted by the air. (The upward lift force that the air exerts on the wings is a reaction to the downward push the wings exert on the air as they move through it). To make the airplane turn, the pilot banks the airplane to one side so that the lift force has a horizontal component, as Fig. 5.15 shows. (The pilot also changes the angle at which the wings "bite" into the air so that the vertical component of lift continues to balance the weight). The bank angle is related to the airplane's speed υ and the radius R of the turn by the expression: $\tan\beta=\upsilon^2/gR$. For an airplane to make a tight turn (small R) at high speed (large υ), $\tan\beta$ must be large and the required bank angle β must approach 90º.

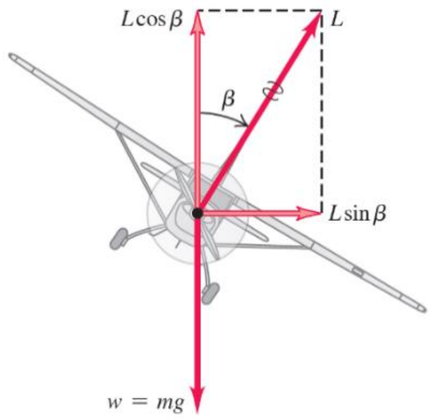


Figure 5.15 - An airplane banks to one side in order to turn in that direction. Thevertical component of the lift force $\vec{L}$ balancesthe force of gravity; the horizontal component of $\vec{L}$ causes the acceleration υ^2/R

EXAMPLE 5.4 Uniform circular motion in a vertical circle

A passenger on a funfair Ferris wheel moves in a vertical circle of radius R with constant speed υ. The seat remains upright during the motion. Find expressions for the force the seat exerts on the passenger when at the top of the circle and when at the bottom.

IDENTIFY and SET UP

The target variables are n_T, the upward normal force the seat applies to the passenger at the top of the circle, and n_B, the normal force at the bottom. We'll find these by using Newton's second law and the uniform circular motion equations.

Figure 5.16 shows the passenger's velocity and acceleration at the two positions. The acceleration always points toward the center of the circle - downward at the top of the circle and upward at the bottom of the circle. At each position the only forces acting are vertical: the upward normal force and the downward force of gravity. Hence we need only the vertical component of Newton's second law. Figures 5.16b and 5.16c show free-body diagrams for the two positions. We take the positive y-direction as upward in both cases (that is, *opposite* the direction of the acceleration at the top of the circle).

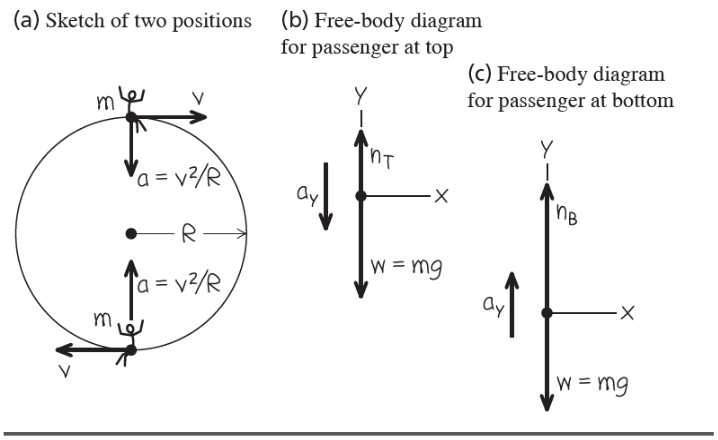


Figure 5.16 - Sketch for this problem

EXECUTE

At the top the acceleration has magnitude υ^2/R, but its verticalcomponent is negative because its direction is downward. Hence $a_y = -\upsilon^2/R$ and Newton's second law tells us that

$$\text{Top: } \sum F_y = n_T + (-mg) = -m\frac{\upsilon^2}{R} \text{ or}$$

$$n_T = mg\left(1 - \frac{\upsilon^2}{gR}\right)$$

At the bottom the acceleration is upward, so $a_y = +\upsilon^2/R$ and Newton'ssecond law says

$$\text{Bottom: } \sum F_y = n_B + (-mg) = +m\frac{\upsilon^2}{R} \text{ or}$$

$$n_B = mg\left(1 + \frac{\upsilon^2}{gR}\right)$$

EVALUATE

Our result for n_T tells us that at the top of the Ferris wheel, the upward force the seat applies to the passenger is *smaller* in magnitude than the passenger's weight $w = mg$. If the ride goes fast enough that $g - \upsilon^2/R$ becomes zero, the seat applies *no* force, and the passenger is about to become airborne. If υ becomes still larger, n_T becomes negative; this means that a *downward* force (such as from a seat belt) is needed to keep the passenger in the seat. By contrast, the normal force n_B at the bottom is always *greater* than the passenger's weight. You feel the seat pushing up on you more firmly than when you are at rest. You can see that n_T and n_B are the values of the passenger's *apparent weight* at the top and bottom of the circle.

KEYCONCEPT. Even when an object moves with varying speed along a circular path, at any point along the path the net force component toward the center of the circle equals the object's mass times its radial acceleration.

When we tie a string to an object and whirl it in a vertical circle, the analysis in Example 5.4 isn't directly applicable. The reason is that υ is *not* constant in this case; except at the top and bottom of the circle, the net force (and hence the acceleration) does *not* point toward the center of the circle (Fig. 5.17). So both $\sum \vec{F}$ and $\vec{a}$ have a component tangent to the circle, which means that the speed changes. Hence this is a case of *nonuniform*circular motion. Even worse, we can't use the constant-acceleration formulas to relate the speeds at various points because *neither* the magnitude nor the direction of the acceleration is constant. The speed relationships we need are best obtained by using the concept of energy.

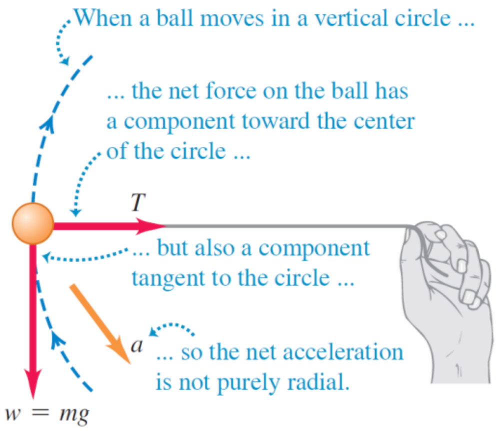


Figure 5.17 - A ball moving in a vertical circle

5.5 The Fundamental Forces of Nature

We have discussed several kinds of forces - including weight, tension, friction, fluid resistance, and the normal force - and we'll encounter others as we continue our study of physics. How many kinds of forces are there? Our best understanding is that all forces are expressions of just four distinct classes of *fundamental* forces, or interactions between particles (**Fig. 5.18**). Two are familiar in everyday experience. The other two involve interactions between subatomic particles that we cannot observe with the unaided senses.

Gravitational interactions include the familiar force of your *weight*, which results from the earth's gravitational attraction acting on you. The mutual gravitational attraction of various parts of the earth for each other holds our planet together, and likewise for the other planets (Fig. 5.18a). Newton recognized that the sun's gravitational attraction for the earth keeps our planet in its nearly circular orbit around the sun.

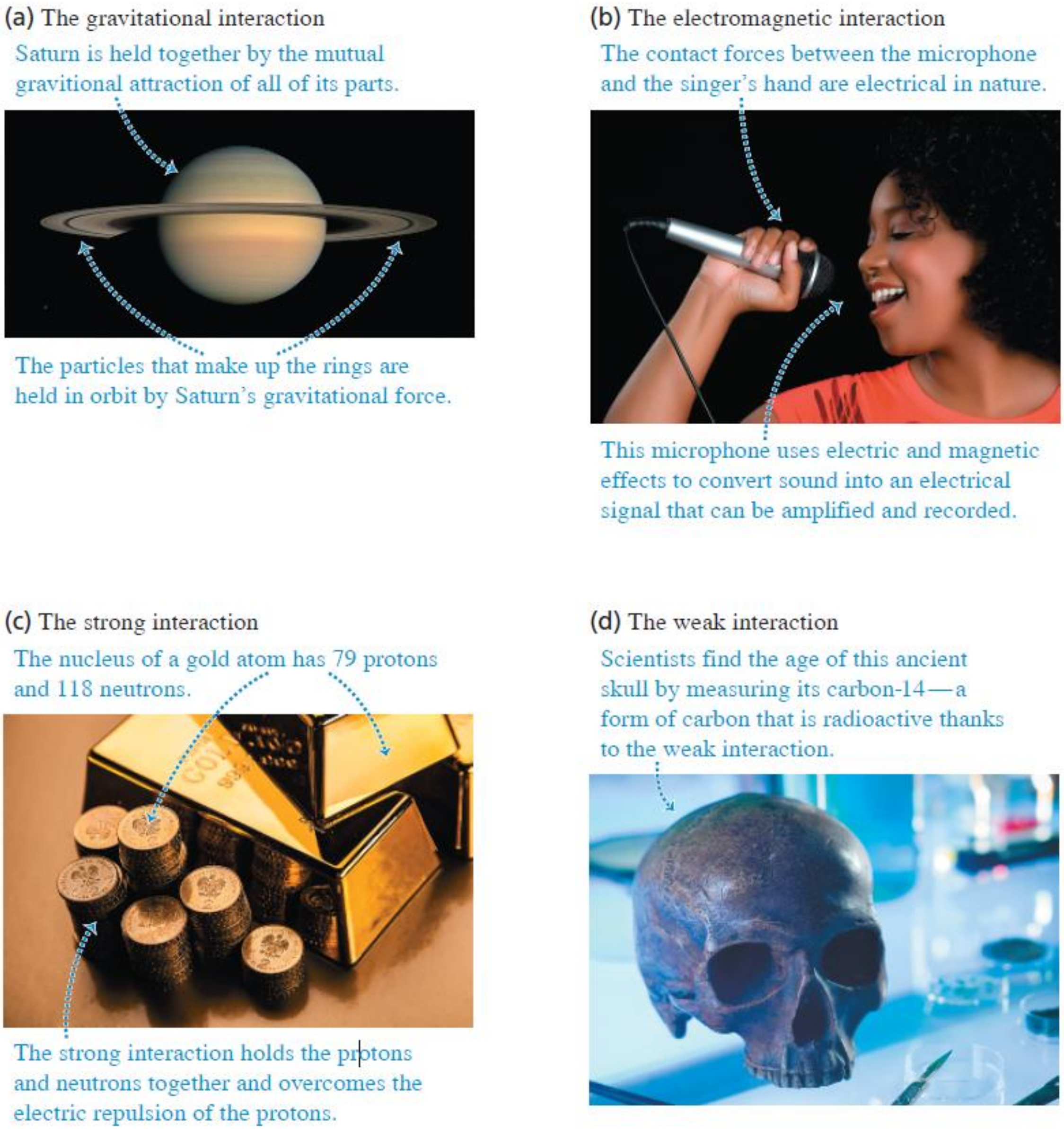


Figure 5.18 - Graphs of the motion of an object falling without fluid resistance and with fluid resistance proportional to the speed

The second familiar class of forces, **electromagnetic interactions**, includes electric and magnetic forces. If you run a comb through your hair, the comb ends up with an electric charge; you can use the electric force exerted by this charge to pick up bits of paper. All atoms contain positive and negative electric charge, so atoms and molecules can exert electric forces on one another. Contact forces, including the normal force, friction, and fluid resistance, are the result of electrical interactions between atoms on the surface of an object and atoms in its surroundings (Fig. 5.18b). *Magnetic* forces, such as those between magnets or between a magnet and a piece of iron, are actually the result of electric charges in motion. For example, an electromagnet causes magnetic interactions because electric charges move through its wires.

On the atomic or molecular scale, gravitational forces play no role because electric forces are enormously stronger: The electrical repulsion between two protons is stronger than their gravitational attraction by a factor of about 10^{35}. But in objects of astronomical size, positive and negative charges are usually present in nearly equal amounts, and the resulting electrical interactions nearly cancel out. Gravitational interactions are thus the dominant influence in the motion of planets and in the internal structure of stars.

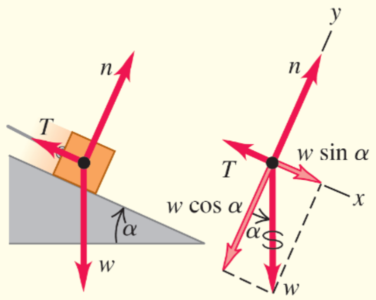
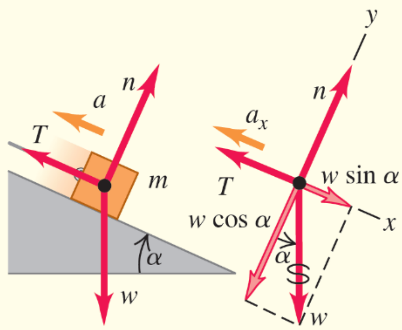
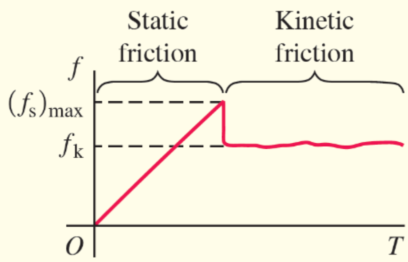
The other two classes of interactions are less familiar. One, the **strong interaction**, is responsible for holding the nucleus of an atom together (Fig. 5.18c). Nuclei contain electrically neutral neutrons and positively charged protons. The electric force between charged protons tries to push them apart; the strong attractive force between nuclear particles counteracts this repulsion and makes the nucleus stable. In this context the strong interaction is also called the *strong nuclear force*. It has much shorter range than electrical interactions, but within its range it is much stronger.

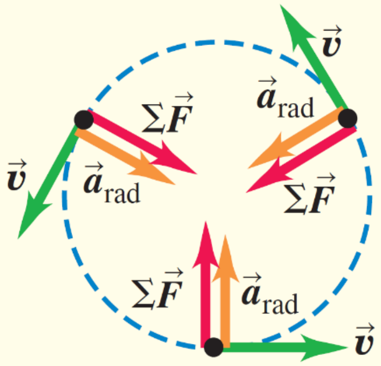
Without the strong interaction, the nuclei of atoms essential to life, such as carbon (six protons, six neutrons) and oxygen (eight protons, eight neutrons), would not exist and you would not be reading these words!

Finally, there is the **weak interaction**. Its range is so short that it plays a role only on the scale of the nucleus or smaller. The weak interaction is responsible for a common form of radioactivity called beta decay, in which a neutron in a radioactive nucleus is transformed into a proton while ejecting an electron and a nearly massless particle called an antineutrino. The weak interaction between the antineutrino and ordinary matter is so feeble that an antineutrino could easily penetrate a wall of lead a million kilometers thick!

An important application of the weak interaction is *radiocarbon dating*, a technique that enables scientists to determine the ages of many biological specimens (Fig. 5.18d). Naturally occurring carbon includes atoms of both carbon-12 (with six protons and six neutrons in the nucleus) and carbon-14 (with two additional neutrons). Living organisms take in carbon atoms of both kinds from their environment but stop doing so when they die. The weak interaction makes carbon-14 nuclei unstable—one of the neutrons changes to a proton, an electron, and an antineutrino—and these nuclei decay at a known rate. Bymeasuring the fraction of carbon-14 that is left in an organism’s remains, scientists candetermine how long ago the organism died.

In the 1960s physicists developed a theory that described the electromagnetic andweak interactions as aspects of a single *electroweak* interaction. This theory has passedevery experimental test to which it has been put. Encouraged by this success, physicistshave made similar attempts to describe the strong, electromagnetic, and weak interactionsin terms of a single *grand unified theory* (GUT) and have taken steps toward apossible unification of all interactions into a *theory of everything* (TOE). Such theoriesare still speculative, and there are many unanswered questions in this very active field ofcurrent research.

CHAPTER 5: SUMMARY		
Using Newton's first law: When an object is in equilibrium in an inertial frame of reference—that is, either at rest or moving with constant velocity—the vector sum of forces acting on it must be zero (Newton's first law). Freebody diagrams are essential in identifying the forces that act on the object being considered. Newton's third law (action and reaction) is also frequently needed in equilibrium problems. The two forces in an action–reaction pair *never* act on the same object. The normal force exerted on an object by a surface is not always equal to the object's weight	Vector form: $\sum \vec{F} = 0$ Component form: $\sum F_x = 0 \quad \sum F_y = 0$	
Using Newton's second law: If the vector sum of forceson an object is *not* zero, the object accelerates. The accelerationis related to the net force by Newton's second law. Just as for equilibrium problems, free-body diagramsare essential for solving problems involving Newton's secondlaw, and the normal force exerted on an object is notalways equal to its weight	Vector form: $\sum \vec{F} = m\vec{a}$ Component form: $\sum F_x = ma_x \quad \sum F_y = ma_y$	
Friction and fluid resistance: The contact force betweentwo objects can always be represented in terms of a normalforce $\vec{n}$ perpendicular to the surface of contact and a frictionforce $\vec{f}$ parallel to the surface. When an object is sliding over the surface, the frictionforce is called *kinetic* friction. Its magnitude f_k is approximatelyequal to the normal force magnitude n multiplied bythe coefficient of kinetic friction μ_k. When an object is *not* moving relative to a surface, the friction force is called *static* friction. The *maximum*possible static friction force is approximately equal to the magnitude *n* of the normal force multiplied by thecoefficient of static friction μ_s. The *actual* static friction force may be anything from zero to this maximum value,depending on the situation. Usually μ_s is greater than μ_k for a given pair of	Magnitude of kinetic friction force: $f_k = \mu_k n$ Magnitude of static friction force: $f_s \le (f_s)_{max} = \mu_s n$	

surfaces in contact. Rolling friction is similar to kinetic friction, but the force of fluid resistance depends on the speed of anobject through a fluid.		
Forces in circular motion: In uniform circular motion,the acceleration vector is directed toward the center of thecircle. The motion is governed by Newton's second law, $\sum \vec{F} = m\vec{a}$	Acceleration in uniform circular motion: $a_{\text{rad}} = \dfrac{\upsilon^2}{R} = \dfrac{4\pi^2 R}{T^2}$	

6 WORK AND KINETIC ENERGY

Suppose you try to find the speed of an arrow that has been shot from a bow. You apply Newton's laws and all the problem-solving techniques that we've learned, but you run across a major stumbling block: After the archer releases the arrow, the bow string exerts a *varying* force that depends on the arrow's position. As a result, the simple methods that we've learned aren't enough to calculate the speed. Never fear; we aren't by any means finished with mechanics, and there are other methods for dealing with such problems.

The new method that we're about to introduce uses the ideas of *work* and *energy*. The importance of the energy idea stems from the *principle of conservation of energy*: Energy is a quantity that can be converted from one form to another but cannot be created or destroyed. In an automobile engine, chemical energy stored in the fuel is converted partially to the energy of the automobile's motion and partially to thermal energy. In a microwave oven, electromagnetic energy obtained from your power company is converted to thermal energy of the food being cooked. In these and all other processes, the *total* energy - the sum of all energy present in all different forms - remains the same. No exception has ever been found.

We'll use the energy idea throughout the rest of this book to study a tremendous range of physical phenomena. This idea will help you understand how automotive engines work, how a camera's flash unit can produce a short burst of light, and the meaning of Einstein's famous equation $E = mc^2$.

In this chapter, though, our concentration will be on mechanics. We'll learn about one important form of energy called *kinetic energy*, or energy of motion, and how it relates to the concept of *work*. We'll also consider *power*, which is the time rate of doing work. In Chapter 7 we'll expand these ideas into a deeper understanding of the concepts of energy and the conservation of energy.

6.1 Work

You'd probably agree that it's hard work to pull a heavy sofa across the room, to lift a stack of encyclopedias from the floor to a high shelf, or to push a stalled car off the road. Indeed, all of these examples agree with the everyday meaning of "work" - any activity that requires muscular or mental effort.

In physics, work has a much more precise definition. By making use of this definition we'll find that in any motion, no matter how complicated, the total work done on a particle by all forces that act on it equals the change in its *kinetic energy* - a quantity that's related to the particle's mass and speed. This relationship holds even when the forces acting on the particle aren't constant. The ideas of work and kinetic energy enable us to solve problems in mechanics that we could not have attempted before.

Figure 6.1 - These people are doing work as they push on the car because they exert a force on the car as it moves

In this section we'll see how work is defined and how to calculate work in a variety of situations involving *constant* forces. Later in this chapter we'll relate work and kinetic energy, and then apply these ideas to problems in which the forces are *not* constant.

The three examples of work described above - pulling a sofa, lifting encyclopedias, and pushing a car - have something in common. In each case you do work by exerting a *force* on an object while that object *moves* from one place to another - that is, undergoes a *displacement* (**Fig. 6.1**).

You do more work if the force is greater (you push harder on the car) or if the displacement is greater (you push the car farther down the road).

The physicist's definition of work is based on these observations. Consider an object that undergoes a displacement of magnitude s along a straight line. (For now, we'll assume that any object we discuss can be treated as a particle so that we can ignore any rotation or changes in shape of the object). While the object moves, a constant force $\vec{F}$ acts on it in the same direction as the displacement $\vec{s}$

(**Fig. 6.2**). We define the **work** W done by this constant force under these circumstances as the product of the force magnitude F and the displacement magnitude s:

$$W = Fs \quad \text{(constant force in direction of straight-line displacement)}. \tag{6.1}$$

The work done on the object is greater if either the force F or the displacement s is greater, in agreement with our observations above.

CAUTION! **Work = *W*, weight = *w*.** Don't confuse uppercase W (work) with lowercase w (weight). Though the symbols are similar, work and weight are different quantities.

The SI unit of work is the **joule** (abbreviated J, pronounced "jool," and named in honor of the 19th-century English physicist James Prescott Joule). From Eq. (6.1) we see that in any system of units, the unit of work is the unit of force multiplied by the unit of distance. In SI units the unit of force is the newton and the unit of distance is the meter, so 1 joule is equivalent to 1 *newton-meter* $(\mathrm{N \cdot m})$:

$$1\ \text{joule} = (1\ \text{newton})(1\ \text{meter}) \quad \text{or} \quad 1\ \mathrm{J} = 1\ \mathrm{N \cdot m}.$$

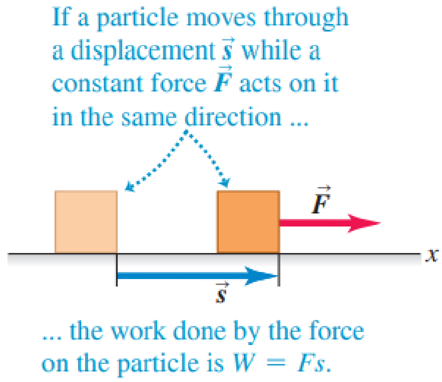


Figure 6.2 - The work done by a constant force acting in the same direction as the displacement

If you lift an object with a weight of 1 N (about the weight of a medium-sized apple) a distance of 1 m at a constant speed, you exert a 1 N force on the object in the same direction as its 1 m displacement and so do 1 J of work on it.

As an illustration of Eq. (6.1), think of a person pushing a stalled car. If he pushes the car through a displacement $\vec{s}$ with a constant force $\vec{F}$ in the direction of motion, the amount of work he does on the car is given by Eq. (6.1): $W = Fs$. But what if the person pushes at an angle ϕ to the car's displacement (Fig. 6.3)? Then $\vec{F}$ has a component $F_{\parallel} = F\cos\phi$ in the direction of the displacement $\vec{s}$ and a component $F_{\perp} = F\sin\phi$ that acts perpendicular to $\vec{s}$. (Other forces must act on the car so that it moves along $\vec{s}$, notin the direction of $\vec{F}$. We're interested in only the work that the person does, however, so we'll consider only the force he exerts). Only the parallel component $F_{\parallel}$ is effective in moving the car, so we define the work as the product of this force component and the magnitude of the displacement. Hence $W = F_{\parallel}s = (F\cos\phi)s$, or

Work done on a particle by **constant force $\vec{F}$** during **straight-line displacement $\vec{s}$**

$$W = Fs\cos\phi \tag{6.2}$$

Magnitude of $\vec{F}$; Magnitude of $\vec{s}$; Angle between $\vec{F}$ and $\vec{s}$

If $\phi = 0$, so that $\vec{F}$ and $\vec{s}$ are in the same direction, then $\cos\phi = 1$ and we are back to Eq. (6.1).

Equation (6.2) has the form of the *scalar product* of two vectors: $\vec{A}\cdot\vec{B} = AB\cos\phi$. You may want to review that definition. Hence we can write Equation (6.2) more compactly as

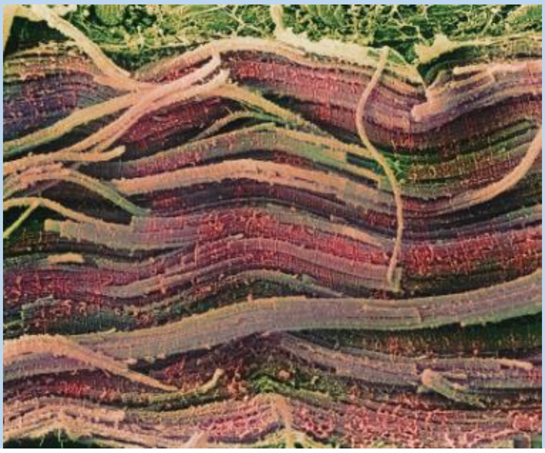
Work done on a particle by **constant force $\vec{F}$** during **straight-line displacement $\vec{s}$**

$$W = \vec{F}\cdot\vec{s} \tag{6.3}$$

Scalar product (dot product) of vectors $\vec{F}$ and $\vec{s}$

CAUTION! **Work is a scalar.** An essential point: Work is a *scalar* quantity, even though it's calculated from two vector quantities (force and displacement). A 5 N force toward the east acting on an object that moves 6 m to the east does the same amount of work as a 5 N force toward the north acting on an object that moves 6 m to the north.

Bio APPLICATION. Work and Muscle Fibers
Our ability to do work with our bodies comes from our skeletal muscles. The fiberlike cells of skeletal muscle, shown in this micrograph, can shorten, causing the muscle as a whole to contract and to exert force on the tendons to which it attaches. Muscle can exert a force of about 0.3 N per square millimeter of cross-sectional area: The greater the cross-sectional area, the more fibers the muscle has and the more force it can exert when it contracts



EXAMPLE 6.1 Work done by a constant force

(a) Steve exerts a steady force of magnitude 210 N on the stalled car in **Fig. 6.3** as he pushes it a distance of 18 m. The car also has a flat tire, so to make the car track straight Steve must push at an angle of 30° to the direction of motion. How much work does Steve do? (b) In a helpful mood, Steve pushes a second stalled car with a steady force $\vec{F} = (160\text{ N})\hat{i} - (40\text{ N})\hat{j}$. The displacement of the car is $\vec{s} = (14\text{ m})\hat{i} + (11\text{ m})\hat{j}$. How much work does Steve do in this case?

IDENTIFY and SET UP

In both parts (a) and (b), the target variable is the work W done by Steve. In each case the force is constant and the displacement is along a straight line, so we can use Eq. (6.2) or (6.3). The angle between $\vec{F}$ and $\vec{s}$ is given in part (a), so we can apply Eq. (6.2) directly. In part (b) both $\vec{F}$ and $\vec{s}$ are given in terms of components, so it's best to calculate the scalar product as follows:

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z .$$

EXECUTE

(a) From Eq. (6.2),

$$W = Fs\cos\phi = (210\text{ N})(18\text{ m})\cos 30^\circ = 3.3\times10^3\text{ J}.$$

(b) The components of $\vec{F}$ are $F_x = 160\text{ N}$ and $F_x = -40\text{ N}$, and the components of $\vec{s}$ are $x = 14\text{ m}$ and $y = 11\text{ m}$. (There are no *z*-components for either vector). Hence, using Eq. (6.3), we have

$$W = \vec{F}\cdot\vec{s} = F_x x + F_y y = (160\text{ N})(14\text{ m}) + (-40\text{ N})(11\text{ m}) = 1.8\times10^3\text{ J}.$$

EVALUATE

In each case the work that Steve does is more than 1000 J. This shows that 1 joule is a rather small amount of work.

KEYCONCEPT. To find the work W done by a constant force $\vec{F}$ acting on an object that undergoes a straight-line displacement $\vec{s}$, calculate the scalar product of these two vectors: $W = \vec{F} \cdot \vec{s}$.

Work: Positive, Negative, or Zero

In Example 6.1 the work done in pushing the cars was positive. But it's important to understand that work can also be negative or zero. This is the essential way in which work as defined in physics differs from the "everyday" definition of work. When the force has a component in the *same direction* as the displacement (ϕ between 0° and 90°), $\cos\phi$ in Eq. (6.2) is positive and the work W is *positive* (Fig. 6.4a). When the force has a component *opposite* to the displacement (ϕ between 90° and 180°), $\cos\phi$ is negative and the work is negative (Fig. 6.4b). When the force is *perpendicular* to the displacement, ϕ = 90° and the work done by the force is *zero*(Fig. 6.4c). The cases of zero work and negative work bear closer examination, so let's look at some examples.

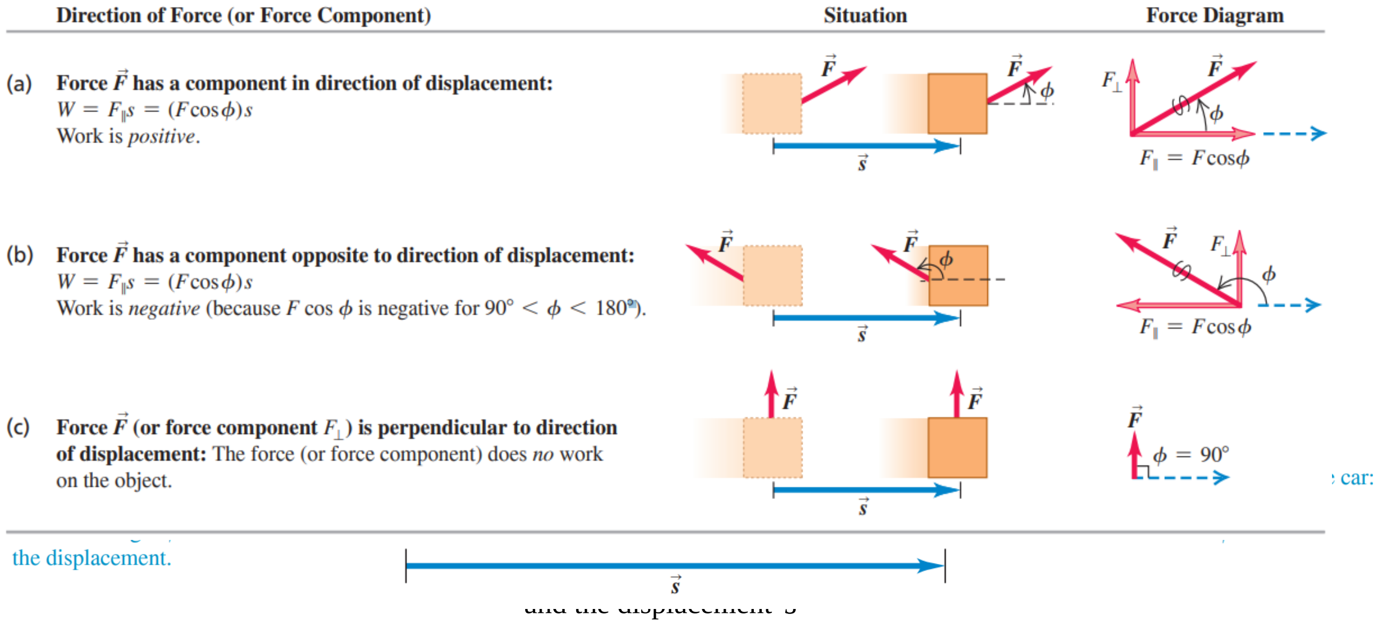


Figure 6.3 - The work done by a constant force acting at an angle to the displacement

There are many situations in which forces act but do zero work. You might think it's "hard work" to hold a barbell motionless in the air for 5 minutes (**Fig. 6.5**). But in fact, you aren't doing any work on the barbell because there is no displacement. (Holding the barbell requires you to keep the muscles of your arms contracted, and this consumes energy stored in carbohydrates and fat within your body. As these energy stores are used up, your muscles feel fatigued even though you do no work on the barbell). Even when you carry a book while you walk with constant velocity on a level floor, you do no work on the book. It has a displacement, but the (vertical) supporting force that you exert on the book has no component in the direction of the (horizontal) motion. Then ϕ = 90° in Eq. (6.2), and $\cos\phi$ = 0. When an object slides along a surface, the

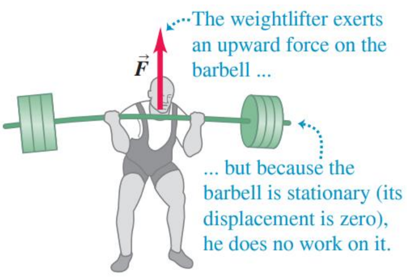


Figure 6.5 - A weightlifter does no work on a barbell as long as he holds it stationary

work done on the object by the normal force is zero; and when a ball on a string moves in uniform circular motion, the work done on the ball by the tension in the string is also zero. In both cases the work is zero because the force has no component in the direction of motion.

What does it mean to do *negative* work? The answer comes from Newton's third law of motion. When a weightlifter lowers a barbell as in **Fig. 6.6a**, his hands and the barbell move together with the same displacement $\vec{s}$. The barbell exerts a force $\vec{F}_{\text{barbell on hands}}$ on hands on his hands in the same direction as the hands' displacement, so the work done by the *barbell* on his *hands* is positive (Fig. 6.6b). But by Newton's third law the weightlifter's hands exert an equal and opposite force $\vec{F}_{\text{hands on barbell}} = -\vec{F}_{\text{barbell on hands}}$ on the barbell (Fig. 6.6c). This force, which keeps the barbell from crashing to the floor, acts opposite to the barbell's displacement. Thus the work done by his *hands* on the *barbell* is negative.

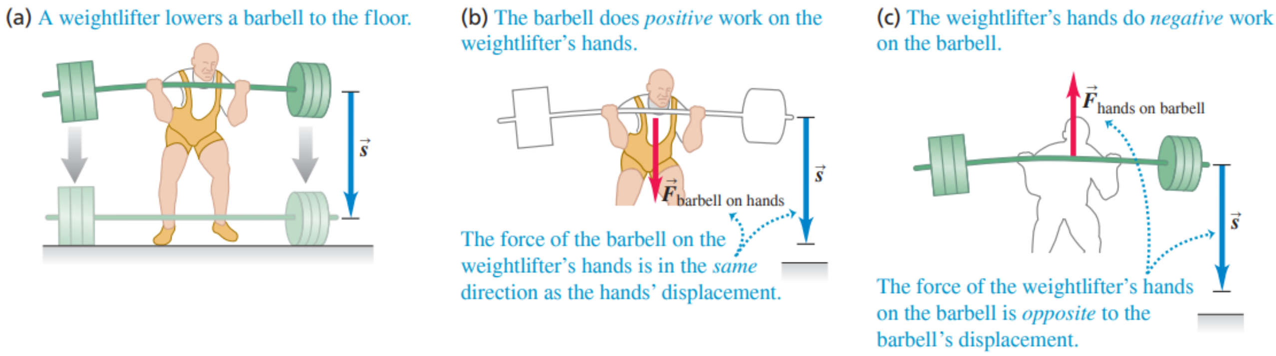


Figure 6.6 - This weightlifter's hands do negative work on a barbell as the barbell does positive work on his hands

Because the weightlifter's hands and the barbell have the same displacement, the work that his hands do on the barbell is just the negative of the work that the barbell does on his hands. In general, when one object does negative work on a second object, the second object does an equal amount of *positive* work on the first object.

As a final note, you should review Fig. 6.4 to help remember when work is positive, when it is zero, and when it is negative.

CAUTION! **Keep track of who's doing the work**. We always speak of work done on a particular object *by* a specific force. Always specify exactly what force is doing the work. When you lift a book, you exert an upward force on it and the book's displacement is upward, so the work done by the lifting force on the book is positive. But the work done by the *gravitational* force (weight) on a book being lifted is *negative* because the downward gravitational force is opposite to the upward displacement.

Total Work

How do we calculate work when several forces act on an object? One way is to use Eq. (6.2) or (6.3) to compute the work done by each separate force. Then, because work is a scalar quantity, the *total* work W_{tot} done on the object by all the forces is the algebraic sum of the quantities of work done by the individual forces. An alternative way to find the total work W_{tot} is to compute the vector sum of the forces (that is, the net force) and then use this vector sum as $\vec{F}$ in Eq. (6.2) or (6.3).

6.2 Kinetic Energy and the Work-Energy Theorem

The total work done on an object by external forces is related to the object's displacement - that is, to changes in its position. But the total work is also related to changes in the *speed* of the object. To see this, consider **Fig. 6.7**, which shows a block sliding on a frictionless table. The forces acting on the block are its weight $\vec{w}$, the normal force $\vec{n}$, and the force $\vec{F}$ exerted on it by the hand.

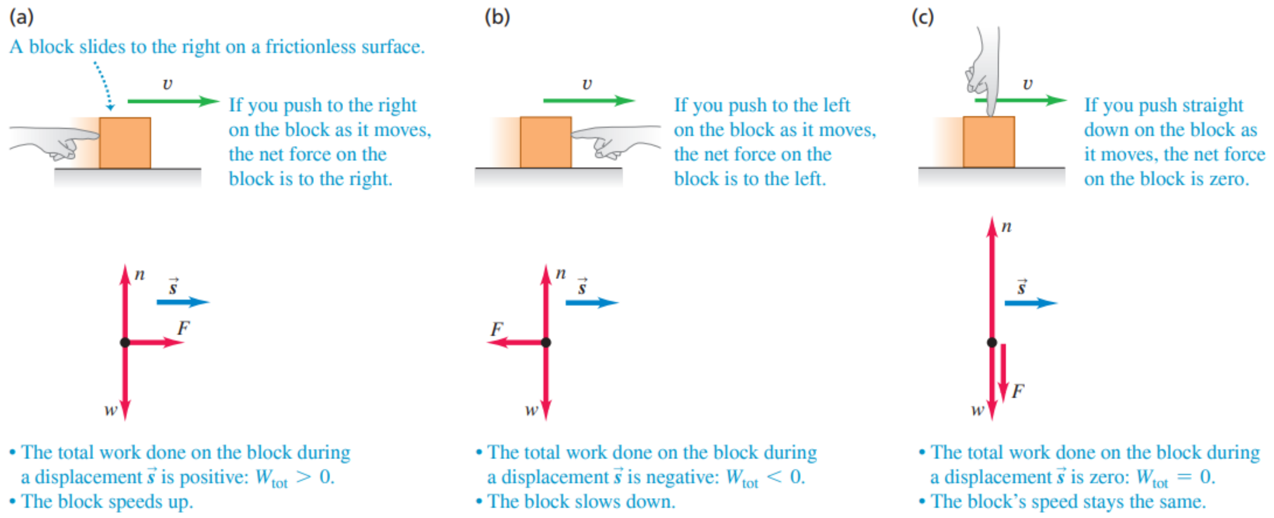


Figure 6.7 - The relationship between the total work done on an object and the change in velocity of the object

In Fig. 6.7a the net force on the block is in the direction of its motion. From Newton's second law, this means that the block speeds up; from Eq. (6.1), this also means that the total work W_{tot} done on the block is positive. The total work is *negative* in Fig. 6.7b because the net force opposes the displacement; in this case the block slows down. The net force is zero in Fig. 6.7c, so the speed of the block stays the same and the total work done on the block is zero. We can conclude that *when a particle undergoes a displacement, it speeds up if* $W_{tot} > 0$, *slows down if* $W_{tot} < 0$, *and maintains the same speed if* $W_{tot} = 0$.

Let's make this more quantitative. In **Fig. 6.8** a particle with mass m moves along the x-axis under the action of a constant net force with magnitude F that points in the positive x-direction. The particle's acceleration is constant and given by Newton's second law: $F = ma_x$. As the particle moves from point x_1 to x_2, it undergoes a displacement $s = x_2 - x_1$ and its speed changes from υ_1 to υ_2. Using a constant-acceleration equation and replacing υ_{0x} by υ_1, υ_x by υ_2, and $(x - x_0)$ by s, we have

$$\upsilon_2^2 = \upsilon_1^2 + 2a_x s,$$

$$a_x = \frac{\upsilon_2^2 - \upsilon_1^2}{2s}.$$

When we multiply this equation by m and equate ma_x to the net force F, we find

$$F = ma_x = m\frac{\upsilon_2^2 - \upsilon_1^2}{2s} \quad \text{and}$$

$$Fs = \frac{1}{2}m\upsilon_2^2 - \frac{1}{2}m\upsilon_1^2. \qquad (6.4)$$

In Eq. (6.4) the product Fs is the work done by the net force F and thus is equal to the total work W_{tot} done by all the forces acting on the particle. The

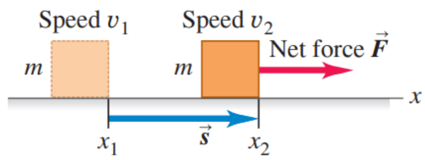


Figure 6.8 - A constant net force $\vec{F}$ does work on a moving object

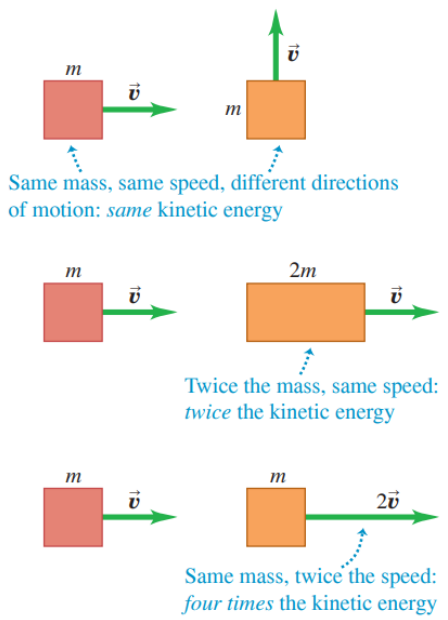


Figure 6.9 - Comparing the kinetic energy $K = \frac{1}{2}m\upsilon^2$ of different objects

quantity $\frac{1}{2}m\upsilon^2$ is called the **kinetic energy** K of the particle:

Kinetic energy of a particle · Mass of particle · Speed of particle

$$K = \tfrac{1}{2}mv^2 \tag{6.5}$$

Like work, the kinetic energy of a particle is a scalar quantity; it depends on only the particle's mass and speed, not its direction of motion (Fig. 6.9). Kinetic energy can never be negative, and it is zero only when the particle is at rest.

We can now interpret Eq. (6.4) in terms of work and kinetic energy. The first term on the right side of Eq. (6.4) is $K_2 = \frac{1}{2}m\upsilon_2^2$, the final kinetic energy of the particle (that is, after the displacement). The second term is the initial kinetic energy, $K_1 = \frac{1}{2}m\upsilon_1^2$, and the difference between these terms is the *change* in kinetic energy. So Eq. (6.4) says:

Work–energy theorem: Work done by the net force on a particle equals the change in the particle's kinetic energy.

Total work done on particle = work done by net force · Final kinetic energy · Initial kinetic energy · Change in kinetic energy

$$W_{\text{tot}} = K_2 - K_1 = \Delta K \tag{6.6}$$

This **work–energy theorem** agrees with our observations about the block in Fig. 6.7. When W_{tot} is *positive*, the kinetic energy *increases* (the final kinetic energy K_2 is greater than the initial kinetic energy K_1) and the particle is going faster at the end of the displacement than at the beginning. When W_{tot} is *negative*, the kinetic energy *decreases* (K_2 is less than K_1) and the speed is less after the displacement. When $W_{\text{tot}} = 0$, the kinetic energy stays the same ($K_1 = K_2$) and the speed is unchanged. Note that the work–energy theorem by itself tells us only about changes in *speed*, not velocity, since the kinetic energy doesn't depend on the direction of motion.

From Eq. (6.4) or Eq. (6.6), kinetic energy and work must have the same units. Hence the joule is the SI unit of both work and kinetic energy (and, as we'll see later, of all kinds of energy). To verify this, note that in SI the quantity $K = \frac{1}{2}m\upsilon^2$ has units $\text{kg}\cdot(\text{m/s})^2$ or $\text{kg}\cdot\text{m}^2/\text{s}^2$; we recall that $1\ \text{N} = 1\ \text{kg}\cdot\text{m/s}^2$, so

$$1\ \text{J} = 1\ \text{N}\cdot\text{m} = 1\ (\text{kg}\cdot\text{m/s}^2)\cdot\text{m} = 1\ \text{kg}\cdot\text{m}^2/\text{s}^2 .$$

Because we used Newton's laws in deriving the work–energy theorem, we can use this theorem only in an inertial frame of reference. Note that the work–energy theorem is valid in *any* inertial frame, but the values of W_{tot} and $K_2 - K_1$ may differ from one inertial frame to another (because the displacement and speed of an object may be different in different frames).

We've derived the work–energy theorem for the special case of straight-line motion with constant forces, and in the following examples we'll apply it to this special case only. We'll find in the next section that the theorem is valid even when the forces are not constant and the particle's trajectory is curved.

The Meaning of Kinetic Energy

To accelerate a particle of mass m from rest (zero kinetic energy) up to a speed υ, the total work done on it must equal the change in kinetic energy from zero to $K = \frac{1}{2}m\upsilon^2$:

$$W_{\text{tot}} = K - 0 = K .$$

So the kinetic energy of a particle is equal to the total work that was done to accelerate it from rest to its present speed (Fig. 6.13). The definition $K = \frac{1}{2} m\upsilon^2$, Eq. (6.5), wasn't chosen at random; it's the *only* definition that agrees with this interpretation of kinetic energy.

Another interpretation of kinetic energy: *The kinetic energy of a particle is equal to the total work that particle can do in the process of being brought to rest*. This is why you pull your hand and arm backward when you catch a ball. As the ball comes to rest, it does an amount of work (force times distance) on your hand equal to the ball's initial kinetic energy. By pulling your hand back, you maximize the distance over which the force acts and so minimize the force on your hand.

PROBLEM-SOLVING STRATEGY
6.1 Work and Kinetic Energy

IDENTIFY *the relevant concepts:*

The work–energy theorem, $W_{\text{tot}} = K_2 - K_1$, is extremely useful when you want to relate an object's speed υ_1 at one point in its motion to its speed υ_2 at a different point. (It's less useful for problems that involve the *time* it takes an object to go from point 1 to point 2 because the work–energy theorem doesn't involve time at all. For such problems it's usually best to use the relationships among time, position, velocity, and acceleration.

SET UP *the problem*:

- Identify the initial and final positions of the object, and draw afree-body diagram showing all the forces that act on the object.
- Choose a coordinate system. (If the motion is along a straight line, it's usually easiest to have both the initial and final positions lie along one of the axes).
- List the unknown and known quantities, and decide which unknowns are your target variables. The target variable may be the object's initial or final speed, the magnitude of one of the forces acting on the object, or the object's displacement.

EXECUTE *the solution*:

- Calculate the work W done by each force. If the force is constant and the displacement is a straight line, you can use Eq. (6.2) or Eq. (6.3).Be sure to check signs; W must be positive if the force has a component in the direction of the displacement, negative if the force has a component opposite to the displacement, andzero if the force and displacement are perpendicular.
- Add the amounts of work done by each force to find the total work Wtot. Sometimes it's easier to calculate the vector sum of the forces (the net force) and then find the work done by the net force; this value is also equal to Wtot.
- Write expressions for the initial and final kinetic energies, K1 and K2. Note that kinetic energy involves mass, not weight; if you are given the object's weight, use w = mg to find the mass.
- Finally, use Eq. (6.6), $W_{\text{tot}} = K_2 - K_1$, and Eq. (6.5), $K = \frac{1}{2} m\upsilon^2$, to solve for the target variable. Remember that the right-hand side of Eq. (6.6) represents the change of the object's kinetic energy between points 1 and 2; that is, it is the final kinetic energy minus the initial kinetic energy, never the other way around. (If you can predict the sign of Wtot, you can predict whether the object speeds up or slows down).

EVALUATE *your answer:*

Check whether your answer makes sense. Remember that kinetic energy $K = \frac{1}{2}m\upsilon^2$ can never be negative. If you come up with a negative value of *K*, perhaps you interchanged the initial and final kinetic energies in $W_{\text{tot}} = K_2 - K_1$ or made a sign error in one of the work calculations.

Work and Kinetic Energy in Composite Systems

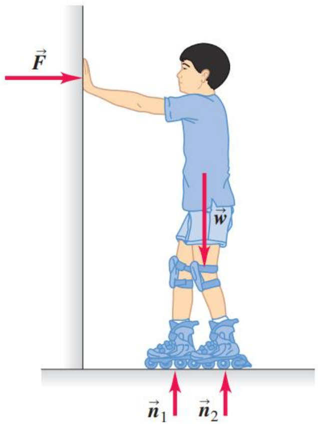


Figure 6.10 - The external forces acting on a skater pushing off a wall. The work done by these forces is zero, but the skater's kinetic energy changes nonetheless

In this section we've been careful to apply the work–energy theorem only to objects that we can represent as *particles* - that is, as moving point masses. New subtleties appear for more complex systems that have to be represented as many particles with different motions. We can't go into these subtleties in detail in this chapter, but here's an example.

Suppose a boy stands on frictionless roller skates on a level surface, facing a rigid wall (**Fig. 6.10**). He pushes against the wall, which makes him move to the right. The forces acting on him are his weight $\vec{w}$, the upward normal forces $\vec{n}_1$ and $\vec{n}_2$ exerted by the ground on his skates, and the horizontal force $\vec{F}$ exerted on him by the wall. There is no vertical displacement, so $\vec{w}$, $\vec{n}_1$, and $\vec{n}_2$ do no work. Force $\vec{F}$ accelerates him to the right, but the parts of his body where that force is applied (the boy's hands) do not move while the force acts. Thus the force $\vec{F}$ also does no work. Where, then, does the boy's kinetic energy come from?

The explanation is that it's not adequate to represent the boy as a single point mass. Different parts of the boy's body have different motions; his hands remain stationary against the wall while his torso is moving away from the wall. The various parts of his body interact with each other, and one part can exert forces and do work on another part.

Therefore the *total* kinetic energy of this *composite* system of body parts can change, even though no work is done by forces applied by objects (such as the wall) that are outside the system. In Chapter 8 we'll consider further the motion of a collection of particles that interact with each other. We'll discover that just as for the boy in this example, the total kinetic energy of such a system can change even when no work is done on any part of the system by anything outside it.

6.3 Work And Energy with Varying Forces

So far we've considered work done by *constant forces* only. But what happens when you stretch a spring? The more you stretch it, the harder you have to pull, so the force you exert is *not* constant as the spring is stretched. We've also restricted our discussion to *straight-line* motion. There are many situations in which an object moves along a curved path and is acted on by a force that varies in magnitude, direction, or both. We need to be able to compute the work done by the force in these more general cases. Fortunately, the work–energy theorem holds true even when forces are varying and when the object's path is not straight.

Work Done by a Varying Force, Straight-Line Motion

To add only one complication at a time, let's consider straight-line motion along the *x*-axis with a force whose *x*-component F_x may change as the object moves. (A real-life example is driving a car along a straight road with stop signs, so the driver has to alternately step on the gas and apply the brakes). Suppose a particle moves along the *x*-axis from point x_1 to x_2 (Fig. 6.11a). Figure 6.11b is a graph of the *x*-component of force as a function of the particle's coordinate *x*. To find the work done by this force, we divide the total displacement into narrow segments Δx_a, Δx_b, and so on (Fig. 6.11c). We approximate the work done by the force during segment Δx_a as the average x-component of force F_{ax} in that segment multiplied by the x-displacement Δx_a. We do this for each segment and then add the results for all the segments. The work done by the force in the total displacement from x_1 to x_2 is approximately

$$W = F_{ax}\Delta x_a + F_{bx}\Delta x_b + \dots$$

In the limit that the number of segments becomes very large and the width of each becomes very small, this sum becomes the *integral* of F_x from x_1 to x_2:

Work done on a particle by a **varying x-component of force** F_x during **straight-line displacement** along x-axis

$$W = \int_{x_1}^{x_2} F_x\,dx \qquad (6.7)$$

Upper limit = final position
Integral of x-component of force
Lower limit = initial position

Note that $F_{ax}\Delta x_a$ represents the *area* of the first vertical strip in Fig. 6.11c and that the integral in Eq. (6.7) represents the area under the curve of Fig. 6.11b between x_1 and x_2. *On such a graph of force as a function of position, the total work done by the force is represented by the area under the curve between the initial and final positions.* Alternatively, the work W equals the average force that acts over the entire displacement, multiplied by the displacement.

In the special case that F_x , the x-component of the force, is constant, we can take it outside the integral in Eq. (6.7):

$$W = \int_{x_1}^{x_2} F_x dx = F_x \int_{x_1}^{x_2} dx = F_x\left(x_2 - x_1\right) \quad \text{(constant force)}.$$

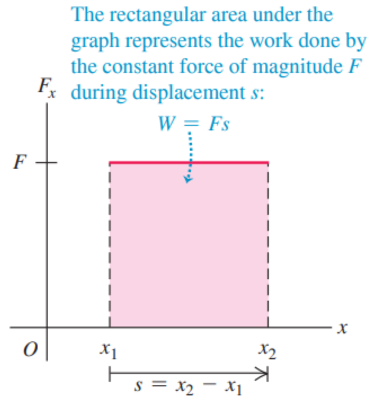


Figure 6.12 - The work done by a constant force F in the x-direction as a particle moves from x_1 to x_2

But $x_2 - x_1 = s$, the total displacement of the particle. So in the case of a constant force F, Eq. (6.7) says that $W = Fs$, in agreement with Eq. (6.1). The interpretation of work as the area under the curve of F_x as a function of x also holds for a constant force: $W = Fs$ is the area of a rectangle of height F and width s (**Fig. 6.12**).

Now let's apply these ideas to the stretched spring. To keep a spring stretched beyond its unstretched length by an amount x, we have to apply a force of equal magnitude at each end (**Fig. 6.13**). If the elongation x is not too great, the force we apply to the right-hand end has an x-component directly proportional to x:

$$F_x = kx \qquad \left(\text{force required to stretch a spring}\right), \qquad (6.8)$$

where k is a constant called the **force constant** (or spring constant) of the spring. The units of k are force divided by distance: N/m in SI units. A floppy toy spring such as a Slinky™ has a force constant of about 1 N/m; for the much stiffer springs in an automobile's suspension, k is about 10^5 N/m. The observation that force is directly proportional to elongation for elongations that are not too great was made by Robert Hooke in 1678 and is known as **Hooke's law**. It really shouldn't be called a "law", since it's a statement about a specific device and not a fundamental law of nature. Real springs don't always obey Eq. (6.8) precisely, but it's still a useful idealized model.

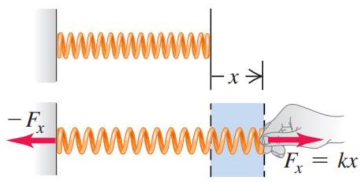


Figure 6.13 - The force needed to stretch an ideal spring is proportional to the spring's elongation: $F_x = kx$

To stretch a spring, we must do work. We apply equal and opposite forces to the ends of the spring and gradually increase the forces. We hold the left end stationary, so the force we apply at this end does no work. The force at the moving end does do work. Figure 6.14 is a graph of F_x as a function of x, the elongation of the spring. The work done by this force when the elongation goes from zero to a maximum value X is

$$W = \int_0^X F_x dx = \int_0^X kx\,dx = \tfrac{1}{2}kX^2. \qquad (6.9)$$

We can also obtain this result graphically. The area of the shaded triangle in Fig. 6.14, representing the total work done by the force, is equal to half the product of the base and altitude, or

$$W = \tfrac{1}{2}(X)(kX) = \tfrac{1}{2}kX^2.$$

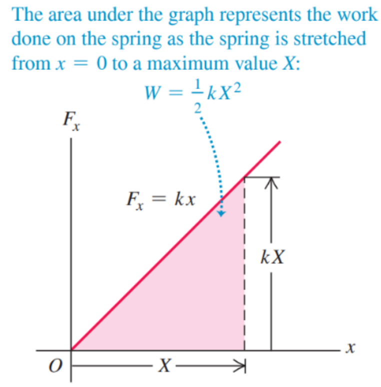


Figure 6.14 - Calculating the work done to stretch a spring by a length X

This equation also says that the work is the *average* force $kX/2$ multiplied by the total displacement X. We see that the total work is proportional to the *square* of the finalelongation X. To stretch an ideal spring by 2 cm, you must do four times as much work as is needed to stretch it by 1 cm.

Equation (6.9) assumes that the spring was originally unstretched. If initially the spring is already stretched a distance x_1, the work we must do to stretch it to a greater elongation x_2 (Fig. 6.15a) is

$$W = \int_{x_1}^{x_2} F_x dx = \int_{x_1}^{x_2} kx\,dx = \tfrac{1}{2}kx_2^2 - \tfrac{1}{2}kx_1^2. \qquad (6.10)$$

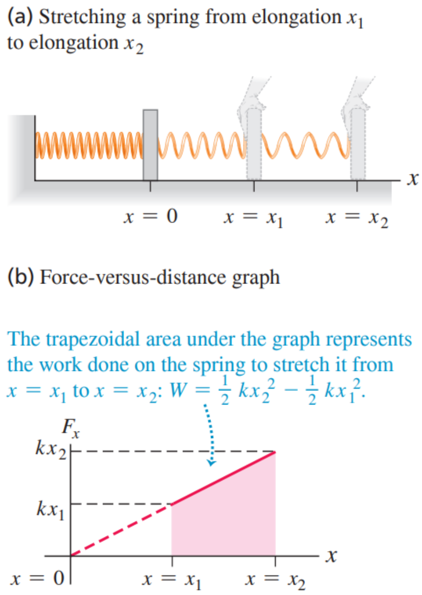


Figure 6.15 - Calculating the work done to stretch a spring from one elongation to a greater one

Use your knowledge of geometry to convince yourself that the trapezoidal area under the graph in Fig. 6.15b is given by the expression in Eq. (6.10).

If the spring has spaces between the coils when it is unstretched, then it can also be compressed, and Hooke's law holds for compression as well as stretching. In this case the force and displacement are in the opposite directions from those shown in Fig. 6.13, so both F_x and x in Eq. (6.8) are negative. Since both F_x and x are reversed, the force again is in the same direction as the displacement, and the work done by F_x is again positive. So the total work is still given by Eq. (6.9) or (6.10), even when X is negative or either or both of x_1 and x_2 are negative.

CAUTION! **Work done *on* a spring vs. work done *by* a spring.** Equation (6.10) gives the work that *you* must do *on* a spring to change its length. If you stretch a spring that's originally relaxed, then $x_1 = 0$, $x_2 > 0$, and $W > 0$: The force you apply to one end of the spring is in the same direction as the displacement, and the work you do is positive. By contrast, the work that the *spring* does on whatever it's attached to is given by the *negative* of Eq. (6.10). Thus, as you pull on the spring, the spring does negative work on you.

Work–Energy Theorem for Straight-Line Motion, Varying Forces

In Section 6.2 we derived the work–energy theorem, $W_{tot} = K_2 - K_1$, for the special case of straight-line motion with a constant net force. We can now prove that this theorem is true even when the force varies with position. As in Section 6.2, let's consider a particle that undergoes a displacement x while being acted on by a net force with x-component F_x, which we now allow to vary. Just as in Fig.

6.11, we divide the total displacement x into a large number of small segments Δx. We can apply the work–energy theorem, Eq. (6.6), to each segment because the value of F_x in each small segment is approximately constant. The change in kinetic energy in segment Δx_a is equal to the work $F_{ax}\Delta x_a$, and so on. The total change of kinetic energy is the sum of the changes in the individual segments, and thus is equal to the total work done on the particle during the entire displacement. So $W_{\text{tot}} = \Delta K$ holds for varying forces as well as for constant ones.

Here's an alternative derivation of the work–energy theorem for a force that may vary with position. It involves making a change of variable from x to υ_x in the work integral. Note first that the acceleration a of the particle can be expressed in various ways, using $a_x = d\upsilon_x/dt$, $\upsilon_x = dx/dt$, and the chain rule for derivatives:

$$a_x = \frac{d\upsilon_x}{dt} = \frac{d\upsilon_x}{dx}\frac{dx}{dt} = \upsilon_x \frac{d\upsilon_x}{dx}. \tag{6.11}$$

From this result, Eq. (6.7) tells us that the total work done by the *net* force F_x is

$$W_{\text{tot}} = \int_{x_1}^{x_2} F_x dx = \int_{x_1}^{x_2} ma_x\, dx = \int_{x_1}^{x_2} m\upsilon_x \frac{d\upsilon_x}{dx} dx. \tag{6.12}$$

Now $(d\upsilon_x/dx)dx$ is the change in velocity $d\upsilon_x$ during the displacement dx, so we can make that substitution in Eq. (6.12). This changes the integration variable from x to υ_x, so we change the limits from x_1 and x_2 to the corresponding x-velocities υ_1 and υ_2:

$$W_{\text{tot}} = \int_{\upsilon_1}^{\upsilon_2} m\upsilon_x d\upsilon_x .$$

The integral of $\upsilon_x d\upsilon_x$ is just $\upsilon_x^2/2$. Substituting the upper and lower limits, we finally find

$$W_{\text{tot}} = \tfrac{1}{2}m\upsilon_2^2 - \tfrac{1}{2}m\upsilon_1^2 . \tag{6.13}$$

This is the same as Eq. (6.6), so the work–energy theorem is valid even without the assumption that the net force is constant.

Work–Energy Theorem for Motion Along a Curve

We can generalize our definition of work further to include a force that varies in direction as well as magnitude, and a displacement that lies along a curved path. **Figure 6.16a** shows a particle moving from P_1 to P_2 along a curve. We divide the curve between these points into many infinitesimal vector displacements, and we call a typical one of these $d\vec{l}$. Each $d\vec{l}$ is tangent to the path at its position. Let $\vec{F}$ be the force at a typical point along the path, and let ϕ be the angle between $\vec{F}$ and $d\vec{l}$ at this point. Then the small element of work dW done on the particle during the displacement $d\vec{l}$ may be written as

$$dW = \vec{F}\cdot d\vec{l} = F\cos\phi\, d\vec{l} = F_{\|}dl ,$$

where $F_{\|} = F\cos\phi$ is the component of $\vec{F}$ in the direction parallel to $d\vec{l}$ (Fig. 6.16b). The work done by $\vec{F}$ on the particle as it moves from P_1 to P_2 is

Upper limit = final position

Scalar product (dot product) of $\vec{F}$ and displacement $d\vec{l}$

Work done on a particle by a varying force $\vec{F}$ along a curved path

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l} = \int_{P_1}^{P_2} F\cos\phi\, dl = \int_{P_1}^{P_2} F_{\parallel}\, dl \qquad (6.14)$$

Lower limit = initial position

Angle between $\vec{F}$ and $d\vec{l}$

Component of $\vec{F}$ parallel to $d\vec{l}$

The integral in Eq. (6.14) (shown in three versions) is called a *line integral*. We'll see shortly how to evaluate an integral of this kind. We can now show that the work–energy theorem, Eq. (6.6), holds true even with varying forces and a displacement along a curved path. The force $\vec{F}$ is essentially constant over any given infinitesimal segment $d\vec{l}$ of the path, so we can apply the work–energy theorem for straight-line motion to that segment. Thus the change in the particle's kinetic energy K over that segment equals the work $dW = F_{\parallel} dl = \vec{F} \cdot d\vec{l}$ done on the particle. Adding up these infinitesimal quantities of work from all the segments along the whole path gives the total work done, Eq. (6.14), which equals the total change in kinetic energy over the whole path. So $W_{tot} = \Delta K = K_2 - K_1$ is true *in general*, no matter what the path and no matter what the character of the forces. This can be proved more rigorously by using steps like those in Eq. (6.11) through (6.13).

Note that only the component of the net force parallel to the path, $F_{\parallel}$, does work on the particle, so only this component can change the speed and kinetic energy of the particle. The component perpendicular to the path, $F_{\perp} = F\sin\phi$, has no effect on the particle's speed; it acts only to change the particle's direction.

To evaluate the line integral in Eq. (6.14) in a specific problem, we need some sort of detailed description of the path and of the way in which $\vec{F}$ varies along the path. We usually express the line integral in terms of some scalar variable, as in the following example.

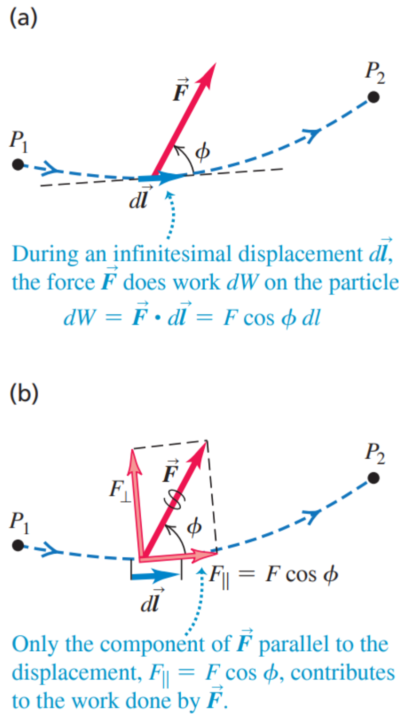


Figure 6.16 - A particle moves along a curved path from point P_1 to P_2, acted on by a force $\vec{F}$ that varies in magnitude and direction

6.4 Power

The definition of work makes no reference to the passage of time. If you lift a barbell weighing 100 N through a vertical distance of 1.0 m at constant velocity, you do $(100\text{ N})(1.0\text{ m}) = 100\text{ J}$ of work whether it takes you 1 second, 1 hour, or 1 year to do it. But often we need to know how quickly work is done. We describe this in terms of *power*. In ordinary conversation the word "power" is often synonymous with "energy" or "force". In physics we use a much more precise definition: **Power** is the time *rate* at which work is done. Like work and energy, power is a scalar quantity.

The average work done per unit time, or **average power** P_{av}, is defined to be

Average power during time interval Δt

$$P_{av} = \frac{\Delta W}{\Delta t} \qquad (6.15)$$

Work done during time interval

Duration of time interval

The rate at which work is done might not be constant. We define **instantaneous power** P as the quotient in Eq. (6.15) as Δt approaches zero:

Instantaneous power

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt} \qquad (6.16)$$

Time rate of doing work

Average power over infinitesimally short time interval

The SI unit of power is the **watt** (W), named for the English inventor James Watt. One watt equals 1 joule per second: 1 W = 1 J/s (**Fig. 6.17**). The kilowatt (1 kW = 10^3 W) and the megawatt (1 MW = 10^6 W) are also commonly used.

The watt is a familiar unit of *electrical* power; a 100 W light bulb converts 100 J of electrical energy into light and heat each second.

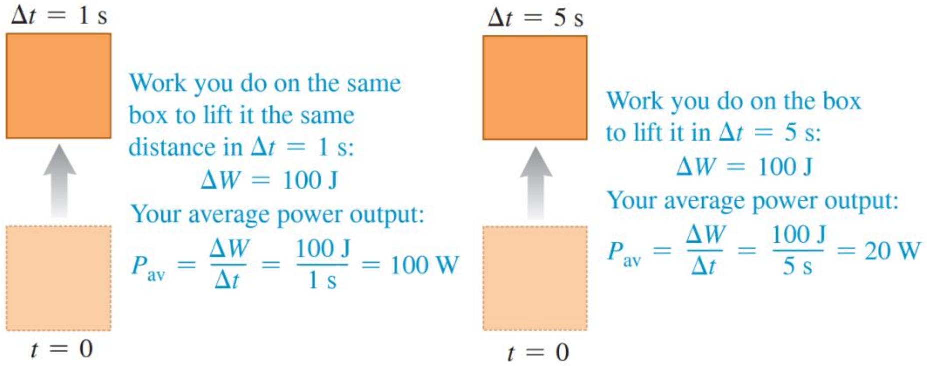


Figure 6.17 - The same amount of work is done in both of these situations, but the power (the rate at which work is done) is different

The *kilowatt-hour* ($\text{kW}\cdot\text{h}$) is the usual commercial unit of electrical energy. One kilowatt-hour is the total work done in 1 hour (3600 s) when the power is 1 kilowatt (10^3 J/s), so

$$1\ \text{kW}\cdot\text{h} = \left(10^3\ \text{J/s}\right)\left(3600\ \text{s}\right) = 3.6\times 10^6\ \text{J} = 3.6\ \text{MJ}.$$

The kilowatt-hour is a unit of *work* or *energy*, not power.

In mechanics we can also express power in terms of force and velocity. Suppose that a force $\vec{F}$ acts on an object while it undergoes a vector displacement $\Delta\vec{s}$. If $F_{\parallel}$ is the component of $\vec{F}$ tangent to the path (parallel to $\Delta\vec{s}$), then the work done by the force is $\Delta W = F_{\parallel}\Delta s$. The average power is

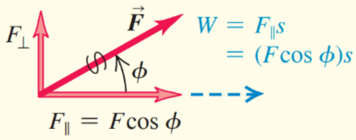
$$P_{av} = \frac{F_{\parallel}\Delta s}{\Delta t} = F_{\parallel}\frac{\Delta s}{\Delta t} = F_{\parallel}\upsilon_{av}. \qquad (6.17)$$

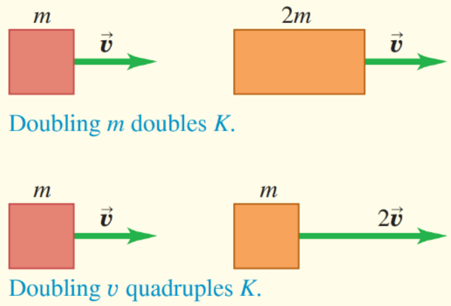
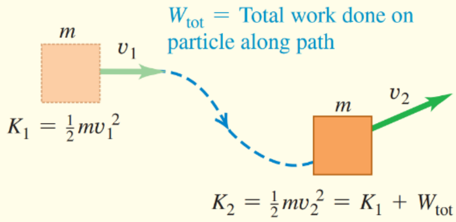
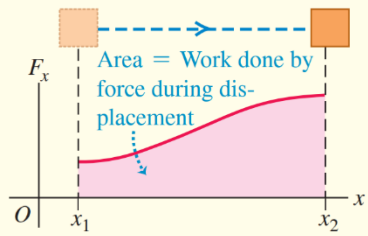
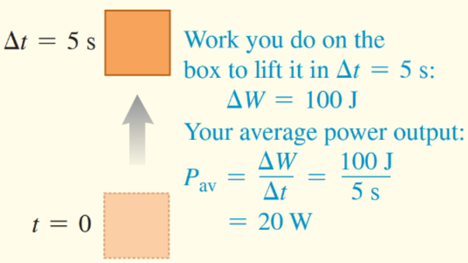
Instantaneous power P is the limit of this expression as $\Delta t \to 0$:

$$P = F_{\parallel}\upsilon, \qquad (6.18)$$

where υ is the magnitude of the instantaneous velocity. We can also express Eq. (6.18) in terms of the scalar product:

Instantaneous power for a force doing work on a particle $\cdots\!\!\rightarrow$ $P = \vec{F}\cdot\vec{\upsilon}$ — Force that acts on particle; Velocity of particle (6.19)

CHAPTER 6: SUMMARY		
Work done by a force: When a constant force $\vec{F}$ acts on a particle that undergoes a straight-line displacement $\vec{s}$, the work done by the force on the particle is defined to be the scalar product of $\vec{F}$ and $\vec{s}$. The unit of work in SI units is 1 joule = 1 newton-meter (1 J = = 1 N·m). Work is a scalar quantity; it can be positive or negative, but it has no direction in space	$W = \vec{F}\cdot\vec{s} = Fs\cos\phi$ ϕ = angle between $\vec{F}$ and $\vec{s}$	

Kinetic energy: The kinetic energy K of a particle equals the amount of work required to accelerate the particle from rest to speed υ. It is also equal to the amount of work the particle can do in the process of being brought to rest. Kinetic energy is a scalar that has no direction in space; it is always positive or zero. Its units are the same as the units of work: $1\ \mathrm{J} = 1\ \mathrm{N \cdot m} = 1\ \mathrm{kg \cdot m^2/s^2}$	$K = \frac{1}{2} m\upsilon^2$	
The work–energy theorem: When forces act on a particle while it undergoes a displacement, the particle's kinetic energy changes by an amount equal to the total work done on the particle by all the forces. This relationship, called the work–energy theorem, is valid whether the forces are constant or varying and whether the particle moves along a straight or curved path. It is applicable only to objects that can be treated as particles	$W_{\text{tot}} = K_2 - K_1 = \Delta K$	
Work done by a varying force or on a curved path: When a force varies during a straight-line displacement, the work done by the force is given by an integral, Eq. (6.7). When a particle follows a curved path, the work done on it by a force $\vec{F}$ is given by an integral that involves the angle ϕ between the force and the displacement. This expression is valid even if the force magnitude and the angle ϕ vary during the displacement	$W = \int_{x_1}^{x_2} F_x\,dx$ $W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l}$ $= \int_{P_1}^{P_2} F\cos\phi\,dl = \int_{P_1}^{P_2} F_{\parallel}\,dl$	
Power: Power is the time rate of doing work. The average power P_{av} is the amount of work ΔW done in time Δt divided by that time. The instantaneous power is the limit of the average power as Δt goes to zero. When a force $\vec{F}$ acts on a particle moving with velocity $\vec{\upsilon}$, the instantaneous power (the rate at which the force does work) is the scalar product of $\vec{F}$ and $\vec{\upsilon}$. Like work and kinetic energy, power is a scalar quantity. The SI unit of power is 1 watt = 1 joule/second	$P_{\text{av}} = \frac{\Delta W}{\Delta t}$ $P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$ $P = \vec{F} \cdot \vec{\upsilon}$	

7 POTENTIAL ENERGY AND ENERGY CONSERVATION

When a diver jumps off a high board into a swimming pool, she hits the water moving pretty fast, with a lot of kinetic energy - energy of *motion*. Where does that energy come from? The answer we learned in Chapter 6 was that the gravitational force does work on the diver as she falls, and her kinetic energy increases by an amount equal to the work done.

However, there's a useful alternative way to think about work and kinetic energy. This new approach uses the idea of *potential energy*, which is associated with the *position* of a system rather than with its motion. In this approach, there is *gravitational potential energy* even when the diver is at rest on the high board. As she falls, this potential energy is *transformed* into her kinetic energy.

If the diver bounces on the end of the board before she jumps, the bent board stores a second kind of potential energy called *elastic potential energy*. We'll discuss elastic potential energy of simple systems such as a stretched or compressed spring. (An important third kind of potential energy is associated with the forces between electrically charged objects).

We'll prove that in some cases the sum of a system's kinetic and potential energies, called the *total mechanical energy* of the system, is constant during the motion of the system. This will lead us to the general statement of the *law of conservation of energy*, one of the most fundamental principles in all of science.

7.1 Gravitational Potential Energy

In many situations it seems as though energy has been stored in a system, to be recovered later. For example, you must do work to lift a heavy stone over your head. It seems reasonable that in hoisting the stone into the air you are storing energy in the system, energy that is later converted into kinetic energy when you let the stone fall.

This example points to the idea of an energy associated with the *position* of objects in a system. This kind of energy is a measure of the *potential* or *possibility* for work to be done; if you raise a stone into the air, there is a potential for the gravitational force to do work on it, but only if you allow the stone to fall to the ground. For this reason, energy associated with position is called **potential energy**. The potential energy associated with an object's weight and its height above the ground is called *gravitational potential energy*.

We now have *two* ways to describe what happens when an object falls without air resistance. One way, which we learned in Chapter 6, is to say that a falling object's kinetic energy increases because the force of the earth's gravity does work on the object. The other way is to say that the kinetic energy increases as the gravitational potential energy decreases. Later in this section we'll use the work–energy theorem to show that these two descriptions are equivalent.

Let's derive the expression for gravitational potential energy. Suppose an object with mass m moves along the (vertical) y-axis, as in **Fig. 7.1.** The forces acting on it are its weight, with magnitude $w = mg$, and possibly some other forces; we call the vector sum (resultant) of all the other forces $\vec{F}_{\text{other}}$. We'll assume that the object stays close enough to the earth's surface that the weight is constant. We want to find the work done by the weight when the object moves downward from a height y_1 above the origin to a lower height y_2 (Fig. 7.1a). The weight and displacement are in the same direction, so the work W_{grav} done on the object by its weight is positive:

$$W_{\text{grav}} = Fs = w\left(y_1 - y_2\right) = mgy_1 - mgy_2 . \tag{7.1}$$

This expression also gives the correct work when the object moves *upward* and y_2 is greater than y_1 (Fig. 7.1b). In that case the quantity $(y_1 – y_2)$ is negative, and W_{grav} is negative because the weight and displacement are opposite in direction.

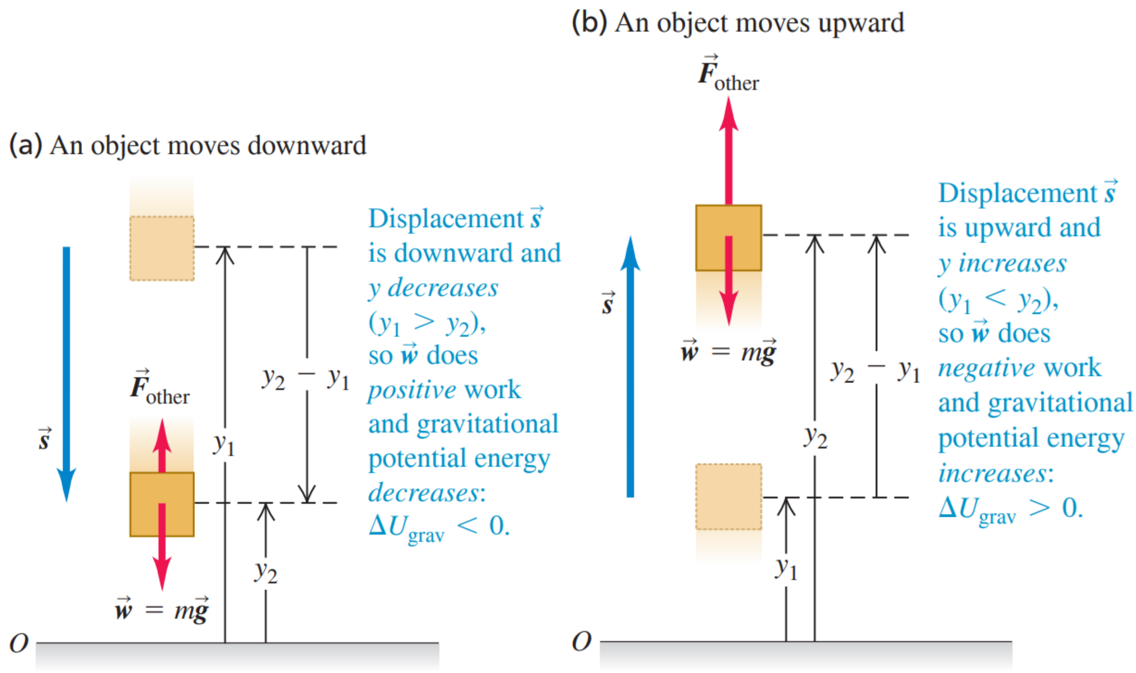


Figure 7.1 - When an object moves vertically from an initial height y_1 to a final height y_2, the gravitational force $\vec{w}$ does work and the gravitational potential energy changes

Equation (7.1) shows that we can express W_{grav} in terms of the values of the quantity mgy at the beginning and end of the displacement. This quantity is called the **gravitational potential energy**, U_{grav}:

Gravitational potential energy associated with a particle

Vertical coordinate of particle (y increases if particle moves upward)

$$U_{grav} = mgy \qquad (7.2)$$

Mass of particle

Acceleration due to gravity

Its initial value is $U_{grav,\,1} = mgy_1$ and its final value is $U_{grav,\,2} = mgy_2$. The change in U_{grav} is the final value minus the initial value, or $\Delta U_{grav} = U_{grav,\,2} - U_{grav,\,1}$. Using Eq. (7.2), we can rewrite Eq. (7.1) for the work done by the gravitational force during the displacement from y_1 to y_2:

$$W_{grav} = U_{grav,\,1} - U_{grav,\,2} = -\left(U_{grav,\,2} - U_{grav,\,1}\right) = -\Delta U_{grav}$$

or

Work done by the gravitational force on a particle ...

... equals the **negative of the change in the gravitational potential energy.**

$$W_{grav} = mgy_1 - mgy_2 = U_{grav,1} - U_{grav,2} = -\Delta U_{grav} \qquad (7.3)$$

Mass of particle

Acceleration due to gravity

Initial and final vertical coordinates of particle

The negative sign in front of ΔU_{grav} is *essential*. When the object moves up, y increases, the work done by the gravitational force is negative, and the gravitational potential energy increases ($\Delta U_{grav} > 0$). When the object moves down, y decreases, the gravitational force does positive work, and the gravitational potential energy decreases ($\Delta U_{grav} < 0$). It's like drawing money out of the bank (decreasing U_{grav}) and spending it (doing positive work). The unit of potential energy is the joule (J), the same unit as is used for work.

CAUTION! **To what object does gravitational potential energy "belong"?** It is *not* correct to call $U_{grav} = mgy$ the "gravitational potential energy of the object". The reason is that U_{grav} is a *shared* property of the object and the earth. The value of U_{grav} increases if the earth stays fixed and the object moves upward, away from the earth; it also increases if the object stays fixed and the earth is moved away from it. Notice that the formula $U_{grav} = mgy$ involves characteristics of both the object (its mass m) and the earth (the value of g).

Conservation of Total Mechanical Energy (Gravitational Forces Only)

To see what gravitational potential energy is good for, suppose an object's weight is the *only* force acting on it, so $\vec{F}_{other} = 0$. The object is then falling freely with no air resistance and can be moving either up or down. Let its speed at point y_1 be υ_1 and let its speed at y_2 be υ_2. The work–energy theorem, Eq. (6.6), says that the total work done on the object equals the change in the object's kinetic energy: $W_{tot} = \Delta K = K_2 - K_1$. If gravity is the only force that acts, then from Eq. (7.3), $W_{tot} = W_{grav} = -\Delta U_{grav} = U_{grav,1} - U_{grav,2}$. Putting these together, we get

$$\Delta K = -\Delta U_{grav} \quad \text{or} \quad K_2 - K_1 = U_{grav,1} - U_{grav,2},$$

which we can rewrite as

If only the gravitational force does work, total mechanical energy is conserved:

Initial kinetic energy $K_1 = \frac{1}{2}mv_1^2$ Initial gravitational potential energy $U_{grav,1} = mgy_1$

$$K_1 + U_{grav,1} = K_2 + U_{grav,2} \tag{7.4}$$

Final kinetic energy $K_2 = \frac{1}{2}mv_2^2$ Final gravitational potential energy $U_{grav,2} = mgy_2$

The sum $K + U_{grav}$ of kinetic and potential energies is called E, the **total mechanical energy of the system**. By "system" we mean the object of mass m and the earth considered together, because gravitational potential energy U is a shared property of both objects. Then $E_1 = K_1 + U_{grav,1}$ is the total mechanical energy at y_1 and $E_2 = K_2 + U_{grav,2}$ is the total mechanical energy at y_2. Equation (7.4) says that when the object's weight is the only force doing work on it, $E_1 = E_2$. That is, E is constant; it has the same value at y_1 and y_2. But since positions y_1 and y_2 are arbitrary points in the motion of the object, the total mechanical energy E has the same value at *all* points during the motion:

$$E = K + U_{grav} = \text{constant} \quad (\text{if only gravity does work}).$$

A quantity that always has the same value is called a *conserved* quantity. *When only the force of gravity does work, the total mechanical energy is constant – that is, it is conserved* (**Fig. 7.2**). This is our first example of the **conservation of total mechanical energy.**

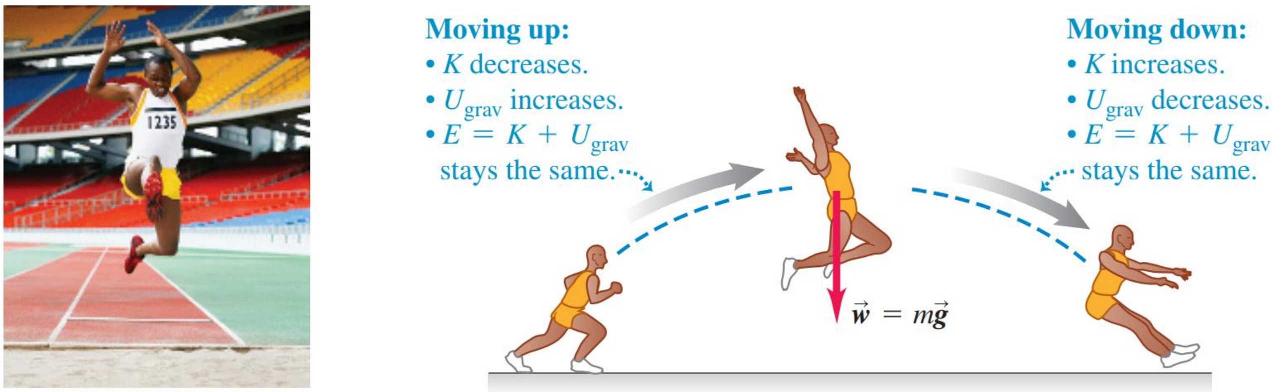


Figure 7.2 – While this athlete is in midair, only gravity does work on him (if we neglect the minor effects of air resistance). Total mechanical energy E – the sum of kinetic and gravitational potential energies – is conserved

When we throw a ball into the air, its speed decreases on the way up as kinetic energy is converted to potential energy: $\Delta K < 0$ and $\Delta U_{grav} > 0$. On the way back down, potential energy is converted back to kinetic energy and the ball's speed increases: $\Delta K > 0$ and $\Delta U_{grav} < 0$. But the *total* mechanical energy (kinetic plus potential) is the same at every point in the motion, provided that no force other than gravity does work on the ball (that is, air resistance must be negligible). It's still true that the gravitational force does work on the object as it moves up or down, but we no longer have to calculate work directly; keeping track of changes in the value of U_{grav} takes care of this completely. Equation (7.4) is also valid if forces other than gravity are present but do not do work.

CAUTION! **Choose "zero height" to be wherever you like.** When working with gravitational potential energy, we may choose any height to be $y = 0$. If we shift the origin for y, the values of y_1 and y_2 change, as do the values of $U_{grav,1}$ and $U_{grav,2}$. But this shift has no effect on the *difference* in height $y_2 - y_1$ or the *difference* in gravitational potential energy $U_{grav,2} - U_{grav,1} = mg\left(y_2 - y_1\right)$. The physically significant quantity is not the value of U_{grav} at a particular point but the *difference* in U_{grav} between two points. We can define U_{grav} to be zero at whatever point we choose.

EXAMPLE 7.1 Height of a baseball from energy conservation

You throw a 0.145 kg baseball straight up, giving it an initial velocity of magnitude 20.0 m/s. Find how high it goes, ignoring air resistance.

IDENTIFY and SET UP

After the ball leaves your hand, only gravity does work on it. Hence total mechanical energy is conserved, and we can use Eq. (7.4). We take point 1 to be where the ball leaves your hand and point 2 to be where it reaches its maximum height. As in Fig. 7.1, we take the positive y-direction to be upward. The ball's speed at point 1 is $\upsilon_1 = 20.0$ m/s; at its maximum height it is instantaneously at rest, so $\upsilon_2 = 0$. We take the origin at point 1, so $y_1 = 0$ (Fig.7.3). Our target variable, the distance the ball moves vertically between the two points, is the displacement $y_2 - y_1 = y_2 - 0 = y_2$.

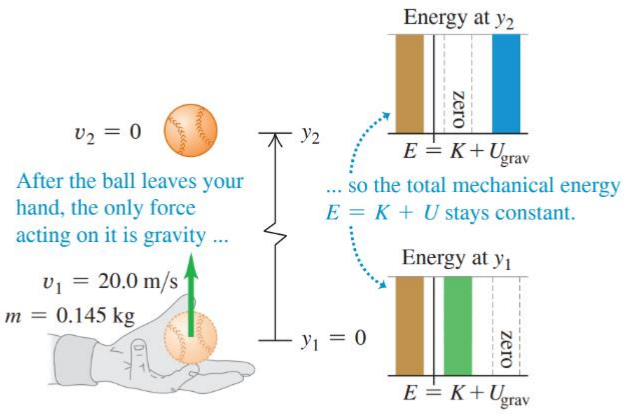


Figure 7.3 - After a baseball leaves your hand, total mechanical energy $E = K + U$ is conserved

EXECUTE

We have $y_1 = 0$, $U_{grav,1} = mgy_1 = 0$, and $K_2 = \frac{1}{2}m\upsilon_2^2 = 0$. Then Eq. (7.4), $K_1 + U_{grav,1} = K_2 + U_{grav,2}$, becomes

$$K_1 = U_{grav,2}.$$

As the energy bar graphs in Fig. 7.3 show, this equation says that the kinetic energy of the ball at point 1 is completely converted to gravitational potential energy at point 2. We substitute $K_1 = \frac{1}{2}m\upsilon_1^2$ and $U_{grav,2} = mgy_2$ and solve for y_2:

$$\tfrac{1}{2}m\upsilon_1^2 = mgy_2; \quad y_2 = \frac{\upsilon_1^2}{2g} = \frac{\left(20.0 \text{ m/s}\right)^2}{2\left(9.80 \text{ m/s}^2\right)} = 20.4 \text{ m}.$$

EVALUATE

As a check, use the given value of υ_1 and our result for y_2 to calculate the kinetic energy at point 1 and the gravitational potential energy at point 2. You should find that these are equal: $K_1 = \frac{1}{2}m\upsilon_1^2 = 29.0\ \text{J}$ and $U_{\text{grav,2}} = mgy_2 = 29.0\ \text{J}$. Note that we could have found the result $y_2 = \upsilon_1^2/2g$ by using Eq. (2.13) in the form $\upsilon_{2y}^2 = \upsilon_{1y}^2 - 2g(y_2 - y_1)$.

What if we put the origin somewhere else—for example, 5.0 m below point 1, so that $y_1 = 5.0\ \text{m}$? Then the total mechanical energy at point 1 is part kinetic and part potential; at point 2 it's still purely potential because $\upsilon_2 = 0$. You'll find that this choice of origin yields $y_2 = 25.4\ \text{m}$, but again $y_2 - y_1 = 20.4\ \text{m}$. In problems like this, you are free to choose the height at which $U_{\text{grav}} = 0$. The physics doesn't depend on your choice.

KEYCONCEPT: Total mechanical energy (the sum of kinetic energy and gravitational potential energy) is conserved when only the force of gravity does work.

When Forces Other Than Gravity Do Work

If other forces act on the object in addition to its weight, then $\vec{F}_{\text{other}}$ other in Fig. 7.2 is *not* zero. For the example of pile driver, the force applied by the hoisting cable and the friction with the vertical guide rails are examples of forces that might be included in $\vec{F}_{\text{other}}$. The gravitational work W_{grav} is still given by Eq. (7.3), but the total work W_{tot} is then the sum of W_{grav} and the work done by $\vec{F}_{\text{other}}$. We'll call this additional work W_{other}, so the total work done by all forces is $W_{\text{tot}} = W_{\text{grav}} + W_{\text{other}}$. Equating this to the change in kinetic energy, we have

$$W_{\text{other}} + W_{\text{grav}} = K_2 - K_1. \qquad (7.5)$$

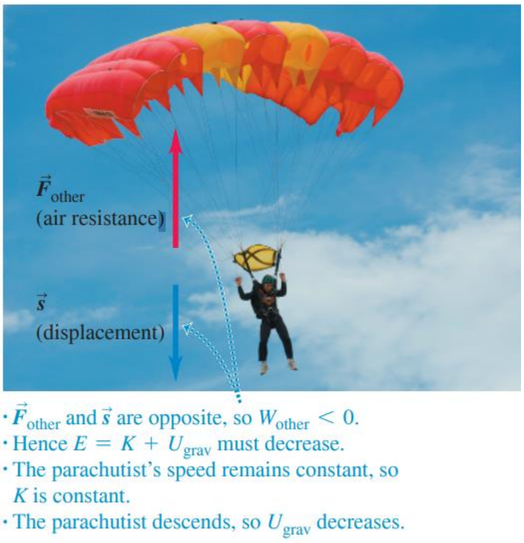


Figure 7.4 - As this parachutist moves downward at a constant speed, the upward force of air resistance does negative work W_{other} on him. Hence the total mechanical energy $E = K + U$ decreases

Also, from Eq. (7.3), $W_{\text{grav}} = U_{\text{grav,1}} - U_{\text{grav,2}}$, so Eq. (7.5) becomes

$$W_{\text{other}} + U_{\text{grav,1}} - U_{\text{grav,2}} = K_2 - K_1,$$

which we can rearrange in the form

$$K_1 + U_{\text{grav,1}} + W_{\text{other}} = K_2 + U_{\text{grav,2}} \quad \text{(if forces other than gravity do work)}. \qquad (7.6)$$

We can use the expressions for the various energy terms to rewrite Eq. (7.6):

$$\tfrac{1}{2}m\upsilon_1^2 + mgy_1 + W_{\text{other}} = \tfrac{1}{2}m\upsilon_2^2 + mgy_2 \quad \text{(if forces other than gravity do work)}. \qquad (7.7)$$

The meaning of Eqs. (7.6) and (7.7) is this: *The work done by all forces other than the gravitational force equals the change in the total mechanical energy* $E = K + U_{\text{grav}}$ *of the system, where* U_{grav} *is the gravitational potential energy*. When W_{other} is positive, E increases $K_2 + U_{\text{grav},2}$ and is greater than $K_1 + U_{\text{grav},1}$. When W_{other} is negative, E decreases (Fig. 7.4). In the special case in which no forces other than the object's weight do work, $W_{\text{other}} = 0$. The total mechanical energy is then constant, and we are back to Eq. (7.4).

PROBLEM-SOLVING STRATEGY
7.1 Problems Using Total Mechanical Energy I

IDENTIFY *the relevant concepts:*
Decide whether the problem should be solved by energy methods, by using $\sum \vec{F} = m\vec{a}$ directly, or by a combination of these. The energy approach is best when the problem involves varying forces or motion along a curved path (discussed later in this section). If the problem involves elapsed time, the energy approach is usually *not* the best choice because it doesn't involve time directly.

SET UP *the problem*:

- When using the energy approach, first identify the initial and final states (the positions and velocities) of the objects in question. Use the subscript 1 for the initial state and the subscript 2 for the final state. Draw sketches showing these states.
- Define a coordinate system, and choose the level at which $y = 0$. Choose the positive y-direction to be upward. (The equations in this section require this).
- Identify any forces that do work on each object and that *cannot* be described in terms of potential energy. (So far, this means any forces other than gravity. Work done by an ideal spring can also be expressed as a change in potential energy). Sketch a free-body diagram for each object.
- List the unknown and known quantities, including the coordinates and velocities at each point. Identify the target variables.

EXECUTE *the solution*:
Write expressions for the initial and final kinetic and potential energies K_1, K_2, $U_{\text{grav},1}$, and $U_{\text{grav},2}$. If no other forces do work, use Eq. (7.4). If there are other forces that do work, use Eq. (7.6). Draw bar graphs showing the initial and final values of K, $U_{\text{grav},1}$, and $E = K + U_{\text{grav}}$. Then solve to find your target variables

EVALUATE *your answer:*
Check whether your answer makes physical sense. Remember that the gravitational work is included in ΔU_{grav}, so do not include it in W_{other}.

Gravitational Potential Energy for Motion Along a Curved Path

In our first two examples the object moved along a straight vertical line. What happens when the path is slanted or curved (Fig. 7.5a)? The object is acted on by the gravitational force $\vec{w} = m\vec{g}$ and possibly by other forces whose resultant we call $\vec{F}_{\text{other}}$. To find the work W_{grav} done by the gravitational force during this displacement, we divide the path into small segments $\Delta\vec{s}$; Fig. 7.5b shows a typical segment. The work done by the gravitational force over this segment is the scalar product of the force and the displacement. In terms of unit vectors, the force is $\vec{w} = m\vec{g} = -mg\hat{j}$ and the displacement is $\Delta\vec{s} = \Delta x\hat{i} + \Delta y\hat{j}$, so

$$W_{\text{grav}} = \vec{w} \cdot \Delta\vec{s} = -mg\hat{j} \cdot \left(\Delta x\hat{i} + \Delta y\hat{j}\right) = -mg\Delta y .$$

The work done by gravity is the same as though the object had been displaced vertically a distance Δy, with no horizontal displacement. This is true for every segment, so the *total* work done by the gravitational force is $-mg$ multiplied by the *total* vertical displacement $\left(y_2 - y_1\right)$:

$$W_{\text{grav}} = -mg\left(y_2 - y_1\right) = mgy_1 - mgy_2 = U_{\text{grav},1} - U_{\text{grav},2} .$$

This is the same as Eq. (7.1) or (7.3), in which we assumed a purely vertical path. So even if the path an object follows between two points is curved, the total work done by the gravitational force depends on only the difference in height between the two points of the path. This work is unaffected by any horizontal motion that may occur. So *we can use the same expression for gravitational potential energy whether the object's path is curved or straight*.

CAUTION! **With gravitational potential energy, only the change in height matters.** The change in gravitational potential energy along a curved path depends only on the difference between the final and initial heights, not on the shape of the path. If gravity is the only force that does work along a curved path, then the total mechanical energy is conserved.

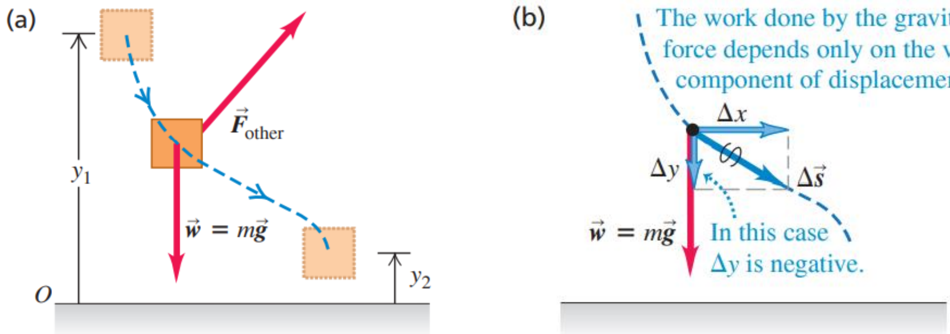


Figure 7.5 - Calculating the change in gravitational potential energy for a displacement along a curved path

7.2 Elastic Potential energy

In many situations we encounter potential energy that is not gravitational in nature. One example is a rubber-band slingshot. Work is done on the rubber band by the force that stretches it, and that work is stored in the rubber band until you let it go. Then the rubber band gives kinetic energy to the projectile.

Do work on the system to store energy, which can later be converted to kinetic energy. We'll describe the process of storing energy in a deformable object such as a spring or rubber band in terms of *elastic potential energy*. An object is called *elastic* if it returns to its original shape and size after being deformed.

To be specific, we'll consider storing energy in an ideal spring, like the ones we discussed in Section 6.3. To keep such an ideal spring stretched by a distance *x*, we must exert a force $F = kx$, where *k* is the force constant of the spring. Many elastic objects show this same direct proportionality between force $\vec{F}$ and displacement *x*, provided that *x* is sufficiently small.

Let's proceed just as we did for gravitational potential energy. We begin with the work done by the elastic (spring) force and then combine this with the work–energy theorem. The difference is that gravitational potential energy is a shared property of an object and the earth, but elastic potential energy is stored in just the spring (or other deformable object).

Figure 7.6 shows the ideal spring with its left end held stationary and its right end attached to a block with mass *m* that can move along the *x*-axis. In Fig. 7.6a the block is at $x = 0$ when the spring is neither stretched nor compressed. We move the block to one side, thereby stretching or compressing the spring, then let it go. As the block moves from a different position x_1 to a different position x_2, how much work does the elastic (spring) force do on the block?

We found in Section 6.3 that the work we must do *on* the spring to move one end from an elongation x_1 to a different elongation x_2 is

$$W = \tfrac{1}{2}kx_2^2 - \tfrac{1}{2}kx_1^2 \quad \text{(work done } on \text{ a spring)}, \tag{7.8}$$

where *k* is the force constant of the spring. If we stretch the spring farther, we do positive work on the spring; if we let the spring relax while holding one end, we do negative work on it. This expression for work is also correct when the spring is compressed such that x_1, x_2, or both are negative. Now, from Newton's third law the work done *by* the spring is just the negative of the work done *on* the spring. So by changing the signs in Eq. (7.8), we find that in a displacement from x_1 to x_2 the spring does an amount of work W_{el} given by

$$W_{el} = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 \quad \text{(work done } by \text{ a spring)}. \tag{7.9}$$

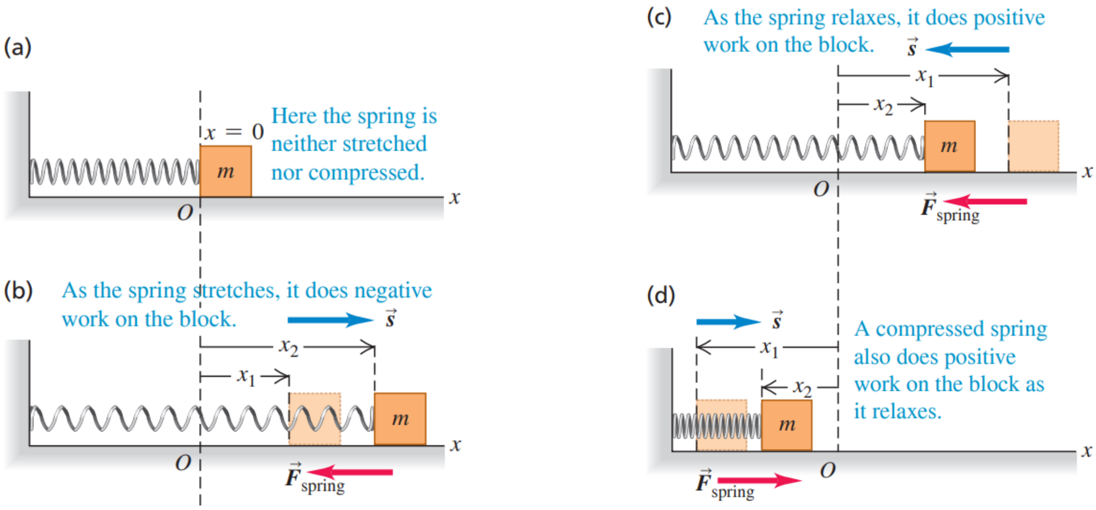


Figure 7.6 - Calculating the work done by a spring attached to a block on a horizontal surface. The quantity *x* is the extension or compression of the spring

The subscript "el" stands for *elastic*. When both x_1 and x_2 are positive and $x_2 > x_1$ (Fig. 7.6b), the spring does negative work on the block, which moves in the +*x*-direction while the spring pulls on it in the –*x*-direction. The spring stretches farther, and the block slows down. When both x_1 and x_2 are positive and $x_2 < x_1$ (Fig. 7.6c), the spring does positive work as it relaxes and the block speeds up. If the spring can be compressed as well as stretched, x_1, x_2, or both may be negative, but the expression for W_{el} is still valid. In Fig. 7.6d, both x_1 and x_2 are negative, but x_2 is less negative than x_1; the compressed spring does positive work as it relaxes, speeding the block up.

Just as for gravitational work, we can express Eq. (7.9) for the work done by the spring in terms of a quantity at the beginning and end of the displacement. This quantity is $\tfrac{1}{2}kx^2$, and we define it to be the **elastic potential energy**:

Force constant of spring

Elastic potential energy stored in a spring

$$U_{el} = \tfrac{1}{2}kx^2 \tag{7.10}$$

Elongation of spring ($x > 0$ if stretched, $x < 0$ if compressed)

Figure 7.7 is a graph of Eq. (7.10). As for all other energy and work quantities, the unit of U_{el} is the joule (J); to see this from Eq. (7.10), recall that the units of k are N/m and that $1\ \text{N}\cdot\text{m} = 1\ \text{J}$. We can now use Eq. (7.10) to rewrite Eq. (7.9) for the work W_{el} done by the spring:

Work done by the elastic force ... ... equals the negative of the change in elastic potential energy.

$$W_{el} = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 = U_{el,1} - U_{el,2} = -\Delta U_{el} \qquad (7.11)$$

Force constant of spring Initial and final elongations of spring

When a stretched spring is stretched farther, as in Fig. 7.6b, W_{el} is negative and U_{el} increases; more elastic potential energy is stored in the spring. When a stretched spring relaxes, as in Fig. 7.6c, x decreases, W_{el} is positive, and U_{el} decreases; the spring loses elastic potential energy. **Figure 7.7** shows that U_{el} is positive for both positive and negative x values; Eqs. (7.10) and (7.11) are valid for both cases. The more a spring is compressed or stretched, the greater its elastic potential energy.

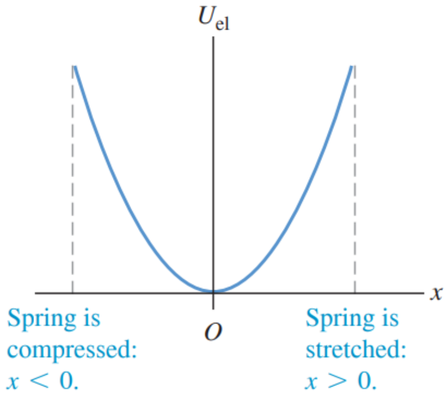


Figure 7.7 - The graph of elastic potential energy for an ideal spring is a parabola: $U_{el} = \frac{1}{2}kx^2$, where x is the extension or compression of the spring. Elastic potential energy U_{el} is never negative

CAUTION! **Gravitational potential energy vs. elastic potential energy.** An important difference between gravitational potential energy $U_{grav} = mgy$ and elastic potential energy $U_{el} = \frac{1}{2}kx^2$ is that we *cannot* choose $x = 0$ to be wherever we wish. In Eq. (7.10), $x = 0$ *must* be the position at which the spring is neither stretched nor compressed. At that position, both its elastic potential energy and the force that it exerts are zero.

The work–energy theorem says that $W_{tot} = K_2 - K_1$, no matter what kind of forces are acting on an object. If the elastic force is the *only* force that does work on the object, then

$$W_{tot} = W_{el} = U_{el,1} - U_{el,2}$$

and so

If only the elastic force does work, total mechanical energy is conserved:

Initial kinetic energy $K_1 = \frac{1}{2}mv_1^2$ Initial elastic potential energy $U_{el,1} = \frac{1}{2}kx_1^2$

$$K_1 + U_{el,1} = K_2 + U_{el,2} \qquad (7.12)$$

Final kinetic energy $K_2 = \frac{1}{2}mv_2^2$ Final elastic potential energy $U_{el,2} = \frac{1}{2}kx_2^2$

In this case the total mechanical energy $E = K + U_{el}$ – the sum of kinetic and *elastic* potential energies – is *conserved*. An example of this is the motion of the block in Fig. 7.6, provided the horizontal surface is frictionless so no force does work other than that exerted by the spring.

For Eq. (7.12) to be strictly correct, the ideal spring that we've been discussing must also be *massless*. If the spring has mass, it also has kinetic energy as the coils of the spring move back and forth. We can ignore the kinetic energy of the spring if its mass is much less than the mass m of the object attached to the spring. For instance, a typical automobile has a mass of 1200 kg or more. The springs in its suspension have masses of only a few kilograms, so their mass can be ignored if we want to study how a car bounces on its suspension.

Situations with Both Gravitational and Elastic Potential Energy

Equation (7.12) is valid when the only potential energy in the system is elastic potential energy. What happens when we have *both* gravitational and elastic forces, such as a block attached to the lower end of a vertically hanging spring? And what if work is also done by other forces that *cannot* be described in terms of potential energy, such as the force of air resistance on a moving block? Then the total work is the sum of the work done by the gravitational force (W_{grav}), the work done by the elastic force (W_{el}), and the work done by other forces (W_{other}): $W_{tot} = W_{grav} + W_{el} + W_{other}$. The work–energy theorem then gives

$$W_{grav} + W_{el} + W_{other} = K_2 - K_1 .$$

The work done by the gravitational force is $W_{grav} = U_{grav,1} - U_{grav,2}$ and the work done by the spring is $W_{el} = U_{el,1} - U_{el,2}$. Hence we can rewrite the work–energy theorem for this most general case as

$$K_1 + U_{grav,1} + U_{el,1} + W_{other} = K_2 + U_{grav,2} + U_{el,2} \text{ (valid in general)} \qquad (7.13)$$

or, equivalently,

General relationship for kinetic energy and potential energy:

Initial kinetic energy; Final kinetic energy

$$K_1 + U_1 + W_{other} = K_2 + U_2 \qquad (7.14)$$

Initial potential energy of all kinds; Work done by other forces (not associated with potential energy); Final potential energy of all kinds

where $U = U_{grav} + U_{el} = mgy + \frac{1}{2}kx^2$ is the *sum* of gravitational potential energy and elastic potential energy. We call U simply "the potential energy".

Equation (7.14) is *the most general statement*of the relationship among kinetic energy, potential energy, and work done by other forces. It says:

The work done by all forces other than the gravitational force or elastic force equals the change in the total mechanical energy $E = K + U$ of the system.

The "system" is made up of the object of mass m, the earth with which it interacts through the gravitational force, and the spring of force constant k.

If W_{other} is positive, $E = K + U$ increases; if W_{other} is negative, E decreases. If the gravitational and elastic forces are the *only* forces that do work on the object, then $W_{other} = 0$ and the total mechanical energy $E = K + U$ is conserved. [Compare Eq. (7.14) to Eqs. (7.6) and (7.7), which include gravitational potential energy but not elastic potential energy.]

Trampoline jumping involves transformations among kinetic energy, elastic potential energy, and gravitational potential energy. As the jumper descends through the air from the high point of the bounce, gravitational potential energy U_{grav} decreases and kinetic energy K increases. Once the jumper touches the trampoline, some of the total mechanical energy goes into elastic potential energy U_{el} stored in the trampoline's springs. At the lowest point of the trajectory (U_{grav} is minimum), the jumper comes to a momentary halt ($K = 0$) and the springs are maximally stretched (U_{el} is maximum). The springs then convert their energy back into K and U_{grav}, propelling the jumper upward.

PROBLEM-SOLVING STRATEGY

7.2 Problems Using Total Mechanical Energy II

Problem-Solving Strategy 7.1 (Section 7.1) is useful in solving problems that involve elastic forces as well as gravitational forces. The only new wrinkle is that the potential energy U now includes the elastic potential energy $U_{el} = \frac{1}{2}kx^2$, where x is the displacement of the spring from its unstretched length. The work done by the gravitational and elastic forces is accounted for by their potential energies; the work done by other forces, Wother, must still be included separately.

7.3 Conservative and Nonconservative Forces

In our discussions of potential energy we have talked about "storing" kinetic energy by converting it to potential energy, with the idea that we can retrieve it again as kinetic energy. For example, when you throw a ball up in the air, it slows down as kinetic energy is converted to gravitational potential energy. But on the way down the ball speeds up as potential energy is converted back to kinetic energy. If there is no air resistance, the ball is moving just as fast when you catch it as when you threw it.

Another example is a glider moving on a frictionless horizontal air track that runs into a spring bumper. The glider compresses the spring and then bounces back. If there is no friction, the glider ends up with the same speed and kinetic energy it had before the collision. Again, there is a two-way conversion from kinetic to potential energy and back. In both cases the total mechanical energy, kinetic plus potential, is constant or *conserved* during the motion.

Conservative Forces

A force that offers this opportunity of two-way conversion between kinetic and potential energies is called a **conservative force**. We have seen two examples of conservative forces: the gravitational force and the spring force. (Later in this book we'll study another conservative force, the electric force between charged objects). An essential feature of conservative forces is that their work is always *reversible*. Anything that we deposit in the energy "bank" can later be withdrawn without loss. Another important aspect of conservative forces is that if an object follows different paths from point 1 to point 2, the work done by a conservative force is the same for all of these paths (**Fig. 7.8**). For example, if an object stays close to the surface of the earth, the gravitational force $m\vec{g}$ is independent of height, and the work done by this force depends on only the change in height. If the object moves around a closed path, ending at the same height where it started, the total work done by the gravitational force is always zero.

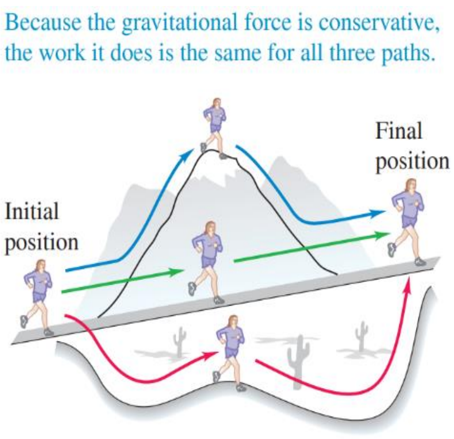


Figure 7.8 The work done by a conservative force such as gravity depends on only the endpoints of a path, not the specific path taken between those points

In summary, the work done by a conservative force has four properties:

1. It can be expressed as the difference between the initial and final values of a *potential-energy* function.
2. It is reversible.
3. It is independent of the path of the object and depends on only the starting and ending points.
4. When the starting and ending points are the same, the total work is zero.

When the *only* forces that do work are conservative forces, the total mechanical energy $E = K + U$ is constant.

Nonconservative Forces

Not all forces are conservative. Consider the friction force acting on the crate sliding on a ramp. When the crate slides up and then back down to the starting point, the total work done on it by the friction force is *not* zero. When the direction of motion reverses, so does the friction force, and friction does *negative* work in *both* directions. Friction also acts when a car with its brakes locked skids with decreasing speed (and decreasing kinetic energy). The lost kinetic energy can't be recovered by reversing the motion or in any other way, and total mechanical energy is *not* conserved. So there is *no* potential-energy function for the friction force.

In the same way, the force of fluid resistance (see Section 5.3) is not conservative. If you throw a ball up in the air, air resistance does negative work on the ball while it's rising *and* while it's descending. The ball returns to your hand with less speed and less kinetic energy than when it left, and there is no way to get back the lost mechanical energy.

A force that is not conservative is called a **nonconservative force**. The work done by a nonconservative force *cannot* be represented by a potential-energy function. Some nonconservative forces, like kinetic friction or fluid resistance, cause mechanical energy to be lost or dissipated; a force of this kind is called a **dissipative force**. There are also nonconservative forces that *increase* mechanical energy. The fragments of an exploding firecracker fly off with very large kinetic energy, thanks to a chemical reaction of gunpowder with oxygen. The forces unleashed by this reaction are nonconservative because the process is not reversible. (The fragments never spontaneously reassemble themselves into a complete firecracker!)

The Law of Conservation of Energy

Nonconservative forces cannot be represented in terms of potential energy. But we can describe the effects of these forces in terms of kinds of energy other than kinetic or potential energy. When a car with locked brakes skids to a stop, both the tires and the road surface become hotter. The energy associated with this change in the state of the materials is called **internal energy**. Raising the temperature of an object increases its internal energy; lowering the object's temperature decreases its internal energy.

Figure 7.9 - The battery pack in this radiocontrolled helicopter contains 2.4×10^4 J of electric energy. When this energy is used up, the internal energy of the battery pack decreases by this amount, so $\Delta U_{\text{int}} = -2.4 \times 10^4$ J. This energy can be converted to kinetic energy to make the rotor blades and helicopter go faster, or to gravitational potential energy to make the helicopter climb

To see the significance of internal energy, let's consider a block sliding on a rough surface. Friction does *negative* work on the block as it slides, and the change in internal energy of the block and surface (both of which get hotter) is *positive*. Careful experiments show that the increase in the internal energy is *exactly* equal to the absolute value of the work done by friction. In other words,

$$\Delta U_{\text{int}} = -W_{\text{other}},$$

where ΔU_{int} is the change in internal energy. We substitute this into Eq. (7.14)

$$K_1 + U_1 - \Delta U_{\text{int}} = K_2 + U_2 .$$

Writing $\Delta K = K_2 - K_1$ and $\Delta U = U_2 - U_1$, we can finally express this as

Law of conservation of energy:

$$\underbrace{\Delta K}_{\text{Change in kinetic energy}} + \underbrace{\Delta U}_{\text{Change in potential energy}} + \underbrace{\Delta U_{\text{int}}}_{\text{Change in internal energy}} = 0 \qquad (7.15)$$

This remarkable statement is the general form of the **law of conservation of energy.** In a given process, the kinetic energy, potential energy, and internal energy of a system may all change. But the *sum* of those changes is always zero. If there is a decrease in one form of energy, it is made up for by an increase in the other forms (**Fig. 7.9**). When we expand our definition of energy to include internal energy, Eq. (7.15) says: *Energy is never created or destroyed; it only changes form*. No exception to this rule has ever been found.

The concept of work has been banished from Eq. (7.15); instead, it suggests that we think purely in terms of the conversion of energy from one form to another. For example, when you throw a baseball straight up, you convert a portion of the internal energy of your molecules to kinetic energy of the baseball. This is converted to gravitational potential energy as the ball climbs and back to kinetic energy as the ball falls. If there is air resistance, part of the energy is used to heat up the air and the ball and increase their internal energy. Energy is converted back to the kinetic form as the ball falls. If you catch the ball in your hand, whatever energy was not lost to the air once again becomes internal energy; the ball and your hand are now warmer than they were at the beginning.

7.4 Force and Potential Energy

For the two kinds of conservative forces (gravitational and elastic) we have studied, we started with a description of the behavior of the *force* and derived from that an expression for the *potential energy*. For example, for an object with mass *m* in a uniform gravitational field, the gravitational force is $F_y = -mg$. We found that the corresponding potential energy is $U(y) = mgy$. The force that an ideal spring exerts on an object is $F_x = -kx$, and the corresponding potential-energy function is $U(x) = \frac{1}{2}kx^2$.

In studying physics, however, you'll encounter situations in which you are given an expression for the *potential energy* as a function of position and have to find the corresponding *force*. We'll see several examples of this kind when we study electric forces later in this book: It's often far easier to calculate the electric potential energy first and then determine the corresponding electric force afterward.

Here's how we find the force that corresponds to a given potential-energy expression. First let's consider motion along a straight line, with coordinate *x*. We denote the *x*-component of force, a function of *x*, by $F_x(x)$ and the potential energy as $U(x)$. This notation reminds us that both F_x and *U* are *functions* of *x*. Now we recall that in any displacement, the work *W* done by a conservative force equals the negative of the change ΔU in potential energy:

$$W = -\Delta U \,.$$

Let's apply this to a small displacement Δx. The work done by the force $F_x(x)$ during this displacement is approximately equal to $F_x(x)\Delta x$. We have to say "approximately" because $F_x(x)$ may vary a little over the interval Δx.
So

$$F_x(x)\Delta x = -\Delta U \quad \text{and} \quad F_x(x) = -\frac{\Delta U}{\Delta x}\,.$$

You can probably see what's coming. We take the limit as $\Delta x \to 0$; in this limit, the variation of F_x becomes negligible, and we have the exact relationship

Force from potential energy:
In **one-dimensional motion,** the value of a conservative force at point *x* ...

$$F_x(x) = -\frac{dU(x)}{dx} \qquad (7.16)$$

... is the negative of the derivative at *x* of the associated potential-energy function.

This result makes sense; in regions where *U*(*x*) changes most rapidly with *x* (that is, where $dU(x)/dx$ is large), the greatest amount of work is done during a given displacement, and this corresponds to a large force magnitude. Also, when $F_x(x)$ is in the positive *x*-direction, *U*(*x*) *decreases* with increasing *x*. So $F_x(x)$ and $dU(x)/dx$ should indeed have opposite signs. The physical meaning of Eq. (7.16) is that *a conservative force always acts to push the system toward lower potential energy*.

As a check, let's consider the function for elastic potential energy, $U(x) = \frac{1}{2}kx^2$. Substituting this into Eq. (7.16) yields

$$F_x(x) = -\frac{d}{dx}\left(\tfrac{1}{2}kx^2\right) = -kx\,.$$

which is the correct expression for the force exerted by an ideal spring (Fig. 7.10a). Similarly, for gravitational potential energy we have $U(y) = mgy$; taking care to change x to y for the choice of axis, we get $F_y = -dU/dy = -d(mgy)/dy = -mg$, which is the correct expression for gravitational force (Fig. 7.10b).

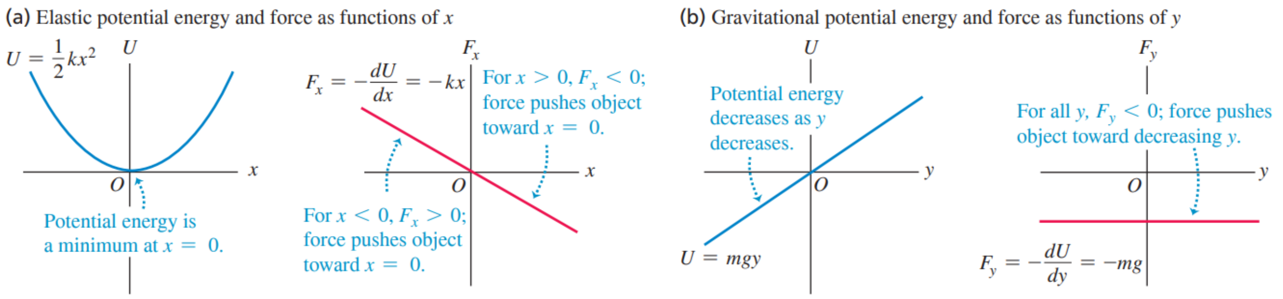


Figure 7.10 - A conservative force is the negative derivative of the corresponding potential energy

EXAMPLE 7.2 An electric force and its potential energy

An electrically charged particle is held at rest at the point $x = 0$; a second particle with equal charge is free to move along the positive x-axis. The potential energy of the system is $U(x) = C/x$, where C is a positive constant that depends on the magnitude of the charges. Derive an expression for the x-component of force acting on the movable particle as a function of its position.

IDENTIFY and SET UP

We are given the potential-energy function $U(x)$. We'll find the corresponding force function by using Eq. (7.16), $F_x(x) = -dU(x)/dx$.

EXECUTE

The derivative of 1/x with respect to x is $-1/x^2$. So for $x > 0$ the force on the movable charged particle is

$$F_x(x) = -\frac{dU(x)}{dx} = -C\left(-\frac{1}{x^2}\right) = \frac{C}{x^2}\,.$$

EVALUATE

The x-component of force is positive, corresponding to a repulsion between like electric charges. Both the potential energy and the force are very large when the particles are close together (small x), and both get smaller as the particles move farther apart (large x). The force pushes the movable particle toward large positive values of x, where the potential energy is lower.

KEYCONCEPT

For motion in one dimension, the force associated with a potential-energy function equals the negative derivative of that function with respect to position.

Force and Potential Energy in Three Dimensions

We can extend this analysis to three dimensions for a particle that may move in the x-, y-, or z-direction, or all at once, under the action of a conservative force that has components F_x, F_y, and F_z. Each component of force may be a function of the coordinates x, y, and z. The potential-energy function U is also a function of all three space coordinates. The potential-energy change ΔU when the particle moves a small distance Δx in the x-direction is again given by $-F_x \Delta x$; it doesn't depend on F_y and F_z, which represent force components that are perpendicular to the displacement and do no work. So we again have the approximate relationship

$$F_x = -\frac{\Delta U}{\Delta x}.$$

We determine the y- and z-components in exactly the same way:

$$F_y = -\frac{\Delta U}{\Delta y} \qquad F_z = -\frac{\Delta U}{\Delta z}.$$

To make these relationships exact, we take the limits $\Delta x \to 0$, $\Delta y \to 0$, and $\Delta z \to 0$ so that these ratios become derivatives. Because U may be a function of all three coordinates, we need to remember that when we calculate each of these derivatives, only one coordinate changes at a time. We compute the derivative of U with respect to x by assuming that y and z are constant and only x varies, and so on. Such a derivative is called a *partial derivative*. The usual notation for a partial derivative is $\partial U/\partial x$ and so on; the symbol ∂ is a modified *d*. So we write

Force from potential energy: In **three-dimensional motion,** the value at a given point of each component of a conservative force ...

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad F_z = -\frac{\partial U}{\partial z} \tag{7.17}$$

... is the negative of the partial derivative at that point of the associated potential-energy function.

We can use unit vectors to write a single compact vector expression for the force $\vec{F}$:

Force from potential energy: The vector value of a conservative force at a given point ...

$$\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k}\right) = -\vec{\nabla} U \tag{7.18}$$

... is the negative of the gradient at that point of the associated potential-energy function.

In Eq. (7.18) we take the partial derivative of U with respect to each coordinate, multiply by the corresponding unit vector, and then take the vector sum. This operation is called the **gradient** of U and is often abbreviated as $\vec{\nabla} U$.

As a check, let's substitute into Eq. (7.18) the function U = mgy for gravitational potential energy:

$$\vec{F} = -\vec{\nabla}(mgy) = -\left(\frac{\partial (mgy)}{\partial x}\hat{\imath} + \frac{\partial (mgy)}{\partial y}\hat{\jmath} + \frac{\partial (mgy)}{\partial z}\hat{k}\right) = (-mg)\,\hat{\jmath}.$$

This is just the familiar expression for the gravitational force.

7.5 Energy Diagrams

When a particle moves along a straight line under the action of a conservative force, we can get a lot of insight into its possible motions by looking at the graph of the potential-energy function $U(x)$. Figure 7.11a shows a glider with mass m that moves along the x-axis on an air track. The spring exerts on the glider a force with x-component $F_x = -kx$. Figure 7.11b is a graph of the corresponding potential-energy function $U(x) = \frac{1}{2}kx^2$. If the elastic force of the spring is the *only* horizontal force acting on the glider, the total mechanical energy $E = K + U$ is constant, independent of x. A graph of E as a function of x is thus a straight horizontal line. We use the term **energy diagram** for a graph like this, which shows both the potential-energy function $U(x)$ and the energy of the particle subjected to the force that corresponds to $U(x)$.

The vertical distance between the U and E graphs at each point represents the difference $E - U$, equal to the kinetic energy K at that point. We see that K is greatest at $x = 0$. It is zero at the values of x where the two graphs cross, labeled A and $-A$ in Fig. 7.11b. Thus the speed υ is greatest at $x = 0$, and it is zero at $x = \pm A$, the points of *maximum* possible displacement from $x = 0$ for a given value of the total energy E. The potential energy U can never be greater than the total energy E; if it were, K would be negative, and that's impossible. The motion is a back-and-forth oscillation between the points $x = A$ and $x = -A$.

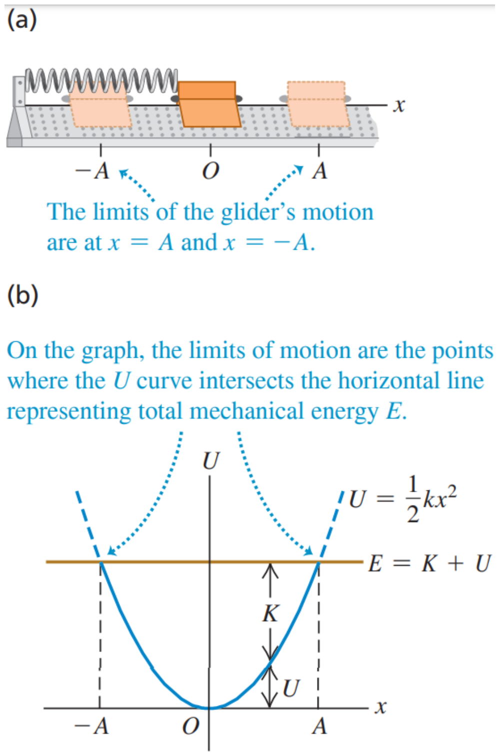


Figure 7.11 – (a) A glider on an air track. The spring exerts a force $F_x = -kx$. (b) The potential-energy function

From Eq. (7.16), at each point the force F_x on the glider is equal to the negative of the slope of the $U(x)$ curve: $F_x = -dU/dx$ (see Fig. 7.10a). When the particle is at $x = 0$, the slope and the force are zero, so this is an *equilibrium* position. When x is positive, the slope of the $U(x)$ curve is positive and the force F_x is negative, directed toward the origin. When x is negative, the slope is negative and F_x is positive, again directed toward the origin. Such a force is called a *restoring force*; when the glider is displaced to either side of $x = 0$, the force tends to "restore" it back to $x = 0$. An analogous situation is a marble rolling around in a round-bottomed bowl. We say that $x = 0$ is a point of **stable equilibrium**. More generally, *any minimum in a potential-energy curve is a stable equilibrium position*.

Figure 7.12a shows a hypothetical but more general potential-energy function $U(x)$. Figure 7.12b shows the corresponding force $F_x = -dU/dx$. Points x_1 and x_3 are stable equilibrium points. At both points, F_x is zero because the slope of the $U(x)$ curve is zero. When the particle is displaced to either side, the force pushes back toward the equilibrium point. The slope of the $U(x)$ curve is also zero at points x_2 and x_4, and these are also equilibrium points. But when the particle is displaced a little to the right of either point, the slope of the $U(x)$ curve becomes negative, corresponding to a positive F_x that tends to push the particle still farther from the point. When the particle is displaced a little to the left, F_x is negative, again pushing away from equilibrium. This is analogous to a marble rolling on the top of a bowling ball. Points x_2 and x_4 are called **unstable equilibrium** points; *any maximum in a potential-energy curve is an unstable equilibrium position*.

CAUTION! **Potential energy and the direction of a conservative force.** The direction of the force on an object is *not* determined by the sign of the potential energy U. Rather, it's the sign of $F_x = -dU/dx$ that matters. The physically significant quantity is the *difference* in the values of U between two points (Section 7.1), which is what the derivative $F_x = -dU/dx$ measures. You can always add a constant to the potential-energy function without changing the physics.

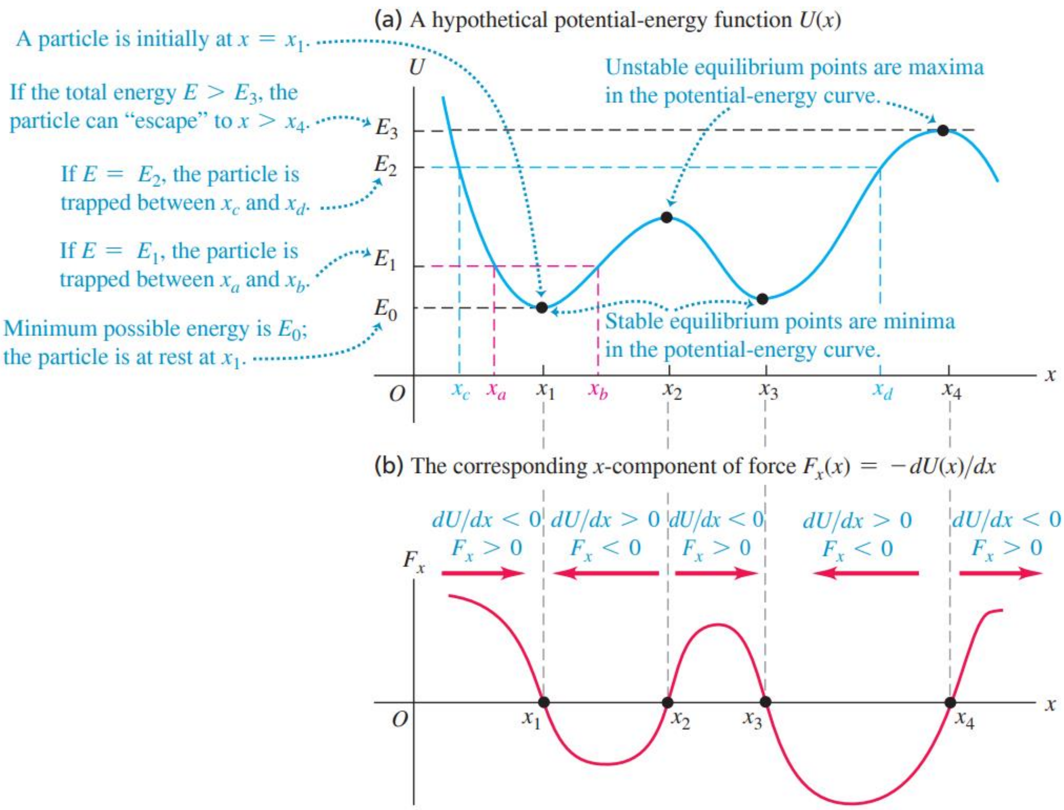
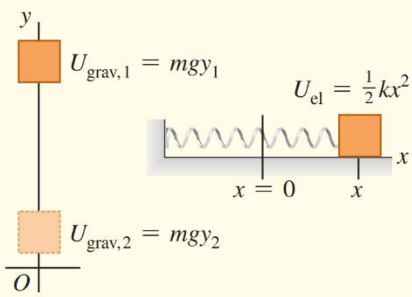


Figure 7.12The maxima and minima of a potential-energy function $U(x)$ correspond to points where $F_x = 0$

If the total energy is E_1 and the particle is initially near x_1, it can move only in the region between x_a and x_b determined by the intersection of the E_1 and U graphs (Fig. 7.12a). Again, U cannot be greater than E_1 because K can't be negative. We speak of the particle as moving in a *potential well*, and x_a and x_b are the *turning points* of the particle's motion (since at these points, the particle stops and reverses direction). If we increase the total energy to the level E_2, the particle can move over a wider range, from x_c to x_d. If the total energy is greater than E_3, the particle can "escape" and move to indefinitely large values of x. At the other extreme, E_0 represents the minimum total energy the system can have.

<table>
<tr><th colspan="3">CHAPTER 7: SUMMARY</th></tr>
<tr>
<td>Gravitational potential energy and elastic potential energy:The work done on a particle by a constant gravitational force can be represented as a change in the gravitational potential energy, $U_{\text{grav}} = mgy$. This energy is a shared property of the particle and the earth. A potential energy is also associated with the elastic force $F_x = -kx$ exerted by an ideal spring, where x is the amount of stretch or compression. The work done by this force can be represented as a change in the elastic potential energy of the spring, $U_{\text{el}} = \frac{1}{2}kx^2$</td>
<td>$W_{\text{grav}} = mgy_1 - mgy_2$
$= U_{\text{grav},1} - U_{\text{grav},2}$
$= -\Delta U_{\text{grav}}$
$W_{\text{el}} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2$
$= U_{\text{el},1} - U_{\text{el},2} = -\Delta U_{\text{el}}$</td>
<td>
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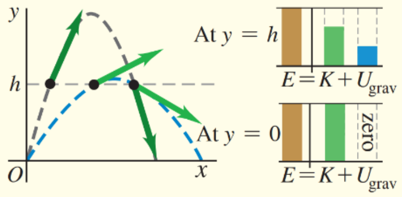
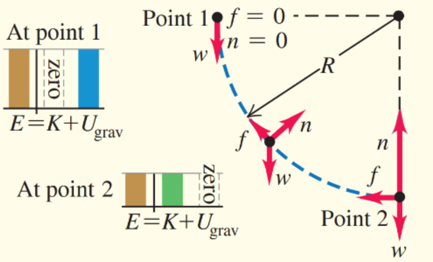
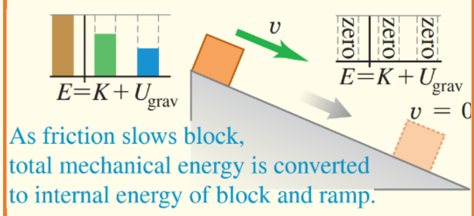
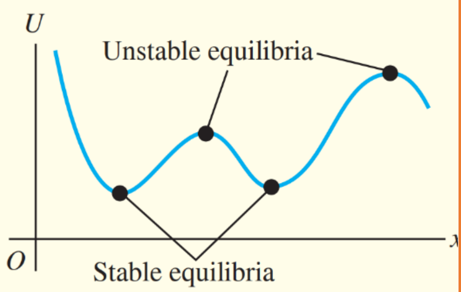
When total mechanical energy is conserved: The total potential energy U is the sum of the gravitational and elastic potential energies: $U = U_{grav} + U_{el}$. If no forces other than the gravitational and elastic forces do work on a particle, the sum of kinetic and potential energies is conserved. This sum $E = K + U$ is called the total mechanical energy	$K_1 + U_1 = K_2 + U_2$.	
When total mechanical energy is not conserved: When forces other than the gravitational and elastic forces do work on a particle, the work W_{other} done by these other forces equals the change in total mechanical energy (kinetic energy plus total potential energy)	$K_1 + U_1 + W_{other} = K_2 + U_2$.	
Conservative forces, nonconservative forces, and the law of conservation of energy: All forces are either conservative or nonconservative. A conservative force is one for which the work–kinetic energy relationship is completely reversible. The work of a conservative force can always be represented by a potential-energy function, but the work of a nonconservative force cannot. The work done by nonconservative forces manifests itself as changes in the internal energy of objects. The sum of kinetic, potential, and internal energies is always conserved	$\Delta K + \Delta U + \Delta U_{int} = 0$.	
Determining force from potential energy: For motion along a straight line, a conservative force $F_x(x)$ is the negative derivative of its associated potential-energy function U. In three dimensions, the components of a conservative force are negative partial derivatives of U	$F_x(x) = -\frac{dU(x)}{dx}$ $F_x = -\frac{\partial U}{\partial x}$ $F_y = -\frac{\partial U}{\partial y}$ $F_z = -\frac{\partial U}{\partial z}$ $\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right)$ $= -\vec{\nabla} U$	

8 MOMENTUM, IMPULSE, AND COLLISIONS

Many questions involving forces can't be answered by directly applying Newton's second law, $\sum \vec{F} = m\vec{a}$. For example, when a truck collides head-on with a compact car, what determines which way the wreckage moves after the collision? In playing pool, how do you decide how to aim the cue ball in order to knock the eight ball into the pocket? And when a meteorite collides with the earth, how much of the meteorite's kinetic energy is released in the impact?

All of these questions involve forces about which we know very little: the forces between the car and the truck, between the two pool balls, or between the meteorite and the earth. Remarkably, we'll find in this chapter that we don't have to know *anything* about these forces to answer questions of this kind!

Our approach uses two new concepts, *momentum* and *impulse*, and a new conservation law, *conservation of momentum*. This conservation law is every bit as important as the law of conservation of energy. The law of conservation of momentum is valid even in situations in which Newton's laws are inadequate, such as objects moving at very high speeds (near the speed of light) or objects on a very small scale (such as the constituents of atoms). Within the domain of Newtonian mechanics, conservation of momentum enables us to analyze many situations that would be very difficult if we tried to use Newton's laws directly. Among these are *collision* problems, in which two objects collide and can exert very large forces on each other for a short time. We'll also use momentum ideas to solve problems in which an object's mass changes as it moves, including the important special case of a rocket (which loses mass as it expends fuel).

8.1 Momentum and Impulse

In Section 6.2 we re-expressed Newton's second law for a particle, $\sum \vec{F} = m\vec{a}$, in terms of the work–energy theorem. This theorem helped us tackle a great number of problems and led us to the law of conservation of energy. Let's return to $\sum \vec{F} = m\vec{a}$ and see yet another useful way to restate this fundamental law.

Newton's Second Law in Terms of Momentum

Consider a particle of constant mass *m*. Because $\vec{a} = d\vec{v}/dt$, we can write Newton's second law for this particle as

$$\sum \vec{F} = m\frac{d\vec{v}}{dt} = \frac{d}{dt}\left(m\vec{v}\right). \tag{8.1}$$

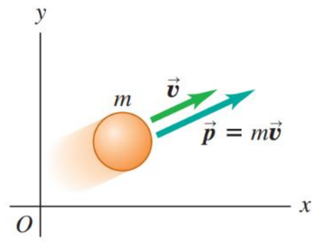
We can move the mass m inside the derivative because it is constant. Thus Newton's second law says that the net external force $\sum \vec{F}$ acting on a particle equals the time rate of change of the product of the particle's mass and velocity. We'll call this product the **momentum**, or **linear momentum**, of the particle:

Momentum of a particle (a vector quantity) $\quad \vec{\boldsymbol{p}} = m\vec{\boldsymbol{v}}\quad$ Particle mass; Particle velocity (8.2)

The greater the mass *m* and speed v of a particle, the greater is its magnitude of momentum mv. Keep in mind that momentum is a *vector* quantity with the same direction as the particle's velocity (**Fig. 8.1**). A car driving north at 20 m/s and an identical car driving east at 20 m/s have the same *magnitude* of momentum (mv) but different momentum *vectors* ($m\vec{v}$) because their directions are different.

We often express the momentum of a particle in terms of its components. If the particle has velocity components υ_x, υ_y, and υ_z, then its momentum components p_x, p_y, and p_z (which we also call the *x-momentum*, *y-momentum*, and *z-momentum*) are

$$p_x = m\upsilon_x \qquad p_y = m\upsilon_y \qquad p_z = m\upsilon_z . \tag{8.3}$$



Momentum $\vec{p}$ is a vector quantity; a particle's momentum has the same direction as its velocity $\vec{\upsilon}$.

Figure 8.1 - The velocity and momentum vectors of a particle

These three component equations are equivalent to Eq. (8.2).

The units of the magnitude of momentum are units of mass times speed; the SI units of momentum are $\text{kg} \cdot \text{m/s}$. The plural of momentum is "momenta." Let's now substitute the definition of momentum, Eq. (8.2), into Eq. (8.1):

Newton's second law in terms of momentum: The net external force acting on a particle ...

$$\sum \vec{F} = \frac{d\vec{p}}{dt} \tag{8.4}$$

... equals the rate of change of the particle's momentum.

The net external force (vector sum of all forces) acting on a particle equals the time rate of change of momentum of the particle. This, not $\sum \vec{F} = m\vec{a}$, is the form in which Newton originally stated his second law (although he called momentum the "quantity of motion"). This law is valid only in inertial frames of reference (see Section 4.2). As Eq. (8.4) shows, a rapid change in momentum requires a large net external force, while a gradual change in momentum requires a smaller net external force.

The Impulse–Momentum Theorem

Both a particle's momentum $\vec{p} = m\vec{\upsilon}$ and its kinetic energy $K = \frac{1}{2}m\upsilon^2$ depend on the mass and velocity of the particle. What is the fundamental difference between these two quantities? A purely mathematical answer is that momentum is a vector whose magnitude is proportional to speed, while kinetic energy is a scalar proportional to the speed squared. But to see the *physical* difference between momentum and kinetic energy, we must first define a quantity closely related to momentum called *impulse*.

Let's first consider a particle acted on by a *constant* net external force $\sum \vec{F}$ during a time interval Δt from t_1 to t_2. The **impulse** of the net external force, denoted by $\vec{J}$, is defined to be the product of the net external force and the time interval:

Impulse of a **constant net external force**

$$\vec{J} = \sum \vec{F}(t_2 - t_1) = \sum \vec{F}\Delta t \tag{8.5}$$

Constant net external force

Time interval over which net external force acts

Impulse is a vector quantity; its direction is the same as the net external force $\sum \vec{F}$. The SI unit of impulse is the newton-second ($\text{N} \cdot \text{s}$). Because $1\ \text{N} = 1\ \text{kg} \cdot \text{m/s}^2$, an alternative set of units for impulse is $\text{kg} \cdot \text{m/s}$, the same as for momentum.

To see what impulse is good for, let's go back to Newton's second law as restated in terms of momentum, Eq. (8.4). If the net external force $\sum \vec{F}$ is constant, then $d\vec{p}/dt$ is also constant. In that case, $d\vec{p}/dt$ is equal to the *total* change in momentum $\vec{p}_2 - \vec{p}_1$ during the time interval $t_2 - t_1$, divided by the interval:

$$\sum \vec{F} = \frac{\vec{p}_2 - \vec{p}_1}{t_2 - t_1} .$$

Multiplying this equation by $(t_2 - t_1)$, we have

$$\sum \vec{F}\,(t_2 - t_1) = \vec{p}_2 - \vec{p}_1 \,.$$

Comparing with Eq. (8.5), we end up with

Impulse–momentum theorem: The impulse of the net external force on a particle during a time interval equals the change in momentum of that particle during that interval:

$$\underbrace{\vec{J}}_{\text{Impulse of net external force over a time interval}} = \underbrace{\vec{p}_2}_{\text{Final momentum}} - \underbrace{\vec{p}_1}_{\text{Initial momentum}} = \underbrace{\Delta\vec{p}}_{\text{Change in momentum}} \qquad (8.6)$$

The impulse–momentum theorem also holds when forces are not constant. To see this, we integrate both sides of Newton's second law $\sum \vec{F} = d\vec{p}/dt$ over time between the limits t_1 and t_2:

$$\int_{t_1}^{t_2} \sum \vec{F}\, dt = \int_{t_1}^{t_2} \frac{d\vec{p}}{dt}\, dt = \int_{p_1}^{p_2} d\vec{p} = \vec{p}_2 - \vec{p}_1 \,.$$

We see from Eq. (8.6) that the integral on the left is the impulse of the net external force:

$$\underbrace{\vec{J}}_{\substack{\textbf{Impulse of a general net external force}\\ \text{(either constant or varying)}}} = \int_{t_1 \,(\text{Lower limit = initial time})}^{t_2 \,(\text{Upper limit = final time})} \underbrace{\sum \vec{F}\, dt}_{\text{Time integral of net external force}} \qquad (8.7)$$

If the net external force $\sum \vec{F}$ is constant, the integral in Eq. (8.7) reduces to Eq. (8.5). We can define an *average* net external force $\vec{F}_{\text{av}}$ such that even when $\sum \vec{F}$ is not constant, the impulse $\vec{J}$ is given by

$$\vec{J} = \vec{F}_{\text{av}}\,(t_2 - t_1)\,. \qquad (8.8)$$

When $\sum \vec{F}$ is constant, $\sum \vec{F} = \vec{F}_{\text{av}}$ and Eq. (8.8) reduces to Eq. (8.5).

Figure 8.2a shows the x-component of net external force $\sum F_x$ as a function of time during a collision. This might represent the force on a football that is in contact with a player's foot from time t_1 to t_2. The x-component of impulse during this interval is represented by the red area under the curve between t_1 and t_2. This area is equal to the green rectangular area bounded by t_1, t_2, and $(F_{\text{av}})_x$, so $(F_{\text{av}})_x (t_2 - t_1)$ is equal to the impulse of the actual time-varying force during the same interval. Note that a large force acting for a short time can have the same impulse as a smaller force acting for a longer time if the areas under the force–time curves are the same (Fig. 8.2b). A small force acting for a relatively long time (as when you land with your legs bent) has the same effect as a larger force acting for a short time (as when you land stiff-legged). Automotive air bags use the same principle.

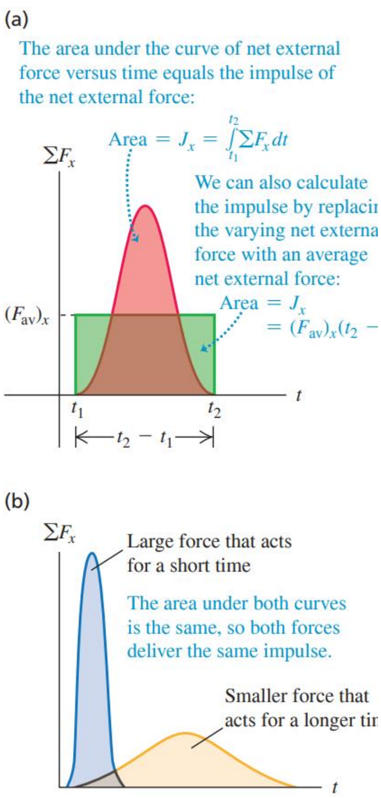


Figure 8.2 - The meaning of the area under a graph of $\sum F_x$ versus t

Both impulse and momentum are vector quantities, and Eqs. (8.5)–

(8.8) are vector equations. It's often easiest to use them in component form:

$$
\begin{aligned}
J_x &= \int_{t_1}^{t_2} \sum F_x dt = \left(F_{\text{av}}\right)_x \left(t_2 - t_1\right) = p_{2x} - p_{1x} = m\upsilon_{2x} - m\upsilon_{1x} \\
J_y &= \int_{t_1}^{t_2} \sum F_y dt = \left(F_{\text{av}}\right)_y \left(t_2 - t_1\right) = p_{2y} - p_{1y} = m\upsilon_{2y} - m\upsilon_{1y}
\end{aligned}
\qquad (8.9)
$$

and similarly for the *z*-component.

Momentum and Kinetic Energy Compared

We can now see the fundamental difference between momentum and kinetic energy. The impulse–momentum theorem, $\vec{J} = \vec{p}_2 - \vec{p}_1$, says that changes in a particle's momentum are due to impulse, which depends on the *time* over which the net external force acts. By contrast, the work–energy theorem, $W_{\text{tot}} = K_2 - K_1$, tells us that kinetic energy changes when work is done on a particle; the total work depends on the *distance* over which the net external force acts.

Let's consider a particle that starts from rest at t_1 so that $\vec{\upsilon}_1 = 0$. Its initial momentum is $\vec{p}_1 = m\vec{\upsilon}_1 = 0$, and its initial kinetic energy is $K_1 = \frac{1}{2} m\upsilon_1^2 = 0$. Now let a constant net external force equal to $\vec{F}$ act on that particle from time t_1 until time t_2. During this interval, the particle moves a distance s in the direction of the force. From Eq. (8.6), the particle's momentum at time t_2 is

$$\vec{p}_2 = \vec{p}_1 + \vec{J} = \vec{J} \,,$$

where $\vec{J} = \vec{F}\left(t_2 - t_1\right)$ is the impulse that acts on the particle. So *the momentum of a particle equals the impulse that accelerated it from rest to its present speed*; impulse is the product of the net external force that accelerated the particle and the time required for the acceleration. By comparison, the kinetic energy of the particle at t_2 is $K_2 = W_{\text{tot}} = Fs$, the total *work* done on the particle to accelerate it from rest. The total work is the product of the net external force and the *distance* required to accelerate the particle.

Here's an application of the distinction between momentum and kinetic energy. Which is easier to catch: a 0.50 kg ball moving at 4.0 m/s or a 0.10 kg ball moving at 20 m/s? Both balls have the same magnitude of momentum, $p = m\upsilon = \left(0.50\text{ kg}\right)\times\left(4.0\text{ m/s}\right) = \left(0.10\text{ kg}\right)\left(20\text{ m/s}\right) = 2.0\text{ kg}\cdot\text{m/s}$. However, the two balls have different values of kinetic energy $K = \frac{1}{2} m\upsilon^2$: The large, slow-moving ball has K = 4.0 J, while the small, fast-moving ball has K = 20 J. Since the momentum is the same for both balls, both require the same *impulse* to be brought to rest. But stopping the 0.10 kg ball with your hand requires five times more *work* than stopping the 0.50 kg ball because the smaller ball has five times more kinetic energy. For a given force that you exert with your hand, it takes the same amount of time (the duration of the catch) to stop either ball, but your hand and arm will be pushed back five times farther if you choose to catch the small, fast-moving ball. To minimize arm strain, you should choose to catch the 0.50 kg ball with its lower kinetic energy.

Both the impulse–momentum and work–energy theorems rest on the foundation of Newton's laws. They are *integral* principles, relating the motion at two different times separated by a finite interval. By contrast, Newton's second law itself (in either of the forms $\sum \vec{F} = m\vec{a}$ or $\sum \vec{F} = d\vec{p}/dt$) is a *differential* principle that concerns the rate of change of velocity or momentum at each instant.

8.2 Conservation of Momentum

The concept of momentum is particularly important in situations in which we have two or more objects that *interact*. To see why, let's consider first an idealized system of two objects that interact with each other but not with anything else—for example, two astronauts who touch each other as they float freely in the zero-gravity environment of outer space (**Fig. 8.3**). Think of the astronauts as particles. Each particle exerts a force on the other; according to Newton's third law, the two forces are always equal in magnitude and opposite in direction. Hence, the *impulses* that act on the two particles are equal in magnitude and opposite in direction, as are the changes in momentum of the two particles.

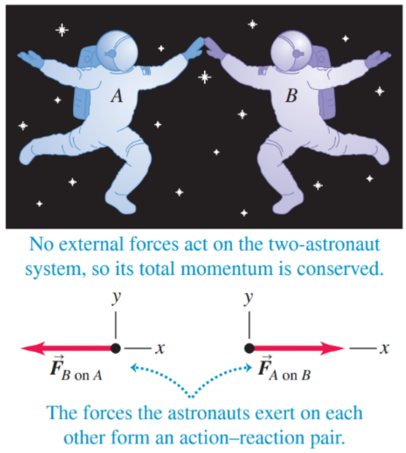


Figure 8.3 - Two astronauts push each other as they float freely in the zero-gravity environment of space

Let's go over that again with some new terminology. For any system, the forces that the particles of the system exert on each other are called **internal forces**. Forces exerted on any part of the system by some object outside it are called **external forces**. For the system shown in Fig. 8.3, the internal forces are $\vec{F}_{B \text{ on } A}$, exerted by particle B on particle A, and $\vec{F}_{A \text{ on } B}$, exerted by particle A on particle B. There are no external forces; when this is the case, we have an **isolated system.**

The net external force on particle A is $\vec{F}_{B \text{ on } A}$ and the net external force on particle B is $\vec{F}_{A \text{ on } B}$, so from Eq. (8.4) the rates of change of the momenta of the two particles are

$$\vec{F}_{B \text{ on } A} = \frac{d\vec{p}_A}{dt} \qquad \vec{F}_{A \text{ on } B} = \frac{d\vec{p}_B}{dt}. \tag{8.10}$$

The momentum of each particle changes, but these changes are related to each other by Newton's third law: Forces $\vec{F}_{B \text{ on } A}$ and $\vec{F}_{A \text{ on } B}$ are always equal in magnitude and opposite in direction. That is, $\vec{F}_{B \text{ on } A} = -\vec{F}_{A \text{ on } B}$, so $\vec{F}_{B \text{ on } A} + \vec{F}_{A \text{ on } B} = 0$. Adding together the two equations in Eq. (8.10), we have

$$\vec{F}_{B \text{ on } A} + \vec{F}_{A \text{ on } B} = \frac{d\vec{p}_A}{dt} + \frac{d\vec{p}_B}{dt} = \frac{d\left(\vec{p}_A + \vec{p}_B\right)}{dt} = 0. \tag{8.11}$$

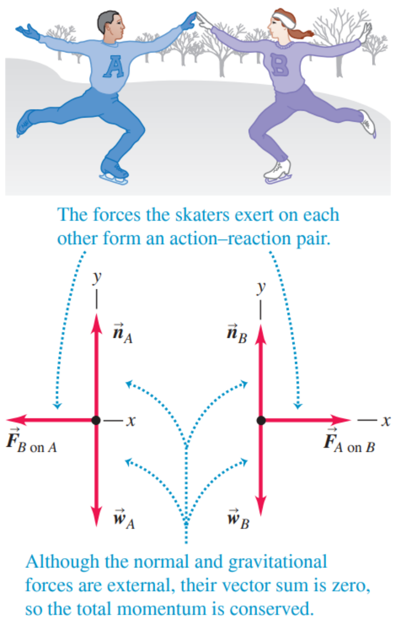


Figure 8.4 - Two ice skaters push each other as they skate on a frictionless, horizontal surface. (Compare to Fig. 8.3)

The rates of change of the two momenta are equal and opposite, so the rate of change of the vector sum $\vec{p}_A + \vec{p}_B$ is zero. We define the **total momentum** $\vec{P}$ of the system of two particles as the vector sum of the momenta of the individual particles; that is,

$$\vec{P} = \vec{p}_A + \vec{p}_B. \tag{8.12}$$

Then Eq. (8.11) becomes

$$\vec{F}_{B \text{ on } A} + \vec{F}_{A \text{ on } B} = \frac{d\vec{P}}{dt} = 0. \tag{8.13}$$

The time rate of change of the total momentum P S is zero. Hence the total momentum of the system is constant, even though the individual momenta of the particles that make up the system can change.

If external forces are also present, they must be included on the left side of Eq. (8.13) along with the internal forces. Then the total momentum is, in general, not constant. But if the vector sum of the external forces is zero, as in Fig. 8.4, these forces have no effect on the left side of Eq. (8.13), and dP S >dt is again zero. Thus we have the following general result:

CONSERVATION OF MOMENTUM: If the vector sum of the external forces on a system is zero, the total momentum of the system is constant.

This is the simplest form of the **principle of conservation of momentum**. This principle is a direct consequence of Newton's third law. What makes this principle useful is that it doesn't depend on the detailed nature of the internal forces that act between members of the system. This means that we can apply conservation of momentum even if (as is often the case) we know very little about the internal forces. We have used Newton's second law to derive this principle, so we have to be careful to use it only in inertial frames of reference.

We can generalize this principle for a system that contains any number of particles *A, B, C*, … interacting only with one another, with total momentum

Total momentum of a system of particles *A*, *B*, *C*, ...

$$\vec{P} = \vec{p}_A + \vec{p}_B + \cdots = m_A\vec{v}_A + m_B\vec{v}_B + \cdots \qquad (8.14)$$

... equals vector sum of momenta of all particles in the system.

We make the same argument as before: The total rate of change of momentum of the system due to each action–reaction pair of internal forces is zero. Thus the total rate of change of momentum of the entire system is zero whenever the vector sum of the external forces acting on it is zero. The internal forces can change the momenta of individual particles but not the *total* momentum of the system.

CAUTION! **Conservation of momentum means conservation of its components.** When you apply the conservation of momentum to a system, remember that momentum is a *vector* quantity. Hence you must use vector addition to compute the total momentum of a system (Fig. 8.5). Using components is usually the simplest method. If p_{Ax}, p_{Ay}, and p_{Az} are the components of momentum of particle A, and similarly for the other particles, then Eq. (8.14) is equivalent to the component equations

$$P_x = p_{Ax} + p_{Bx} + \ldots, \quad P_y = p_{Ay} + p_{By} + \ldots, \quad P_z = p_{Az} + p_{Bz} + \ldots \quad (8.15)$$

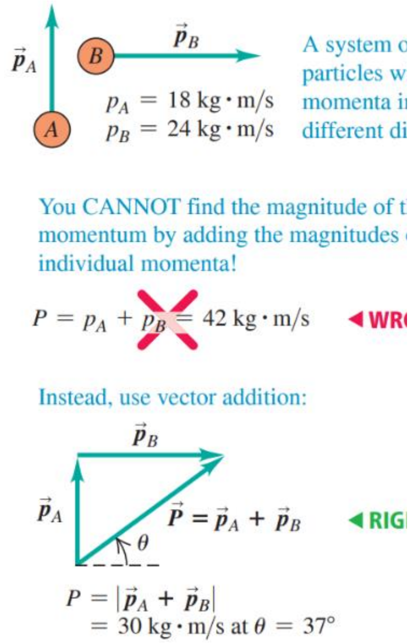


Figure 8.5 - When applying conservation of momentum, remember that momentum is a vector quantity!

If the vector sum of the external forces on the system is zero, then P_x, P_y, and P_z are all constant.

In some ways the principle of conservation of momentum is more general than the principle of conservation of total mechanical energy. For example, total mechanical energy is conserved only when the internal forces are *conservative* – that is, when the forces allow two-way conversion between kinetic and potential energies. But conservation of momentum is valid even when the internal forces are *not* conservative. In this chapter we'll analyze situations in which both momentum and total mechanical energy are conserved, and others in which only momentum is conserved. These two principles play a fundamental role in all areas of physics, and we'll encounter them throughout our study of physics.

PROBLEM-SOLVING STRATEGY

8.1 Conservation of Momentum

IDENTIFY *the relevant concepts:*

Confirm that the vector sum of the external forces acting on the system of particles is zero. If it isn't zero, you can't use conservation of momentum.

SET UP *the problem*:

- Treat each object as a particle. Draw "before" and "after" sketches, including velocity vectors. Assign algebraic symbols to each magnitude, angle, and component. Use letters to label each particle and subscripts 1 and 2 for "before" and "after" quantities. Include any given values.
- Define a coordinate system and show it in your sketches; define the positive direction for each axis.
- Identify the target variables.

EXECUTE *the solution*:

- Write an equation in symbols equating the total initial and final x-components of momentum, using $p_x = m\upsilon_x$ for each particle. Write a corresponding equation for the *y*-components. Components can be positive or negative, so be careful with signs!
- In some problems, energy considerations (discussed in Section 8.4) give additional equations relating the velocities.
- Solve your equations to find the target variables.

EVALUATE *your answer:*

Does your answer make physical sense? If your target variable is a certain object's momentum, check that the direction of the momentum is reasonable.

8.3 Momentum Conservation and Collisions

To most people the term "collision" is likely to mean some sort of automotive disaster. We'll broaden the meaning to include any strong interaction between objects that lasts a relatively short time. So we include not only car accidents but also balls colliding on a billiard table, neutrons hitting atomic nuclei in a nuclear reactor, and a close encounter of a spacecraft with the planet Saturn.

If the forces between the colliding objects are much larger than any external forces, as is the case in most collisions, we can ignore the external forces and treat the objects as an *isolated* system. Then momentum is conserved and the total momentum of the system has the same value before and after the collision. Two cars colliding at an icy intersection provide a good example. Even two cars colliding on dry pavement can be treated as an isolated system during the collision if the forces between the cars are much larger than the friction forces of pavement against tires.

Elastic and Inelastic Collisions

If the forces between the objects are also *conservative*, so no mechanical energy is lost or gained in the collision, the total *kinetic* energy of the system is the same after the collision as before. Such a collision is called an **elastic collision**. A collision between two marbles or two billiard balls is almost completely elastic. **Figure 8.6** shows a model for an elastic collision. When the gliders collide, their springs are momentarily compressed and some of the original kinetic energy is momentarily converted to elastic potential energy. Then the gliders bounce apart, the springs expand, and this potential energy is converted back to kinetic energy.

A collision in which the total kinetic energy after the collision is *less* than before the collision is called an **inelastic collision**. A meatball landing on a plate of spaghetti and a bullet embedding itself in a block of wood are examples of inelastic collisions. An inelastic collision in which the colliding objects stick together and move as one object after the collision is called a **completely inelastic collision**. **Figure 8.7** shows an example; we have replaced the spring bumpers in Fig. 8.6 with Velcro®, which sticks the two objects together.

CAUTION! **An inelastic collision doesn't have to be completely inelastic.** Inelastic collisions include many situations in which the objects do not stick. If two cars bounce off each other in a "fender bender," the work done to deform the fenders cannot be recovered as kinetic energy of the cars, so the collision is inelastic.

Remember this rule: **In any collision in which external forces can be ignored, momentum is conserved and the total momentum before equals the total momentum after; in elastic collisions *only*, the total kinetic energy before equals the total kinetic energy after**.

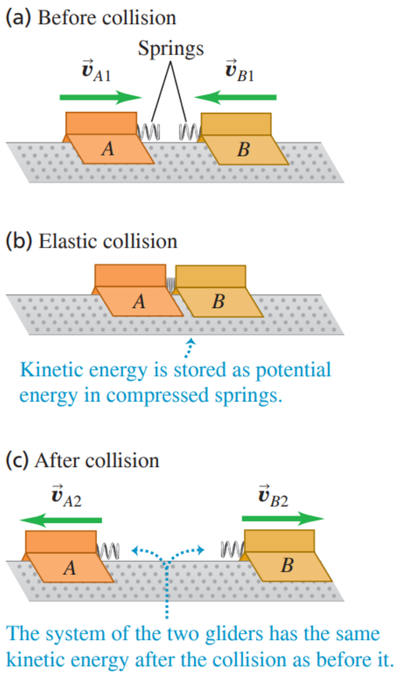


Figure 8.6 - Two gliders undergoing an elastic collision on a frictionless surface. Each glider has a steel spring bumper that exerts a conservative force on the other glider

Completely Inelastic Collisions

Let's look at what happens to momentum and kinetic energy in a *completely* inelastic collision of two objects (*A* and *B*), as in Fig. 8.7. Because the two objects stick together after the collision, they have the same final velocity $\vec{\upsilon}_2$:

$$\vec{\upsilon}_{A2} = \vec{\upsilon}_{B2} = \vec{\upsilon}_2 .$$

Conservation of momentum gives the relationship

$$m_A\vec{\upsilon}_{A1} + m_B\vec{\upsilon}_{B1} = (m_A + m_B)\vec{\upsilon}_2 \text{ (completely inelastic collision)}. \tag{8.16}$$

If we know the masses and initial velocities, we can compute the common final velocity $\vec{\upsilon}_2$.

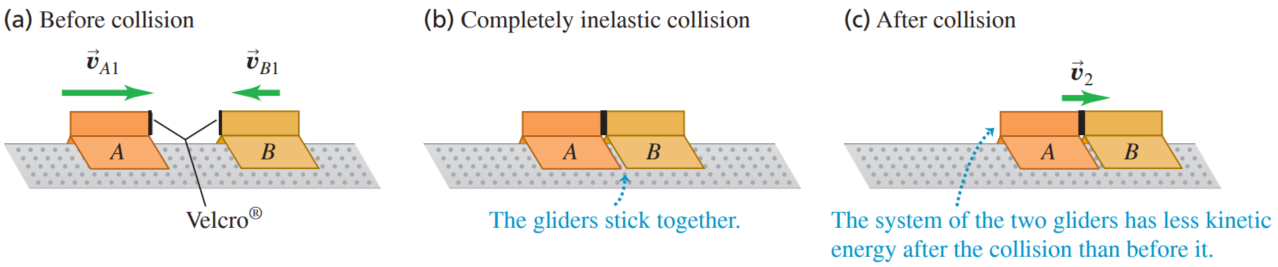


Figure 8.7 – Two gliders undergoing a completely inelastic collision. The spring bumpers on the gliders are replaced by Velcro, so the gliders stick together after collision

Suppose, for example, that an object with mass m_A and initial *x*-component of velocity υ_{A1x} collides inelastically with an object with mass m_B that is initially at rest ($\upsilon_{B1x} = 0$). From Eq. (8.16) the common *x*-component of velocity υ_{2x} of both objects after the collision is

$$\upsilon_{2x} = \frac{m_A}{m_A + m_B}\upsilon_{A1x} \quad \text{(completely inelastic collision, } B \text{ initially at rest)}. \tag{8.17}$$

Let's verify that the total kinetic energy after this completely inelastic collision is less than before the collision. The motion is purely along the x-axis, so the kinetic energies K_1 and K_2 before and after the collision, respectively, are

$$K_1 = \tfrac{1}{2} m_A \upsilon_{A1x}^2$$

$$K_2 = \tfrac{1}{2}\left(m_A + m_B\right)\upsilon_{2x}^2 = \tfrac{1}{2}\left(m_A + m_B\right)\left(\frac{m_A}{m_A + m_B}\right)^2 \upsilon_{A1x}^2 .$$

The ratio of final to initial kinetic energy is

$$\frac{K_2}{K_1} = \frac{m_A}{m_A + m_B} \quad \text{(completely inelastic collision, } B \text{ initially at rest)}. \tag{8.18}$$

The right side is always less than unity because the denominator is always greater than the numerator. Even when the initial velocity of m_B is not zero, the kinetic energy after a completely inelastic collision is always less than before.

Please note: Don't memorize Eq. (8.17) or (8.18)! We derived them only to prove that kinetic energy is always lost in a completely inelastic collision.

Classifying Collisions

It's important to remember that we can classify collisions according to energy considerations (Fig. 8.8). A collision in which kinetic energy is conserved is called elastic. (We'll explore this type in more depth in the next section). A collision in which the total kinetic energy decreases is called *inelastic*. When the two objects have a common final velocity, we say that the collision is *completely inelastic*. There are also cases in which the final kinetic energy is *greater* than the initial value. Rifle recoil is an example.

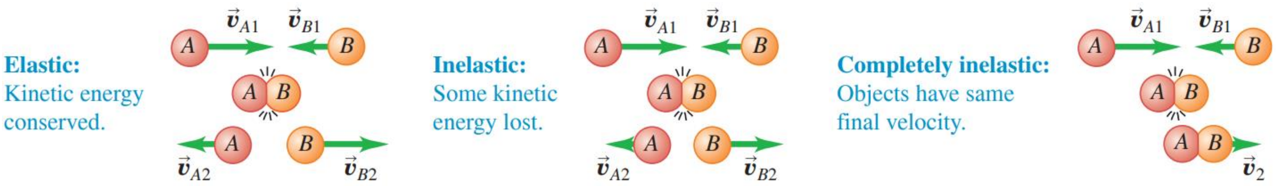


Figure 8.8 - Collisions are classified according to energy considerations

Finally, we emphasize again that we can typically use momentum conservation for collisions even when external forces are acting on the system. That's because the net external force acting on the colliding objects is typically small in comparison with the internal forces during the collision.

8.4 Elastic Collisions

We saw in Section 8.3 that an *elastic collision* in an isolated system is one in which kinetic energy (as well as momentum) is conserved. Elastic collisions occur when the forces between the colliding objects are *conservative*. When two billiard balls collide, they squash a little near the surface of contact, but then they spring back. Some of the kinetic energy is stored temporarily as elastic potential energy, but at the end it is reconverted to kinetic energy.

Let's look at a *one-dimensional* elastic collision between two objects A and B, in which all the velocities lie along the same line. We call this line the x-axis, so each momentum and velocity has only an x-component. We call the x-velocities before the collision υ_{A1x} and υ_{B1x}, and those after the collision υ_{A2x} and υ_{B2x}. From conservation of kinetic energy we have

$$\tfrac{1}{2}m_A\upsilon_{A1x}^2+\tfrac{1}{2}m_B\upsilon_{B1x}^2=\tfrac{1}{2}m_A\upsilon_{A2x}^2+\tfrac{1}{2}m_B\upsilon_{B2x}^2,$$

and conservation of momentum gives

$$m_A\upsilon_{A1x}+m_B\upsilon_{B1x}=m_A\upsilon_{A2x}+m_B\upsilon_{B2x}.$$

If the masses m_A and m_B and the initial velocities υ_{A1x} and υ_{B1x} are known, we can solve these two equations to find the two final velocities υ_{A2x} and υ_{B2x}.

Elastic Collisions, One Object Initially at Rest

The general solution to the above equations is a little complicated, so we'll concentrate on the particular case in which object *B* is at rest before the collision (so $\upsilon_{B1x}=0$). Think of object *B* as a target for object *A* to hit. Then the kinetic energy and momentum conservation equations are, respectively,

$$\tfrac{1}{2}m_A\upsilon_{A1x}^2=\tfrac{1}{2}m_A\upsilon_{A2x}^2+\tfrac{1}{2}m_B\upsilon_{B2x}^2, \tag{8.19}$$

$$m_A\upsilon_{A1x}=m_A\upsilon_{A2x}+m_B\upsilon_{B2x}. \tag{8.20}$$

We can solve for υ_{A2x} and υ_{B2x} in terms of the masses and the initial velocity υ_{A1x}. This involves some fairly strenuous algebra, but it's worth it. No pain, no gain! The simplest approach is somewhat indirect, but along the way it uncovers an additional interesting feature of elastic collisions.

First we rearrange Eqs. (8.19) and (8.20) as follows:

$$m_B\upsilon_{B2x}^2=m_A\left(\upsilon_{A1x}^2-\upsilon_{A2x}^2\right)=m_A\left(\upsilon_{A1x}-\upsilon_{A2x}\right)\left(\upsilon_{A1x}+\upsilon_{A2x}\right), \tag{8.21}$$

$$m_B\upsilon_{B2x}=m_A\left(\upsilon_{A1x}-\upsilon_{A2x}\right). \tag{8.22}$$

Now we divide Eq. (8.21) by Eq. (8.22) to obtain

$$\upsilon_{B2x}=\upsilon_{A1x}+\upsilon_{A2x}. \tag{8.23}$$

We substitute this expression back into Eq. (8.22) to eliminate υ_{B2x} and then solve for υ_{A2x}:

$$\begin{gathered} m_B\left(\upsilon_{A1x}+\upsilon_{A2x}\right)=m_A\left(\upsilon_{A1x}-\upsilon_{A2x}\right) \\ \upsilon_{A2x}=\frac{m_A-m_B}{m_A+m_B}\upsilon_{A1x} \end{gathered} \tag{8.24}$$

Finally, we substitute this result back into Eq. (8.23) to obtain

$$\upsilon_{B2x}=\frac{2m_A}{m_A+m_B}\upsilon_{A1x}. \tag{8.25}$$

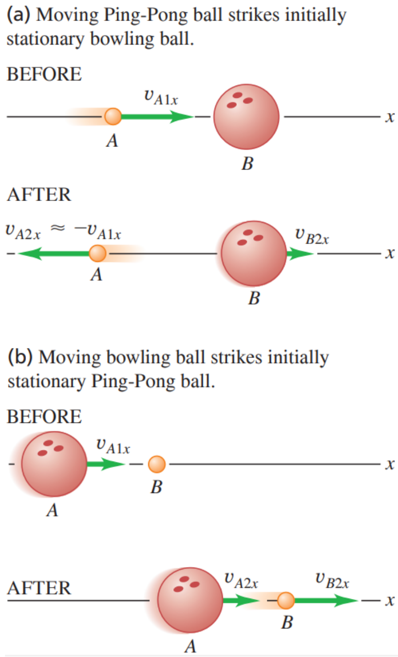


Figure 8.9 - One-dimensional elastic collisions between objects with different masses

Now we can interpret the results. Suppose *A* is a Ping-Pong ball and *B* is a bowling ball. Then we expect *A* to bounce off after the collision with a velocity nearly equal to its original value but in the opposite direction (Fig. 8.9a), and we expect *B*'s velocity to be much less. That's just what the equations predict. When m_A is much smaller than m_B, the fraction in Eq. (8.24) is approximately equal to (–1), so υ_{A2x} is approximately equal to $-\upsilon_{A1x}$. The fraction in Eq. (8.25) is much smaller than unity, so υ_{B2x} is much less than υ_{A1x}. Figure 8.9b shows the opposite case, in which *A* is the bowling ball and *B* the PingPong ball and mA is much larger than mB. What do you expect to happen then? Check your predictions against Eqs. (8.24) and (8.25).

Another interesting case occurs when the masses are equal (**Fig. 8.10**). If $m_A = m_B$, then Eqs. (8.24) and (8.25) give $\upsilon_{A2x} = 0$ and $\upsilon_{B2x} = \upsilon_{A1x}$. That is, the object that was moving stops dead; it gives all its momentum and kinetic energy to the object that was at rest. This behavior is familiar to all pool players.

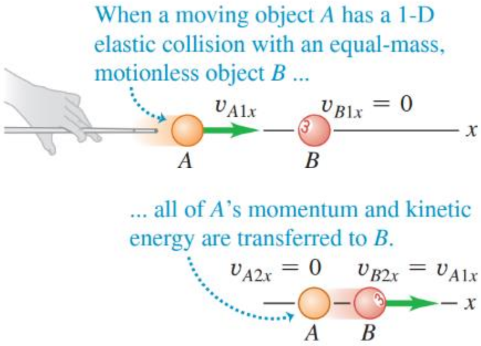


Figure 8.10 - A one-dimensional elastic collision between objects of equal mass

Elastic Collisions and Relative Velocity

Let's return to the more general case in which *A* and *B* have different masses. Equation (8.23) can be rewritten as

$$\upsilon_{A1x} = \upsilon_{B2x} - \upsilon_{A2x}. \tag{8.26}$$

Here $\upsilon_{B2x} - \upsilon_{A2x}$ is the velocity of *B* relative to *Aafter* the collision; from Eq. (8.26), this equals υ_{A1x}, which is the *negative* of the velocity of *B* relative to *Abefore* the collision. The relative velocity has the same magnitude, but opposite sign, before and after the collision. The sign changes because *A* and *B* are approaching each other before the collision but moving apart after the collision. If we view this collision from a second coordinate system moving with constant velocity relative to the first, the velocities of the objects are different but the *relative* velocities are the same. Hence our statement about relative velocities holds for *any* straight-line elastic collision, even when neither object is at rest initially. *In a straight-line elastic collision of two objects, the relative velocities before and after the collision have the same magnitude but opposite sign*. This means that if *B* is moving before the collision, Eq. (8.26) becomes

$$\upsilon_{B2x} - \upsilon_{A2x} = -\left(\upsilon_{B1x} - \upsilon_{A1x}\right). \tag{8.27}$$

It turns out that a *vector* relationship similar to Eq. (8.27) is a general property of *all* elastic collisions, even when both objects are moving initially and the velocities do not all lie along the same line. This result provides an alternative and equivalent definition of an elastic collision: *In an elastic collision, the relative velocity of the two objects has the same magnitude before and after the collision*. Whenever this condition is satisfied, the total kinetic energy is also conserved.

When an elastic two-object collision isn't head-on, the velocities don't all lie along a single line. If they all lie in a plane, then each final velocity has two unknown components, and there are four unknowns in all. Conservation of energy and conservation of the *x*- and *y*-components of momentum give only three equations. To determine the final velocities uniquely, we need additional information, such as the direction or magnitude of one of the final velocities.

8.5 Center of Mass

We can restate the principle of conservation of momentum in a useful way by using the concept of **center of mass**. Suppose we have several particles with masses m_1, m_2, and so on. Let the coordinates of m_1 be $\left(x_1, y_1\right)$, those of m_2 be $\left(x_2, y_2\right)$, and so on. We define the center of mass of the system as the point that has coordinates $\left(x_{\text{cm}}, y_{\text{cm}}\right)$ given by

$$x_{\text{cm}} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$
$$y_{\text{cm}} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{\sum_i m_i y_i}{\sum_i m_i} \quad \text{(center of mass).} \tag{8.28}$$

We can express the position of the center of mass as a vector $\vec{r}_{\text{cm}}$:

Position vectors of individual particles

Position vector of **center of mass** of a system of particles

$$\vec{r}_{\text{cm}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} \tag{8.29}$$

Masses of individual particles

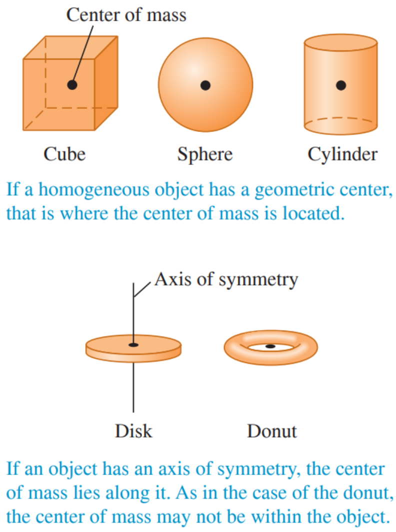


Figure 8.11 - Locating the center of mass of a symmetric object

We say that the center of mass is a *mass-weighted average* position of the particles.

For solid objects, in which we have (at least on a macroscopic level) a continuous distribution of matter, the sums in Eqs. (8.28) have to be replaced by integrals. The calculations can get quite involved, but we can say three general things about such problems (**Fig. 8.11**). First, whenever a homogeneous object has a geometric center, such as a billiard ball, a sugar cube, or a can of frozen orange juice, the center of mass is at the geometric center. Second, whenever an object has an axis of symmetry, such as a wheel or a pulley, the center of mass always lies on that axis. Third, there is no law that says the center of mass has to be within the object. For example, the center of mass of a donut is in the middle of the hole.

Motion of the Center of Mass

To see the significance of the center of mass of a collection of particles, we must ask what happens to the center of mass when the particles move. The *x*- and *y*-components of velocity of the center of mass, $\upsilon_{\text{cm-}x}$ and $\upsilon_{\text{cm-}y}$, are the time derivatives of x_{cm} and y_{cm}. Also, dx_1/dt is the *x*-component of velocity of particle 1, so $dx_1/dt = \upsilon_{1x}$, and so on. Taking time derivatives of Eqs. (8.28), we get

$$\upsilon_{\text{cm-}x} = \frac{m_1\upsilon_{1x} + m_2\upsilon_{2x} + m_3\upsilon_{3x} + \ldots}{m_1 + m_2 + m_3 + \ldots}$$
$$\upsilon_{\text{cm-}y} = \frac{m_1\upsilon_{1y} + m_2\upsilon_{2y} + m_3\upsilon_{3y} + \ldots}{m_1 + m_2 + m_3 + \ldots}. \tag{8.30}$$

These equations are equivalent to the single vector equation obtained by taking the time derivative of Eq. (8.29):

$$\vec{\upsilon}_{\text{cm}} = \frac{m_1\vec{\upsilon}_1 + m_2\vec{\upsilon}_2 + m_3\vec{\upsilon}_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}. \tag{8.31}$$

We denote the *total* mass $m_1 + m_2 + \ldots$ by *M*. We can then rewrite Eq. (8.31) as

Total mass of a system of particles

Momenta of individual particles

Velocity of center of mass

$$M\vec{v}_{\text{cm}} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots = \vec{P} \quad (8.32)$$

Total momentum of system

So the total momentum $\vec{P}$ of a system equals the total mass times the velocity of the center of mass. When you catch a baseball, you are really catching a collection of a very large number of molecules of masses m_1, m_2, m_3,…. The impulse you feel is due to the total momentum of this entire collection. But this impulse is the same as if you were catching a single particle of mass $M = m_1 + m_2 + m_3 + \dots$ moving with $\vec{\upsilon}_{\text{cm}}$, the velocity of the collection's center of mass. So Eq. (8.32) helps us justify representing an extended object as a particle.

For a system of particles on which the net external force is zero, so that the total momentum $\vec{P}$ is constant, the velocity of the center of mass $\vec{\upsilon}_{\text{cm}} = \vec{P}/M$ is also constant.

External Forces and Center-of-Mass Motion

If the net external force on a system of particles is not zero, then total momentum is not conserved and the velocity of the center of mass changes. Let's look at this situation in more detail.

Equations (8.31) and (8.32) give the *velocity* of the center of mass in terms of the velocities of the individual particles. We take the time derivatives of these equations to show that the *accelerations* are related in the same way. Let $\vec{a}_{\text{cm}} = d\vec{\upsilon}_{\text{cm}}/dt$ be the acceleration of the center of mass; then

$$M\vec{a}_{\text{cm}} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \dots \quad (8.33)$$

Now $m_1\vec{a}_1$ is equal to the vector sum of forces on the first particle, and so on, so the right side of Eq. (8.33) is equal to the vector sum $\sum \vec{F}$ of *all* the forces on *all* the particles. Just as we did in Section 8.2, we can classify each force as *external* or *internal*. The sum of all forces on all the particles is then

$$\sum \vec{F} = \sum \vec{F}_{\text{ext}} + \sum \vec{F}_{\text{int}} = M\vec{a}_{\text{cm}}$$

Because of Newton's third law, all of the internal forces cancel in pairs, and $\sum \vec{F}_{\text{int}} = 0$. What survives on the left side is the sum of only the *external* forces:

Net external force on **an object or a collection of particles**

Total mass of object or collection of particles

$$\sum \vec{F}_{\text{ext}} = M\vec{a}_{\text{cm}} \quad (8.34)$$

Acceleration of center of mass

When an object or a collection of particles is acted on by external forces, the center of mass moves as though all the mass were concentrated at that point and it were acted on by a net external force equal to the sum of the external forces on the system.

This result is central to the whole subject of mechanics. In fact, we've been using this result all along; without it, we would not be able to represent an extended object as a point particle when we apply Newton's laws. It explains why only *external* forces can affect the motion of an extended object. If you pull upward on your belt, your belt exerts an equal downward force on your hands; these are *internal* forces that cancel and have no effect on the overall motion of your body.

As an example, suppose that a cannon shell traveling in a parabolic trajectory (ignoring air resistance) explodes in flight, splitting into two fragments with equal mass (**Fig. 8.12**). The fragments follow new parabolic paths, but the center of mass continues on the original parabolic trajectory, as though all the mass were still concentrated at that point.

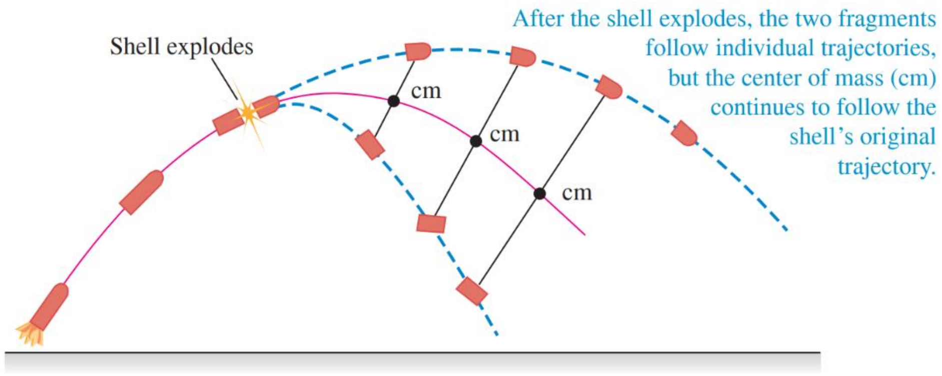


Figure 8.12 - A shell explodes into two fragments in flight. If air resistance is ignored, the center of mass continues on the same trajectory as the shell's path before the explosion

This property of the center of mass is important when we analyze the motion of rigid objects. In Chapter 10 we'll describe the motion of an extended object as a combination of translational motion of the center of mass and rotational motion about an axis through the center of mass. This property also plays an important role in the motion of astronomical objects. It's not correct to say that the moon orbits the earth; rather, both the earth and the moon move in orbits around their common center of mass.

There's one more useful way to describe the motion of a system of particles. Using $\vec{a}_{\text{cm}} = d\vec{\upsilon}_{\text{cm}}/dt$, we can rewrite Eq. (8.33) as

$$M\vec{a}_{\text{cm}} = M\frac{d\vec{\upsilon}_{\text{cm}}}{dt} = \frac{d\left(M\vec{\upsilon}_{\text{cm}}\right)}{dt} = \frac{d\vec{P}}{dt}. \tag{8.35}$$

The total system mass M is constant, so we're allowed to move it inside the derivative. Substituting Eq. (8.35) into Eq. (8.34), we find

$$\sum \vec{F}_{\text{ext}} = \frac{d\vec{P}}{dt} \quad \text{(extended object or system of particles)}. \tag{8.36}$$

This equation looks like Eq. (8.4). The difference is that Eq. (8.36) describes a *system* of particles, such as an extended object, while Eq. (8.4) describes a single particle. The interactions between the particles that make up the system can change the individual momenta of the particles, but the *total* momentum $\vec{P}$ of the system can be changed only by external forces acting from outside the system.

If the net external force is zero, Eqs. (8.34) and (8.36) show that the center-of-mass acceleration $\vec{a}_{\text{cm}}$ is zero (so the center-of-mass velocity $\vec{\upsilon}_{\text{cm}}$ is constant) and the total momentum $\vec{P}$ is constant. This is just our statement from Section 8.3: If the net external force on a system is zero, momentum is conserved.

8.6 Rocket Propulsion

Momentum considerations are particularly useful for analyzing a system in which the masses of parts of the system change with time. In such cases we can't use Newton's second law $\sum \vec{F} = m\vec{a}$ directly because m changes. Rocket propulsion is an important example of this situation. A rocket is propelled forward by rearward ejection of burned fuel that initially was in the rocket (which is why rocket fuel is also called *propellant*). The forward force on the rocket is the reaction to the backward force on the ejected material. The total mass of the system is constant, but the mass of the rocket itself decreases as material is ejected.

For simplicity, let's consider a rocket in outer space, where there is no gravitational force and no air resistance. Let m denote the mass of the rocket, which will change as it expends fuel. We choose our x-axis to be along the rocket's direction of motion. Figure 8.13a shows the rocket at a time t, when its mass is m and its x-velocity relative to our coordinate system is υ. (To simplify, we'll drop the subscript

x in this discussion). The x-component of total momentum at this instant is $P_1 = m\upsilon$. In a short time interval dt, the mass of the rocket changes by an amount dm. This is an inherently negative quantity because the rocket's mass m *decreases* with time. During dt, a *positive* mass $-dm$ of burned fuel is ejected from the rocket. Let υ_{ex} be the exhaust *speed* of this material *relative to the rocket*; the burned fuel is ejected opposite the direction of motion, so its x-component of *velocity* relative to the rocket is $-\upsilon_{\text{ex}}$. The x-velocity υ_{fuel} of the burned fuel relative to our coordinate system is then

$$\upsilon_{\text{fuel}} = \upsilon + (-\upsilon_{\text{ex}}) = \upsilon - \upsilon_{\text{ex}},$$

and the x-component of momentum of the ejected mass $(-dm)$ is

$$(-dm)\upsilon_{\text{fuel}} = (-dm)(\upsilon - \upsilon_{\text{ex}}).$$

Figure 8.13b shows that at the end of the time interval dt, the x-velocity of the rocket and unburned fuel has increased to $\upsilon + d\upsilon$, and its mass has decreased to $m + dm$ (remember that dm is negative). The rocket's momentum at this time is

$$(m + dm)(\upsilon + d\upsilon).$$

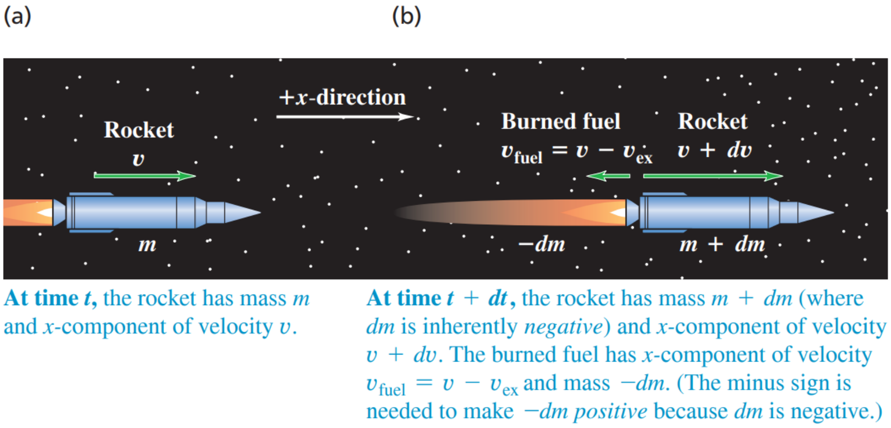


Figure 8.13 - A rocket moving in gravity-free outer space at (a) time t and (b) time $t+dt$

Thus the *total* x-component of momentum P_2 of the rocket plus ejected fuel at time $t + dt$ is

$$P_2 = (m + dm)(\upsilon + d\upsilon) + (-dm)(\upsilon - \upsilon_{\text{ex}}).$$

According to our initial assumption, the rocket and fuel are an isolated system. Thus momentum is conserved, and the total x-component of momentum of the system must be the same at time t and at time $t + dt$: $P_1 = P_2$. Hence

$$m\upsilon = (m + dm)(\upsilon + d\upsilon) + (-dm)(\upsilon - \upsilon_{\text{ex}}).$$

This can be simplified to

$$m\,d\upsilon = -dm\,\upsilon_{\text{ex}} - dm\,d\upsilon.$$

We can ignore the term $(-dm\,d\upsilon)$ because it is a product of two small quantities and thus is much smaller than the other terms. Dropping this term, dividing by dt, and rearranging, we find

$$m\frac{d\upsilon}{dt} = -\upsilon_{\text{ex}}\frac{dm}{dt}\,. \tag{8.37}$$

Now $d\upsilon/dt$ is the acceleration of the rocket, so the left side of Eq. (8.37) (mass times acceleration) equals the net external force *F*, or *thrust*, on the rocket:

$$F = -\upsilon_{\text{ex}}\frac{dm}{dt}\,. \tag{8.38}$$

The thrust is proportional both to the relative speed υ_{ex} of the ejected fuel and to the mass of fuel ejected per unit time, $-dm/dt$. (Remember that dm/dt is negative because it is the rate of change of the rocket's mass, so *F* is positive).

The *x*-component of acceleration of the rocket is

$$a = \frac{d\upsilon}{dt} = -\frac{\upsilon_{\text{ex}}}{m}\frac{dm}{dt}\,. \tag{8.39}$$

This is positive because υ_{ex} is positive (remember, it's the exhaust *speed*) and dm/dt is negative. The rocket's mass *m* decreases continuously while the fuel is being consumed. If υ_{ex} and dm/dt are constant, the acceleration increases until all the fuel is gone.

Equation (8.38) tells us that an effective rocket burns fuel at a rapid rate (large $-dm/dt$) and ejects the burned fuel at a high relative speed (large υ_{ex}). In the early days of rocket propulsion, people who didn't understand conservation of momentum thought that a rocket couldn't function in outer space because "it doesn't have anything to push against." In fact, rockets work *best* in outer space, where there is no air resistance! The launch vehicle is *not* "pushing against the ground" to ascend.

If the exhaust speed υ_{ex} is constant, we can integrate Eq. (8.39) to relate the velocity υ at any time to the remaining mass *m*. At time $t = 0$, let the mass be m_0 and the velocity be υ_0. Then we rewrite Eq. (8.39) as

$$d\upsilon = -\upsilon_{\text{ex}}\frac{dm}{m}\,.$$

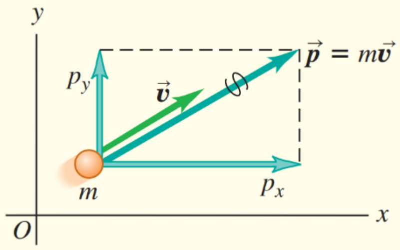
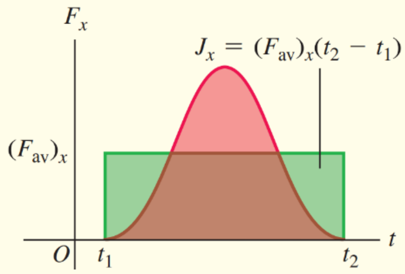
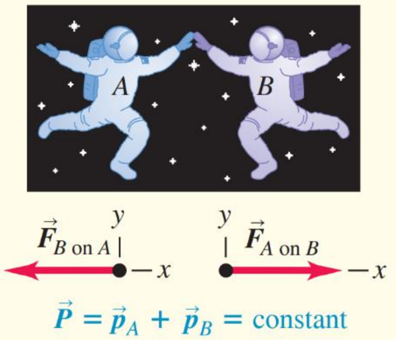
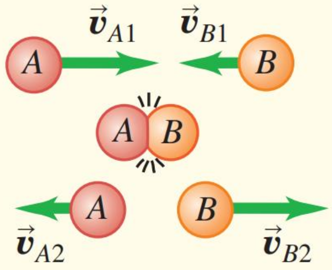
We change the integration variables to υ' and m', so we can use υ and *m* as the upper limits (the final speed and mass). Then we integrate both sides, using limits υ_0 to υ and m_0 to *m*, and take the constant υ_{ex} outside the integral:

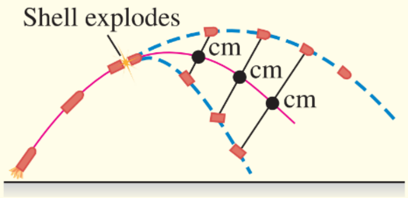
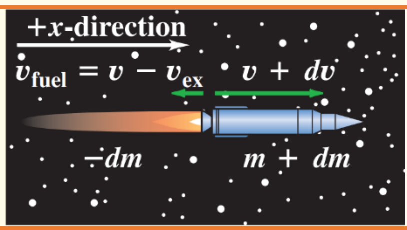
$$\begin{gathered}\int_{\upsilon_0}^{\upsilon} d\upsilon' = -\int_{m_0}^{m} \upsilon_{\text{ex}}\frac{dm'}{m'} = -\upsilon_{\text{ex}}\int_{m_0}^{m}\frac{dm'}{m'}\\ \upsilon - \upsilon_0 = -\upsilon_{\text{ex}}\ln\frac{m}{m_0} = \upsilon_{\text{ex}}\ln\frac{m_0}{m}\end{gathered} \tag{8.40}$$

The ratio m_0/m is the original mass divided by the mass after the fuel has been exhausted. In practical spacecraft this ratio is made as large as possible to maximize the speed gain, which means that the initial mass of the rocket is almost all fuel. The final velocity of the rocket will be greater in magnitude (and is often *much* greater) than the relative speed υ_{ex} if $\ln(m_0/m) > 1$—that is, if $m_0/m > e = 2.71828\ldots$.

We've assumed throughout this analysis that the rocket is in gravity-free outer space. However, gravity must be taken into account when a rocket is launched from the surface of a planet.

CHAPTER 8: SUMMARY

Momentum of a particle: The momentum $\vec{p}$ of a particle is a vector quantity equal to the product of the particle's mass m and velocity $\vec{\upsilon}$. Newton's second law says that the net external force on a particle is equal to the rate of change of the particle's momentum.	$\vec{p} = m\vec{\upsilon}$. $\sum \vec{F} = \frac{d\vec{p}}{dt}$.	
Impulse and momentum: If a constant net external force $\sum \vec{F}$ acts on a particle for a time interval Δt from t_1 to t_2, the impulse $\vec{J}$ of the net external force is the product of the net external force and the time interval. If $\sum \vec{F}$ varies with time, $\vec{J}$ is the integral of the net external force over the time interval. In any case, the change in a particle's momentum during a time interval equals the impulse of the net external force that acted on the particle during that interval. The momentum of a particle equals the impulse that accelerated it from rest to its present speed.	$\vec{J} = \sum \vec{F}(t_2 - t_1) = \sum \vec{F}\Delta t$ $\vec{J} = \int_{t_1}^{t_2} \sum \vec{F} dt$ $\vec{J} = \vec{p}_2 - \vec{p}_1$	
Conservation of momentum: An internal force is a force exerted by one part of a system on another. An external force is a force exerted on any part of a system by something outside the system. If the net external force on a system is zero, the total momentum of the system $\vec{P}$ (the vector sum of the momenta of the individual particles that make up the system) is constant, or conserved. Each component of total momentum is separately conserved.	$\vec{P} = \vec{p}_A + \vec{p}_B + \ldots$ $= m_A\vec{\upsilon}_A + m_B\vec{\upsilon}_B + \ldots$ If $\sum \vec{F} = 0$, then $\vec{P}$ = constant	
Collisions: In typical collisions, the initial and final total momenta are equal. In an elastic collision between two objects, the initial and final total kinetic energies are also equal, and the initial and final relative velocities have the same magnitude. In an inelastic two-object collision, the total kinetic energy is less after the collision than before. If the two objects have the same final velocity, the collision is completely inelastic.		

Center of mass: The position vector of the center of mass of a system of particles, $\vec{r}_{cm}$, is a weighted average of the positions $\vec{r}_1, \vec{r}_2, \ldots$ of the individual particles. The total momentum $\vec{P}$ of a system equals the system's total mass M multiplied by the velocity of its center of mass, $\vec{\upsilon}_{cm}$. The center of mass moves as though all the mass M were concentrated at that point. If the net external force on the system is zero, the center-of-mass velocity $\vec{\upsilon}_{cm}$ is constant. If the net external force is not zero, the center of mass accelerates as though it were a particle of mass M being acted on by the same net external force.	$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} =$ $= \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i}$ $\vec{P} = m_1\vec{\upsilon}_1 + m_2\vec{\upsilon}_2 + m_3\vec{\upsilon}_3 + \ldots =$ $= M\vec{\upsilon}_{cm}$ $\sum \vec{F}_{ext} = M\vec{a}_{cm}$	
Rocket propulsion: In rocket propulsion, the mass of a rocket changes as the fuel is used up and ejected from the rocket. Analysis of the motion of the rocket must include the momentum carried away by the spent fuel as well as the momentum of the rocket itself.		

9 ROTATION OF RIGID BODIES

What do the motions of an airplane propeller, a Blu-ray disc, a Ferris wheel, and a circular saw blade have in common? None of these can be represented adequately as a moving *point*; each involves an object that *rotates* about an axis that is stationary in some inertial frame of reference.

Rotation occurs at all scales, from the motions of electrons in atoms to the motions of entire galaxies. We need to develop some general methods for analyzing the motion of a rotating object. In this chapter and the next we consider objects that have definite size and definite shape, and that in general can have rotational as well as translational motion.

Real-world objects can be very complicated; the forces that act on them can deform them - stretching, twisting, and squeezing them. We'll ignore these deformations for now and assume that the object has a perfectly definite and unchanging shape and size. We call this idealized model a **rigid body**. This chapter and the next are mostly about rotational motion of a rigid body.

We begin with kinematic language for *describing* rotational motion. Next we look at the kinetic energy of rotation, the key to using energy methods for rotational motion. Then in Chapter 10 we'll develop dynamic principles that relate the forces on a body to its rotational motion.

9.1 Angular Velocity and Acceleration

In analyzing rotational motion, let's think first about a rigid body that rotates about a *fixed axis* - an axis that is at rest in some inertial frame of reference and does not change direction relative to that frame. The rotating rigid body might be a motor shaft, a chunk of beef on a barbecue skewer, or a merry-go-round.

Figure 9.1 shows a rigid body rotating about a fixed axis. The axis passes through point *O* and is perpendicular to the plane of the diagram, which we'll call the *xy*-plane. One way to describe the rotation of this body would be to choose a particular point *P* on the body and to keep track of the *x*- and *y*-coordinates of *P*. This isn't very convenient, since it takes two numbers (the two coordinates *x* and *y*) to specify the rotational position of the body. Instead, we notice that the line *OP* is fixed in the body and rotates with it. The angle θ that *OP* makes with the +*x*-axis is a single **angular cooordinate** that completely describes the body's rotational position.

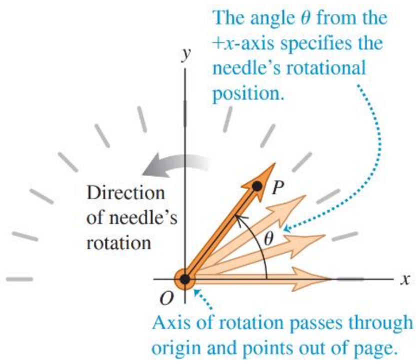


Figure 9.1 - A speedometer needle (an example of a rigid body) rotating counterclockwise about a fixed axis

The angular coordinate θ of a rigid body rotating around a fixed axis can be positive or negative. If we choose positive angles to be measured counterclockwise from the positive *x*-axis, then the angle θ in Fig. 9.1 is positive. If we instead choose the positive rotation direction to be clockwise, then θ in Fig. 9.1 is negative. When we considered the motion of a particle along a straight line, it was essential to specify the direction of positive displacement along that line; when we discuss rotation around a fixed axis, it's just as essential to specify the direction of positive rotation.

The most natural way to measure the angle θ is not in degrees but in **radians**. As Fig. 9.2a shows, one radian (1 rad) is the angle subtended at the center of a circle by an arc with a length equal to the radius of the circle. In Fig. 9.2b an angle θ is subtended by an arc of length *s* on a circle of radius *r*. The value of θ (in radians) is equal to *s* divided by *r*:

$$\theta = \frac{s}{r} \quad \text{or} \quad s = r\theta \quad (\theta \text{ in radians})\,. \qquad (9.1)$$

An angle in radians is the ratio of two lengths, so it is a pure number, without dimensions. If *s* = 3.0 m and *r* = 2.0 m, then θ = 1.5, but we'll often write this as 1.5 rad to distinguish it from an angle measured in degrees or revolutions.

The circumference of a circle (that is, the arc length all the way around the circle) is 2π times the radius, so there are 2π (about 6.283) radians in one complete revolution (360°). Therefore

$$1 \text{ rad} = \frac{360°}{2\pi} = 57.3°.$$

Similarly, $180° = \pi$ rad, $90° = \pi/2$ rad, and so on. If we had measured angle θ in degrees, we would have needed an extra factor of $(2\pi/360)$ on the right-hand side of $s = r\theta$ in Eq. (9.1). By measuring angles in radians, we keep the relationship between angle and distance along an arc as simple as possible.

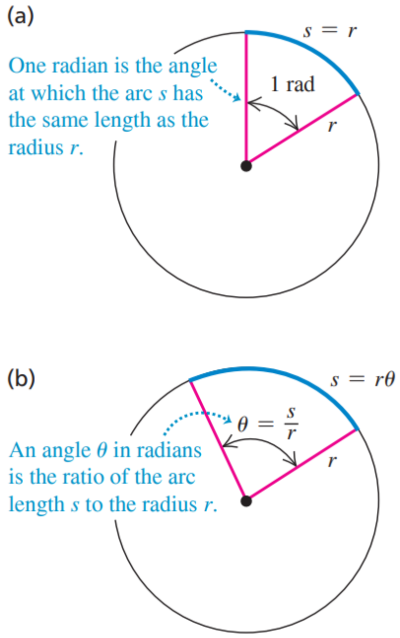


Figure 9.2 - Measuring angles in radians

Angular Velocity

The coordinate θ shown in Fig. 9.1 specifies the rotational position of a rigid body at a given instant. We can describe the rotational *motion* of such a rigid body in terms of the rate of change of θ. In Fig. 9.3a, a reference line *OP* in a rotating body makes an angle θ_1 with the $+x$-axis at time t_1. At a later time t_2 the angle has changed to θ_2. We define the **average angular velocity** $\omega_{\text{av-z}}$ (the Greek letter omega) of the body in the time interval $\Delta t = t_2 - t_1$ as the ratio of the **angular displacement** $\Delta\theta = \theta_2 - \theta_1$ to Δt:

$$\omega_{\text{av-z}} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}. \tag{9.2}$$

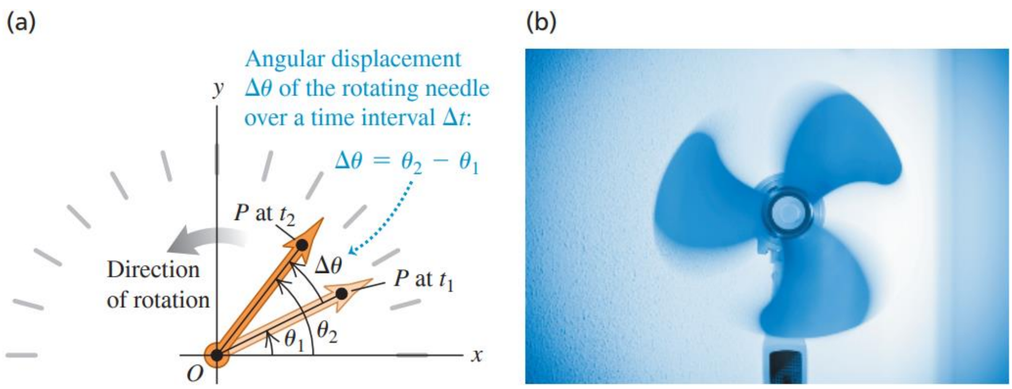


Figure 9.3 - (a) Angular displacement $\Delta\theta$ of a rotating body. (b) Every part of a rotating rigid body has the same average angular velocity $\Delta\theta/\Delta t$

The subscript z indicates that the body in Fig. 9.3a is rotating about the z-axis, which is perpendicular to the plane of the diagram. The **instantaneous angular velocity** ω_z is the limit of $\omega_{\text{az-z}}$ as Δt approaches zero:

The **instantaneous angular velocity** of a rigid body rotating around the z-axis ...

$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \tag{9.3}$$

... equals the limit of the body's average angular velocity as the time interval approaches zero ...

... and equals the instantaneous rate of change of the body's angular coordinate.

When we refer simply to "angular velocity," we mean the instantaneous angular velocity, not the average angular velocity.

The angular velocity ω_z can be positive or negative, depending on the direction in which the rigid body is rotating (**Fig. 9.4**). The angular *speed* ω, which we'll use in Sections 9.3 and 9.4, is the magnitude of angular velocity. Like linear speed υ, the angular speed is never negative.

CAUTION! **Angular velocity vs. linear velocity.** Keep in mind the distinction between *angular* velocity ω_z and *linear* velocity υ_x. If an object has a linear velocity υ_x, the object as a whole is *moving* along the *x*-axis. By contrast, if an object has an angular velocity ω_z, then it is *rotating* around the *z*-axis. We do *not* mean that the object is moving along the *z*-axis.

Different points on a rotating rigid body move different distances in a given time interval, depending on how far each point lies from the rotation axis. But because the body is rigid, all points rotate through the same angle in the same time (Fig. 9.3b). Hence *at any instant, every part of a rotating rigid body has the same angular velocity*.

If angle θ is in radians, the unit of angular velocity is the radian per second (rad/s). Other units, such as the revolution per minute (rev/min or rpm), are often used. Since $1\,\text{rev} = 2\pi\,\text{rad}$, two useful conversions are

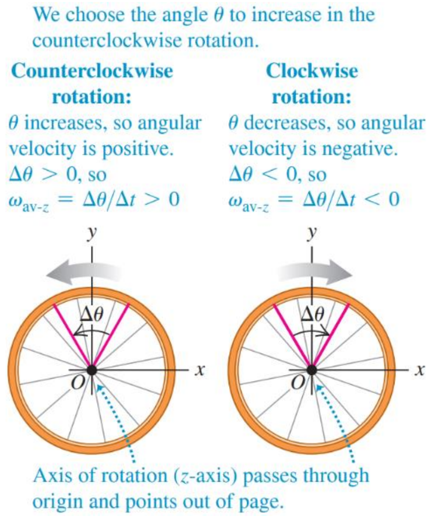


Figure 9.4 - A rigid body's average angular velocity (shown here) and instantaneous angular velocity can be positive or negative

$$1\ \text{rev/s} = 2\pi\ \text{rad/s} \quad \text{and} \quad 1\ \text{rev/min} = 1\ \text{rpm} = \frac{2\pi}{60}\ \text{rad/s}\,.$$

That is, 1 rad/s is about 10 rpm.

Angular Velocity as a Vector

As we have seen, our notation for the angular velocity ω_z about the *z*-axis is reminiscent of the notation υ_x for the ordinary velocity along the *x*-axis. Just as υ_x is the *x*-component of the velocity vector $\vec{\upsilon}$, ω_z is the *z*-component of an angular velocity *vector* $\vec{\omega}$ directed along the axis of rotation. As Fig. 9.5a shows, the direction of $\vec{\omega}$ is given by the right-hand rule. If the rotation is about the *z*-axis, then $\vec{\omega}$ has only a *z*-component. This component is positive if $\vec{\omega}$ is along the positive *z*-axis and negative if $\vec{\omega}$ is along the negative *z*-axis (Fig. 9.5b).

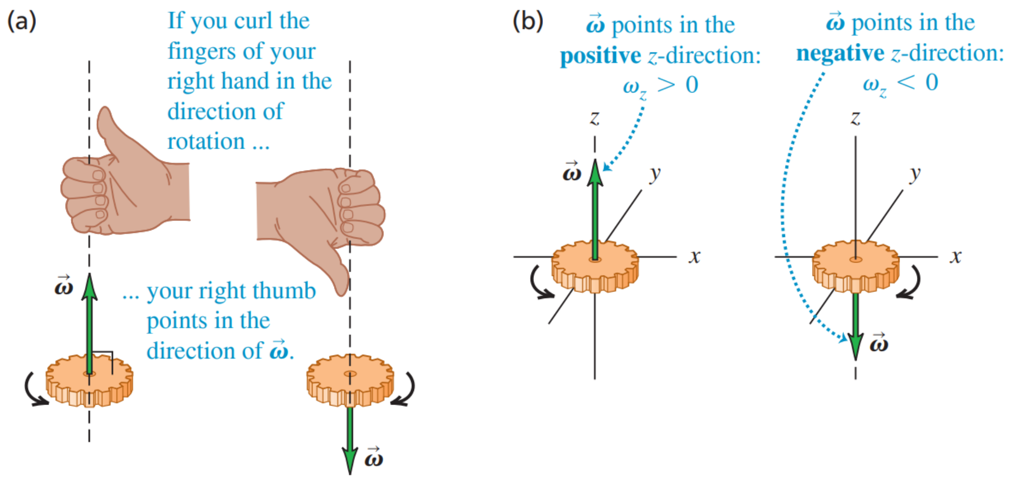


Figure 9.5 - (a) The right-hand rule for the direction of the angular velocity vector $\vec{\omega}$. Reversing the direction of rotation reverses the direction of $\vec{\omega}$. (b) The sign of ω_z for rotation along the *z*-axis

The vector formulation is especially useful when the direction of the rotation axis *changes*. We'll examine such situations briefly at the end of Chapter 10. In this chapter, however, we'll consider only situations in which the rotation axis is fixed. Hence throughout this chapter we'll use "angular velocity" to refer to ω_z, the component of $\vec{\omega}$ along the axis.

CAUTION! **The angular velocity vector is perpendicular to the plane of rotation, not in it.** It's a common error to think that an object's angular velocity vector $\vec{\omega}$ points in the direction in which some particular part of the object is moving. Another error is to think that $\vec{\omega}$ is a "curved vector" that points around the rotation axis in the direction of rotation (like the curved arrows in Figs. 9.1, 9.3, and 9.4). Neither of these is true! Angular velocity is an attribute of the *entire* rotating rigid body, not any one part, and there's no such thing as a curved vector. We choose the direction of $\vec{\omega}$ to be along the rotation axis—*perpendicular* to the plane of rotation—because that axis is common to every part of a rotating rigid body.

Angular Acceleration

A rigid body whose angular velocity changes has an *angular acceleration*. When you pedal your bicycle harder to make the wheels turn faster or apply the brakes to bring the wheels to a stop, you're giving the wheels an angular acceleration.

If ω_{1z} and ω_{2z} are the instantaneous angular velocities at times t_1 and t_2, we define the **average angular acceleration** $\alpha_{\text{av-}z}$ over the interval $\Delta t = t_2 - t_1$ as the change in angular velocity divided by Δt (**Fig. 9.6**):

$$\alpha_{\text{av-}z} = \frac{\omega_{2z} - \omega_{1z}}{t_2 - t_1} = \frac{\Delta\omega_z}{\Delta t}. \tag{9.4}$$

The **instantaneous angular acceleration** α_z is the limit of $\alpha_{\text{av-}z}$ as $\Delta t \to 0$:

The **instantaneous angular acceleration** of a rigid body rotating around the z-axis ...

$$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta\omega_z}{\Delta t} = \frac{d\omega_z}{dt} \tag{9.5}$$

... equals the limit of the body's average angular acceleration as the time interval approaches zero ...

... and equals the instantaneous rate of change of the body's angular velocity.

The usual unit of angular acceleration is the radian per second per second, or rad/s^2. From now on we'll use the term "angular acceleration" to mean the instantaneous angular acceleration rather than the average angular acceleration.

Because $\omega_z = d\theta/dt$, we can also express angular acceleration as the second derivative of the angular coordinate:

$$\alpha_z = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d^2\theta}{dt^2}. \tag{9.6}$$

You've probably noticed that we use Greek letters for angular kinematic quantities: θ for angular position, ω_z for angular velocity, and α_z for angular acceleration. These are analogous to x for position, υ_x for velocity, and a_x for acceleration in straight-line motion. In each case, velocity is the rate of change of position with respect to time and acceleration is the rate of change of velocity with respect to time. We sometimes use the terms "*linear* velocity" for υ_x and "*linear* acceleration" for a_x to distinguish clearly between these and the *angular* quantities introduced in this chapter.

If the angular acceleration α_z is positive, then the angular velocity ω_z is increasing; if α_z is negative, then ω_z is decreasing. The rotation is speeding up if α_z and ω_z have the same sign and slowing down if α_z and ω_z have opposite signs. (These are exactly the same relationships as those between *linear* acceleration a_x and linear velocity υ_x for straight-line motion).

Angular Acceleration as a Vector

Just as we did for angular velocity, it's useful to define an angular acceleration *vector* $\vec{\alpha}$. Mathematically, $\vec{\alpha}$ is the time derivative of the angular velocity vector $\vec{\omega}$. If the object rotates around the fixed *z*-axis, then $\vec{\alpha}$ has only a *z*-component α_z. In this case, $\vec{\alpha}$ is in the same direction as $\vec{\omega}$ if the rotation is speeding up and opposite to $\vec{\omega}$ if the rotation is slowing down (**Fig. 9.7**).

The vector $\vec{\alpha}$ will be particularly useful in Chapter 10 when we discuss what happens when the rotation axis changes direction. In this chapter, however, the rotation axis will always be fixed and we need only the *z*-component α_z.

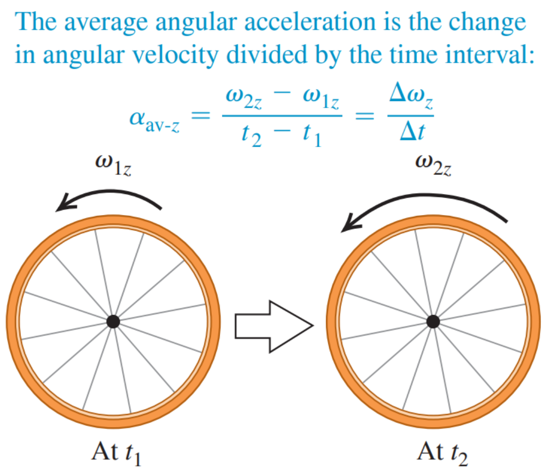


Figure 9.6 - Calculating the average angular acceleration of a rotating rigid body

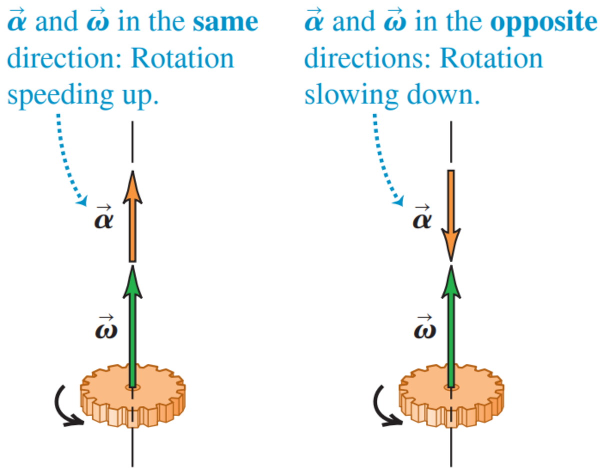


Figure 9.7 - When the rotation axis is fixed, both the angular acceleration and angular velocity vectors lie along that axis

9.2 Rotation with Constant Angular Acceleration

In Chapter 2 we found that straight-line motion is particularly simple when the acceleration is constant. This is also true of rotational motion about a fixed axis. When the angular acceleration is constant, we can derive equations for angular velocity and angular position by using the same procedure that we used for straight-line motion in Section 2.4. In fact, the equations we are about to derive are identical to Eqs. (2.8), (2.12), (2.13), and (2.14) if we replace *x* with θ, υ_x with ω_z, and a_x with α_z. We suggest that you review Section 2.4 before continuing.

Let ω_{0z} be the angular velocity of a rigid body at time $t = 0$ and ω_z be its angular velocity at a later time *t*. The angular acceleration α_z is constant and equal to the average value for any interval. From Eq. (9.4) with the interval from 0 to *t*,

$$\alpha_z = \frac{\omega_z - \omega_{0z}}{t - 0} \quad \text{or}$$

Angular velocity at time *t* of a rigid body with **constant angular acceleration**

Angular velocity of body at time 0

$$\omega_z = \omega_{0z} + \alpha_z t \qquad (9.7)$$

Time

Constant angular acceleration of body

The product $\alpha_z t$ is the total change in ω_z between $t = 0$ and the later time *t*; angular velocity ω_z at time *t* is the sum of the initial value ω_{0z} and this total change.

With constant angular acceleration, the angular velocity changes at a uniform rate, so its average value between 0 and t is the average of the initial and final values:

$$\omega_{\text{av-z}} = \frac{\omega_{0z} + \omega_z}{2}. \tag{9.8}$$

We also know that $\omega_{\text{av-z}}$ is the total angular displacement $(\theta - \theta_0)$ divided by the time interval $(t - 0)$:

$$\omega_{\text{av-z}} = \frac{\theta - \theta_0}{t - 0}. \tag{9.9}$$

When we equate Eqs. (9.8) and (9.9) and multiply the result by t, we get

Angular position at time t of a rigid body **with constant angular acceleration**
Angular position of body at time 0
Time
Angular velocity of body at time 0
Angular velocity of body at time t

$$\theta - \theta_0 = \tfrac{1}{2}(\omega_{0z} + \omega_z)t \tag{9.10}$$

To obtain a relationship between θ and t that doesn't contain ω_z, we substitute Eq. (9.7) into Eq. (9.10):

$$\theta - \theta_0 = \tfrac{1}{2}\left[\omega_{0z} + \left(\omega_{0z} + \alpha_z t\right)\right]t \quad \text{or}$$

Angular position at time t of a rigid body **with constant angular acceleration**
Angular position of body at time 0
Time
Angular velocity of body at time 0
Constant angular acceleration of body

$$\theta = \theta_0 + \omega_{0z}t + \tfrac{1}{2}\alpha_z t^2 \tag{9.11}$$

That is, if at the initial time $t = 0$ the body is at angular position θ_0 and has angular velocity ω_{0z}, then its angular position θ at any later time t is θ_0, plus the rotation $\omega_{0z}t$ it would have if the angular velocity were constant, plus an additional rotation $\frac{1}{2}\alpha_z t^2$ caused by the changing angular velocity.

Following the same procedure as for straight-line motion in Section 2.4, we can combine Eqs. (9.7) and (9.11) to obtain a relationship between θ and ω_z that does not contain t. We invite you to work out the details, following the same procedure we used to get Eq. (9.12). We get

Angular velocity at time t of a rigid body **with constant angular acceleration**
Angular velocity of body at time 0
Constant angular acceleration of body
Angular position of body at time t
Angular position of body at time 0

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0) \tag{9.12}$$

Table 9.1 - Comparison of Linear and Angular Motions with Constant Acceleration

Straight-Line Motion with Constant Linear Acceleration	**Fixed-Axis Rotation with Constant Angular Acceleration**
$a_x = \text{constant}$.	$\alpha_z = \text{constant}$.
$\upsilon_x = \upsilon_{0x} + a_x t$.	$\omega_z = \omega_{0z} + \alpha_z t$.
$x = x_0 + \upsilon_{0x}t + \frac{1}{2}a_x t^2$.	$\theta = \theta_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2$.
$\upsilon_x^2 = \upsilon_{0x}^2 + 2a_x(x - x_0)$.	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$.
$x - x_0 = \frac{1}{2}(\upsilon_{0x} + \upsilon_x)t$.	$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t$.

CAUTION! **Constant angular acceleration.** Keep in mind that all of these results are valid *only* when the angular acceleration α_z is *constant*; do not try to apply them to problems in which α_z is not constant. Table 9.1 shows the analogy between Eqs. (9.7), (9.10), (9.11), and (9.12) for fixed-axis rotation with constant angular acceleration and the corresponding equations for straight-line motion with constant linear acceleration.

9.3 Relating Linear and Angular Kinematics

How do we find the linear speed and acceleration of a particular point in a rotating rigid body? We need to answer this question to proceed with our study of rotation. For example, to find the kinetic energy of a rotating body, we have to start from $K = \frac{1}{2}mv^2$ for a particle, and this requires that we know the speed v for each particle in the body. So it's worthwhile to develop general relationships between the *angular* speed and acceleration of a rigid body rotating about a fixed axis and the *linear* speed and acceleration of a specific point or particle in the body.

Linear Speed in Rigid-Body Rotation

When a rigid body rotates about a fixed axis, every particle in the body moves in a circular path that lies in a plane perpendicular to the axis and is centered on the axis. A particle's speed is directly proportional to the body's angular velocity; the faster the rotation, the greater the speed of each particle. In Fig. 9.8, point *P* is a constant distance *r* from the axis, so it moves in a circle of radius *r*. At any time, Eq. (9.1) relates the angle θ (in radians) and the arc length *s*:

$$s = r\theta .$$

We take the time derivative of this, noting that *r* is constant for any specific particle, and take the absolute value of both sides:

$$\left|\frac{ds}{dt}\right| = r\left|\frac{d\theta}{dt}\right| .$$

Now $|ds/dt|$ is the absolute value of the rate of change of arc length, which is equal to the instantaneous *linear* speed v of the particle. The absolute value of the rate of change of the angle, $|d\theta/dt|$, is the instantaneous **angular speed** ω - that is, the magnitude of the instantaneous angular velocity in rad/s. Thus

Linear speed of a point on a rotating rigid body

$$v = r\omega \qquad (9.13)$$

Angular speed of the rotating rigid body

Distance of that point from rotation axis

The farther a point is from the axis, the greater its linear speed. The *direction* of the linear velocity *vector* is tangent to its circular path at each point (**Fig. 9.8**).

CAUTION! **Speed vs. velocity.** Keep in mind the distinction between the linear and angular *speeds* v and ω, which appear in Eq. (9.13), and the linear and angular *velocities* v_x and ω_z. The quantities without subscripts, v and ω, are never negative; they are the magnitudes of the vectors $\vec{v}$ and $\vec{\omega}$, respectively, and their values tell you only how fast a particle is moving (v) or how fast a body is rotating (ω). The quantities with subscripts, υ_x and ω_z, can be either positive or negative; their signs tell you the direction of the motion.

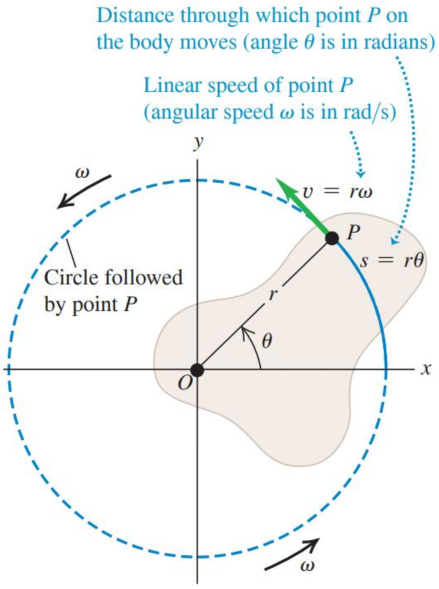


Figure 9.8 - A rigid body rotating about a fixed axis through point O

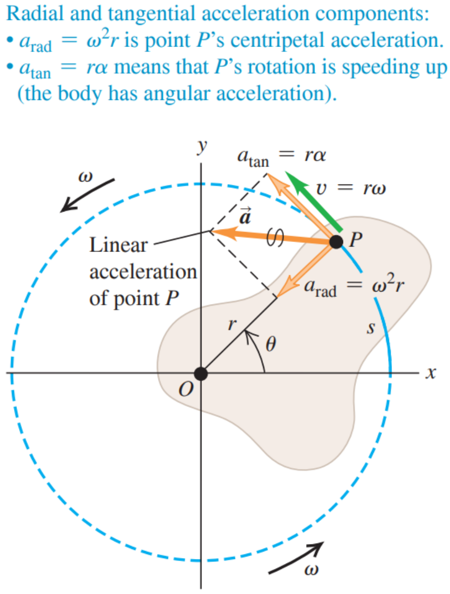


Figure 9.9 - A rigid body whose rotation is speeding up. The acceleration of point P has a component a_{rad} toward the rotation axis (perpendicular to $\vec{\upsilon}$) and a component a_{tan} along the circle that point P follows (parallel to $\vec{\upsilon}$)

Linear Acceleration in Rigid-Body Rotation

We can represent the acceleration $\vec{a}$ of a particle moving in a circle in terms of its centripetal and tangential components, a_{rad} and a_{tan} (**Fig. 9.9**), as we did in Section 3.4. (You should review that section now). We found that the **tangential component of acceleration**a_{tan}, the component parallel to the instantaneous velocity, acts to change the *magnitude* of the particle's velocity (i.e., the speed) and is equal to the rate of change of speed. Taking the derivative of Eq. (9.13), we find

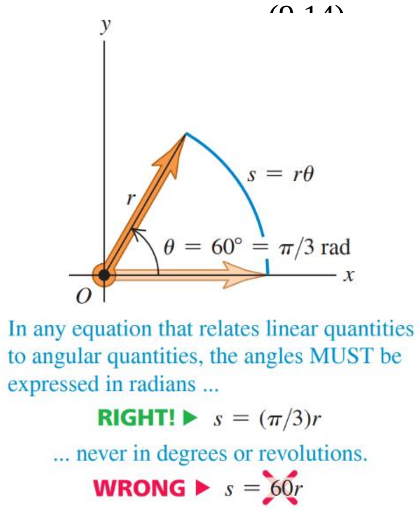
Tangential acceleration of a point on a rotating rigid body

Distance of that point from rotation axis

$$a_{tan} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \qquad (9.14)$$

Rate of change of linear speed of that point

Rate of change of angular speed of body



This component of $\vec{a}$ is always tangent to the circular path of point P (**Fig. 9.10**).

Figure 9.10 - Always use radians when relating linear and angular quantities

The quantity $\alpha = d\omega/dt$ in Eq. (9.14) is the rate of change of the angular *speed*. It is not quite the same as $\alpha_z = d\omega_z/dt$, which is the rate of change of the angular *velocity*. For example, consider a object rotating so that its angular velocity vector points in the –*z*-direction (see Fig. 9.5b). If the body is gaining angular speed at a rate of 10 rad/s per second, then $\alpha = 10\ \text{rad/s}^2$. But ω_z is negative and becoming more negative as the rotation gains speed, so $\alpha = -10\ \text{rad/s}^2$. The rule for rotation about a fixed axis is that α is equal to α_z if ω_z is positive but equal to $-\alpha_z$ if ω_z is negative.

The component of $\vec{a}$ in Fig. 9.9 directed toward the rotation axis, the **centripetal component of acceleration** a_{rad}, is associated with the change of *direction* of the velocity of point *P*. In Section 3.4 we worked out the relationship $a_{\text{rad}} = \upsilon^2/r$. We can express this in terms of ω by using Eq. (9.13):

Centripetal acceleration of a point on a rotating rigid body

Linear speed of that point

Angular speed of body

$$a_{\text{rad}} = \frac{\upsilon^2}{r} = \omega^2 r \qquad (9.15)$$

Distance of that point from rotation axis

This is true at each instant, *even when* ω *and* υ *are not constant*. The centripetal component always points toward the axis of rotation.

CAUTION! **Use angles in radians.** Remember that Eq. (9.1), $s = r\theta$, is valid *only* when θ is measured in radians. The same is true of any equation derived from this, including Eqs. (9.13), (9.14), and (9.15). When you use these equations, you *must* express the angular quantities in radians, not revolutions or degrees (Fig. 9.10).

Equations (9.1), (9.13), and (9.14) also apply to any particle that has the same tangential velocity as a point in a rotating rigid body. For example, when a rope wound around a circular cylinder unwraps without stretching or slipping, its speed and acceleration at any instant are equal to the speed and tangential acceleration of the point at which it is tangent to the cylinder. The same principle holds for situations such as bicycle chains and sprockets, belts and pulleys that turn without slipping, and so on. We'll have several opportunities to use these relationships later in this chapter and in Chapter 10. Note that Eq. (9.15) for the centripetal component a_{rad} is applicable to the rope or chain *only* at points that are in contact with the cylinder or sprocket. Other points do not have the same acceleration toward the center of the circle that points on the cylinder or sprocket have.

9.4 Energy in Rotational Motion

A rotating rigid body consists of mass in motion, so it has kinetic energy. As we'll see, we can express this kinetic energy in terms of the body's angular speed and a new quantity, called *moment of inertia*, that depends on the body's mass and how the mass is distributed.

To begin, we think of a body as being made up of a large number of particles, with masses $m_1, m_2, \ldots$ at distances $r_1, r_2, \ldots$ from the axis of rotation. We label the particles with the index *i*: The mass of the *i*th particle is m_i and r_i is the *perpendicular* distance from the axis to the *i*th particle. (The particles need not all lie in the same plane).

When a rigid body rotates about a fixed axis, the speed υ_i of the *i*th particle is given by Eq. (9.13), $\upsilon_i = r_i\omega$, where ω is the body's angular speed. Different particles have different values of r_i, but ω is the same for all (otherwise, the body wouldn't be rigid). The kinetic energy of the *i*th particle can be expressed as

$$\tfrac{1}{2}m_i\upsilon_i^2 = \tfrac{1}{2}m_i r_i^2\omega^2 .$$

The body's *total* kinetic energy is the sum of the kinetic energies of all its particles:

$$K = \tfrac{1}{2}m_1 r_1^2\omega^2 + \tfrac{1}{2}m_2 r_2^2\omega^2 + \ldots = \sum_i \tfrac{1}{2}m_i r_i^2\omega^2 .$$

Taking the common factor $\frac{1}{2}\omega^2$ out of this expression, we get

$$K = \tfrac{1}{2}\left(m_1 r_1^2 + m_2 r_2^2 + \ldots\right)\omega^2 = \tfrac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 .$$

The quantity in parentheses, obtained by multiplying the mass of each particle by the square of its distance from the axis of rotation and adding these products, is called the **moment of inertia***I* of the body for this rotation axis:

Moment of inertia of a body for a given rotation axis

Masses of the particles that make up the body

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots = \sum_i m_i r_i^2 \qquad (9.16)$$

Perpendicular distances of the particles from rotation axis

"Moment" means that I depends on how the body's mass is distributed in space; it has nothing to do with a "moment" of time. For a body with a given rotation axis and a given total mass, the greater the distances from the axis to the particles that make up the body, the greater the moment of inertia I. In a rigid body, all distances r_i are constant and I is independent of how the body rotates around the given axis. The SI unit of I is the kilogram-meter^2 $\left(\mathrm{kg}\cdot\mathrm{m}^2\right)$.

Using Eq. (9.16), we see that the **rotational kinetic energy***K* of a rigid body is

Rotational kinetic energy of a rigid body rotating around an axis

$$K = \tfrac{1}{2}I\omega^2 \qquad (9.17)$$

Moment of inertia of body for given rotation axis

Angular speed of body

The kinetic energy given by Eq. (9.17) is *not* a new form of energy; it's simply the sum of the kinetic energies of the individual particles that make up the rotating rigid body. To use Eq. (9.17), ω *must* be measured in radians per second, not revolutions or degrees per second, to give K in joules. That's because we used $\upsilon_i = r_i\omega$ in our derivation.

Equation (9.17) gives a simple physical interpretation of moment of inertia: *The greater the moment of inertia, the greater the kinetic energy of a rigid body rotating with a given angular speed* ω. We learned in Chapter 6 that the kinetic energy of an object equals the amount of work done to accelerate that object from rest. So the greater a body's moment of inertia, the harder it is to start the body rotating if it's at rest and the harder it is to stop its rotation if it's already rotating (Fig. 9.11). For this reason, I is also called the *rotational inertia*.

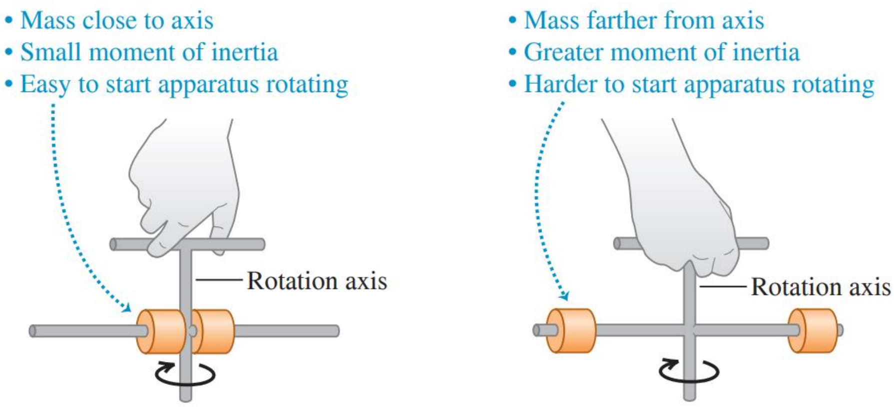
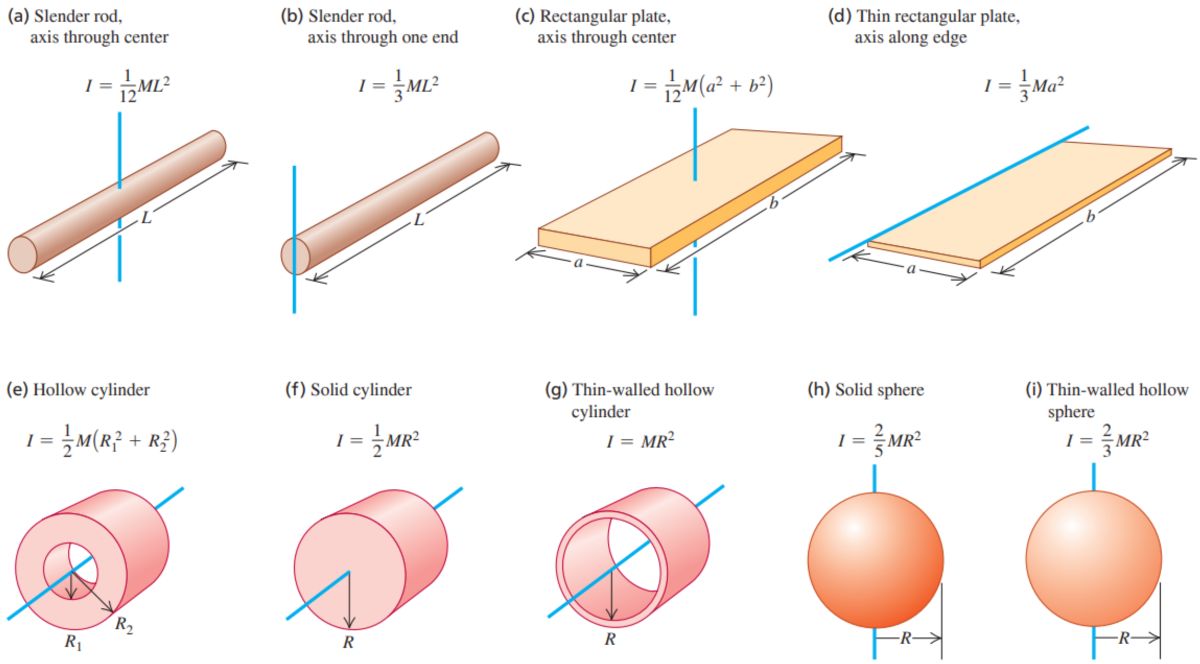


Figure 9.11 - An apparatus free to rotate around a vertical axis. To vary the moment of inertia, the two equal-mass cylinders can be locked into different positions on the horizontal shaft

When the body is a *continuous* distribution of matter, such as a solid cylinder or plate, the sum becomes an integral, and we need to use calculus to calculate the moment of inertia. Table 9.2 gives moments of inertia for several familiar shapes in terms of their masses and dimensions. Each body shown in Table 9.2 is *uniform*; that is, the density has the same value at all points within the solid parts of the body.

Table 9.2 - Moments of Inertia of Various Bodies



CAUTION! **Computing moments of inertia.** You may be tempted to try to compute the moment of inertia of a body by assuming that all the mass is concentrated at the center of mass and multiplying the total mass by the square of the distance from the center of mass to the axis. That doesn't work! For example, when a uniform thin rod of length L and mass M is pivoted about an axis through one end, perpendicular to the rod, the moment of inertia is $I = ML^2/3$ [case (b) in Table 9.2]. If we took the mass as concentrated at the center, a distance $L/2$ from the axis, we would obtain the *incorrect* result $I = M(L/2)^2 = ML^2/4$.

Now that we know how to calculate the kinetic energy of a rotating rigid body, we can apply the energy principles of Chapter 7 to rotational motion.

PROBLEM-SOLVING STRATEGY
9.1 Rotational Energy

IDENTIFY *the relevant concepts:*
You can use work–energy relationships and conservation of energy to find relationships involving the position and motion of a rigid body rotating around a fixed axis. The energy method is usually not helpful for problems that involve elapsed time. In Chapter 10 we'll see how to approach rotational problems of this kind

SET UP *the problem* using Problem-Solving Strategy 7.1 (Section 7.1) with the following additional steps:

- You can use Eqs. (9.13) and (9.14) in problems involving a rope (or the like) wrapped around a rotating rigid body, if the rope doesn't slip. These equations relate the linear speed and tangential acceleration of a point on the body to the body's angular velocity and angular acceleration.
- Use Table 9.2 to find moments of inertia. Use the parallel-axis theorem, Eq. (9.19) (to be derived in Section 9.5), to find moments of inertia for rotation about axes parallel to those shown in the table.

EXECUTE *the solution*:
Write expressions for the initial and final kinetic and potential energies K_1, K_2, U_1, and U_2 and for the nonconservative work W_{other} (if any), where K_1 and K_2 must now include any rotational kinetic energy $K = \frac{1}{2}I\omega^2$. Substitute these expressions into Eq. (7.14), $K_1 + U_1 + W_{\text{other}} = K_2 + U_2$ (if nonconservative work is done), or Eq. (7.12), $K_1 + U_1 = K_2 + U_2$ (if only conservative work is done), and solve for the target variables. It's helpful to draw bar graphs showing the initial and final values of K, U, and $E = K + U$.

EVALUATE *your answer:*
Check whether your answer makes physical sense.

Gravitational Potential Energy for an Extended Body

If we have the cable of negligible mass, we could ignore its kinetic energy as well as the gravitational potential energy associated with it. If the mass is *not* negligible, we need to know how to calculate the *gravitational potential energy* associated with such an extended body. If the acceleration of gravity g is the same at all points on the body, the gravitational potential energy is the same as though all the mass were concentrated at the center of mass of the body. Suppose we take the y-axis vertically upward. Then for a body with total mass M, the gravitational potential energy U is simply

$$U = Mgy_{\text{cm}} \quad (\text{gravitational potential energy for an extended body}). \tag{9.18}$$

where y_{cm} is the y-coordinate of the center of mass. This expression applies to any extended body, whether it is rigid or not.

To prove Eq. (9.18), we again represent the body as a collection of mass elements m_i. The potential energy for element m_i is $m_i g y_i$, so the total potential energy is

$$U = m_1 g y_1 + m_2 g y_2 + \ldots = (m_1 y_1 + m_2 y_2 + \ldots)\, g\ .$$

But from Eq. (8.28), which defines the coordinates of the center of mass,

$$m_1 y_1 + m_2 y_2 + \ldots = \left(m_1 + m_2 + \ldots\right) y_{\text{cm}} = M y_{\text{cm}} .$$

where $M = m_1 + m_2 + \ldots$ is the total mass. Combining this with the above expression for *U*, we find $U = Mgy_{\text{cm}}$ in agreement with Eq. (9.18).

We leave the application of Eq. (9.18) to the problems. In Chapter 10 we'll use this equation to help us analyze rigid-body problems in which the axis of rotation moves.

9.5 Parallel-Axis Theorem

We pointed out in Section 9.4 that a body doesn't have just one moment of inertia. In fact, it has infinitely many, because there are infinitely many axes about which it might rotate. But there is a simple relationship, called the **parallel-axis theorem**, between the moment of inertia of a body about an axis through its center of mass and the moment of inertia about any other axis parallel to the original axis (**Fig. 9.12**):

Parallel-axis theorem:

Moment of inertia of a body for a rotation axis through point *P*

$$I_P = I_{\text{cm}} + Md^2 \qquad (9.19)$$

Moment of inertia of body for a parallel axis through center of mass

Mass of body

Distance between two parallel axes

To prove this theorem, we consider two axes, both parallel to the *z*-axis: one through the center of mass and the other through a point *P* (**Fig. 9.13**). First we take a very thin slice of the body, parallel to the *xy*-plane and perpendicular to the *z*-axis. We take the origin of our coordinate system to be at the center of mass of the body; the coordinates of the center of mass are then $x_{\text{cm}} = y_{\text{cm}} = z_{\text{cm}} = 0$. The axis through the center of mass passes through this thin slice at point *O*, and the parallel axis passes through point *P*, whose *x*- and *y*-coordinates are (a, b). The distance of this axis from the axis through the center of mass is *d*, where $d^2 = a^2 + b^2$.

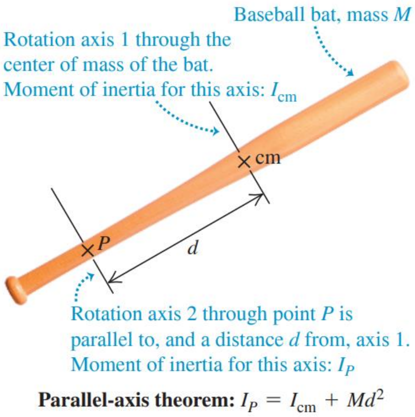


Figure 9.12 - The parallel-axis theorem

We can write an expression for the moment of inertia I_P about the axis through point *P*. Let m_i be a mass element in our slice, with coordinates (x_i, y_i, z_i). Then the moment of inertia I_{cm} of the slice about the axis through the center of mass (at *O*) is

$$I_{\text{cm}} = \sum_i m_i \left(x_i^2 + y_i^2\right).$$

The moment of inertia of the slice about the axis through *P* is

$$I_P = \sum_i m_i \left[\left(x_i - a\right)^2 + \left(y_i - b\right)^2\right].$$

These expressions don't involve the coordinates z_i measured perpendicular to the slices, so we can extend the sums to include *all* particles in *all* slices. Then I_P becomes the moment of inertia of the *entire* body for an axis through *P*. We then expand the squared terms and regroup, and obtain

$$I_P = \sum_i m_i \left(x_i^2 + y_i^2\right) - 2a \sum_i m_i x_i - 2b \sum_i m_i y_i + \left(a^2 + b^2\right) \sum_i m_i .$$

The first sum is I_{cm}. From Eq. (8.28), the definition of the center of mass, the second and third sums are proportional to x_{cm} and y_{cm}; these are zero because we have taken our origin to be the center of mass. The final term is d^2 multiplied by the total mass, or Md^2. This completes our proof that $I_P = I_{cm} + Md^2$.

As Eq. (9.19) shows, a rigid body has a lower moment of inertia about an axis through its center of mass than about any other parallel axis. Thus it's easier to start a body rotating if the rotation axis passes through the center of mass. This suggests that it's somehow most natural for a rotating body to rotate about an axis through its center of mass; we'll make this idea more quantitative in Chapter 10.

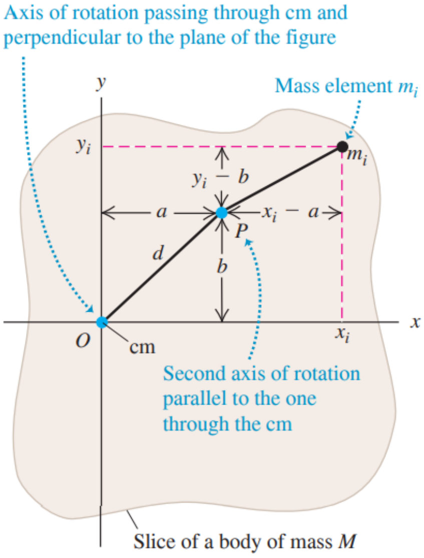


Figure 9.13 - The mass element m_i has coordinates (x_i, y_i) with respect to an axis of rotation through the center of mass (cm) and coordinates $(x_i - a, y_i - b)$ with respect to the parallel axis through point P

9.6 Moment-of-Inertia Calculations

If a rigid body is a continuous distribution of mass - like a solid cylinder or a solid sphere - it cannot be represented by a few point masses. In this case the *sum* of masses and distances that defines the moment of inertia [Eq. (9.16)] becomes an *integral*. Imagine dividing the body into elements of mass *dm* that are very small, so that all points in a particular element are at essentially the same perpendicular distance from the axis of rotation. We call this distance *r*, as before. Then the moment of inertia is

$$I = \int r^2 dm . \tag{9.20}$$

To evaluate the integral, we have to represent *r* and *dm* in terms of the same integration variable. When the body is effectively one-dimensional, such as the slender rods (a) and (b) in Table 9.2, we can use a coordinate *x* along the length and relate *dm* to an increment *dx*. For a three-dimensional body it is usually easiest to express *dm* in terms of an element of volume *dV* and the *density* ρ of the body. Density is mass per unit volume, $\rho = dm/dV$, so we may write Eq. (9.20) as

$$I = \int r^2 \rho dV .$$

This expression tells us that a body's moment of inertia depends on how its density varies within its volume. By measuring small variations in the orbits of satellites, geophysicists can measure the earth's moment of inertia. This tells us how our planet's mass is distributed within its interior. The data show that the earth is far denser at the core than in its outer layers. If the body is uniform in density, then we may take ρ outside the integral:

$$I = \rho \int r^2 dV . \tag{9.21}$$

To use this equation, we have to express the volume element *dV* in terms of the differentials of the integration variables, such as $dV = dx\,dy\,dz$. The element *dV* must always be chosen so that all points within it are at very nearly the same distance from the axis of rotation. The limits on the integral are determined by the shape and dimensions of the body. For regularly shaped bodies, this integration is often easy to do.

CHAPTER 9: SUMMARY

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Rotational kinematics: When a rigid body rotates about a stationary axis (usually called the *z*-axis), the body's position is described by an angular coordinate θ. The angular velocity ω_z is the time derivative of θ, and the angular acceleration α_z is the time derivative of ω_z or the second derivative of θ. If the angular acceleration is constant, then θ, ω_z, and α_z are related by simple kinematic equations analogous to those for straight-line motion with constant linear acceleration

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$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

$$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta\omega_z}{\Delta t} = \frac{d\omega_z}{dt}$$

Constant α_z only:

$$\theta = \theta_0 + \omega_{0z} t + \tfrac{1}{2}\alpha_z t^2$$

$$\theta - \theta_0 = \tfrac{1}{2}\left(\omega_{0z} + \omega_z\right)t$$

$$\omega_z = \omega_{0z} + \alpha_z t$$

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z\left(\theta - \theta_0\right)$$

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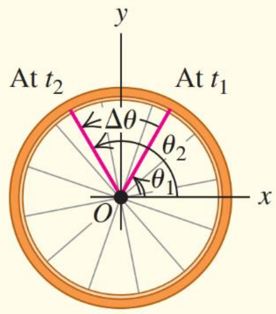

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Relating linear and angular kinematics: The angular speed ω of a rigid body is the magnitude of the body's angular velocity. The rate of change of ω is $\alpha = d\omega/dt$. For a particle in the body a distance r from the rotation axis, the speed υ and the components of the acceleration $\vec{a}$ are related to ω and α

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$$\upsilon = r\omega$$

$$a_{\text{tan}} = \frac{d\upsilon}{dt} = r\frac{d\omega}{dt} = r\alpha$$

$$a_{\text{rad}} = \frac{\upsilon^2}{r} = \omega^2 r$$

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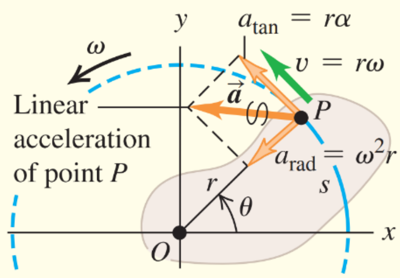

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Moment of inertia and rotational kinetic energy: The moment of inertia *I* of a body about a given axis is a measure of its rotational inertia: The greater the value of *I*, the more difficult it is to change the state of the body rotation. The moment of inertia can be expressed as a sum over the particles m_i that make up the body, each of which is at its own perpendicular distance r_i from the axis. The rotational kinetic energy of a rigid body rotating about a fixed axis depends on the angular speed ω and the moment of inertia *I* for that rotation axis

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$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots$$

$$= \sum_i m_i r_i^2$$

$$K = \tfrac{1}{2} I \omega^2$$

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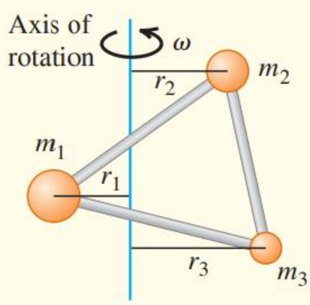

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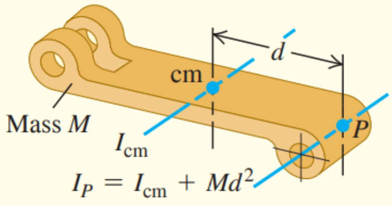
Calculating the moment of inertia: The parallel-axis theorem relates the moments of inertia of a rigid body of mass M about two parallel axes: an axis through the center of mass (moment of inertia I_{cm}) and a parallel axis a distance d from the first axis (moment of inertia I_P). If the body has a continuous mass distribution, the moment of inertia can be calculated by integration	$I_P = I_{cm} + Md^2$.	

10 DYNAMICS OF ROTATIONAL MOTION

We learned in Chapters 4 and 5 that a net force applied to an object gives that object an acceleration. But what does it take to give an object an *angular* acceleration? That is, what does it take to start a stationary object rotating or to bring a spinning object to a halt? A force is required, but it must be applied in a way that gives a twisting or turning action.

In this chapter we'll define a new physical quantity, *torque*, that describes the twisting or turning effort of a force. We'll find that the net torque acting on a rigid body determines its angular acceleration, in the same way that the net force on an object determines its linear acceleration. We'll also look at work and power in rotational motion so as to understand, for example, how energy is transferred by an electric motor. Next we'll develop a new conservation principle, *conservation of angular momentum*, that is tremendously useful for understanding the rotational motion of both rigid and nonrigid bodies. We'll finish this chapter by studying *gyroscopes*, rotating devices that don't fall over when you might think they should—but that actually behave in accordance with the dynamics of rotational motion.

10.1 Torque

We know that forces acting on an object can affect its **translational motion**—that is, the motion of the object as a whole through space. Now we want to learn which aspects of a force determine how effective it is in causing or changing *rotational* motion. The magnitude and direction of the force are important, but so is the point on the object where the force is applied. In **Fig. 10.1** a wrench is being used to loosen a tight bolt. Force $\vec{F}_b$, applied near the end of the handle, is more effective than an equal force $\vec{F}_a$ applied near the bolt. Force $\vec{F}_c$ does no good; it's applied at the same point and has the same magnitude as $\vec{F}_b$, but it's directed along the length of the handle. The quantitative measure of the tendency of a force to cause or change an object's rotational motion is called *torque*; we say that $\vec{F}_a$ applies a torque about point *O* to the wrench in Fig. 10.1, $\vec{F}_b$ applies a greater torque about *O*, and $\vec{F}_c$ applies zero torque about *O*.

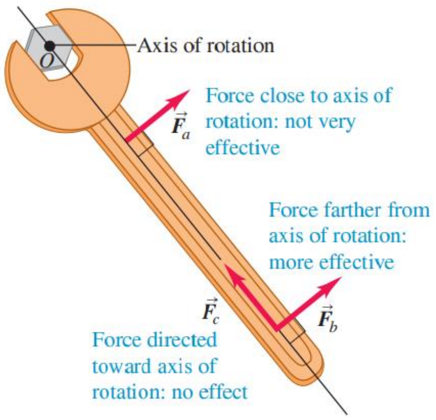


Figure 10.1 - Which of these three equal-magnitude forces is most likely to loosen the tight bolt?

Figure 10.2 shows three examples of how to calculate torque. The object can rotate about an axis that is perpendicular to the plane of the figure and passes through point *O*. Three forces act on the object in the plane of the figure. The tendency of the first of these forces, $\vec{F}_1$, to cause a rotation about *O* depends on its magnitude F_1. It also depends on the *perpendicular* distance l_1 between point *O* and the **line of action** of the force (that is, the line along which the force vector lies). We call the distance l_1 the **lever arm** (or **moment** arm) of force $\vec{F}_1$ about *O*. The twisting effort is directly proportional to both F_1 and l_1, so we define the **torque** (or moment) of the force $\vec{F}_1$ with respect to *O* as the product $F_1 l_1$. We use the Greek letter τ (tau) for torque. If a force of magnitude *F* has a line of action that is a perpendicular distance *l* from *O*, the torque is

$$\tau = Fl. \tag{10.1}$$

Physicists usually use the term "torque," while engineers usually use "moment" (unless they are talking about a rotating shaft).

The lever arm of $\vec{F}_1$ in Fig. 10.2 is the perpendicular distance l_1, and the lever arm of $\vec{F}_2$ is the perpendicular distance l_2. The line of action of $\vec{F}_3$ passes through point *O*, so the lever arm for

$\vec{F}_3$ is zero and its torque with respect to O is zero. In the same way, force $\vec{F}_c$ in Fig. 10.1 has zero torque with respect to point O; $\vec{F}_b$ has a greater torque than $\vec{F}_a$ because its lever arm is greater.

CAUTION! **Torque is always measured about a point** Torque is *always* defined with reference to a specific point. If we shift the position of this point, the torque of each force may change. For example, the torque of force $\vec{F}_3$ in **Fig. 10.2** is zero with respect to point O but not with respect to point A. It's not enough to refer to "the torque of $\vec{F}$"; you must say "the torque of $\vec{F}$ with respect to point X" or "the torque of $\vec{F}$ about point X."

Force $\vec{F}_1$ in Fig. 10.2 tends to cause *counterclockwise* rotation about O, while $\vec{F}_2$ tends to cause *clockwise* rotation. To distinguish between these two possibilities, we need to choose a positive sense of rotation. With the choice *that counterclockwise torques are positive and clockwise torques are negative*, the torques of $\vec{F}_1$ and $\vec{F}_2$ about O are

$$\tau_1 = +F_1 l_1 \; \tau_2 = -F_2 l_2 .$$

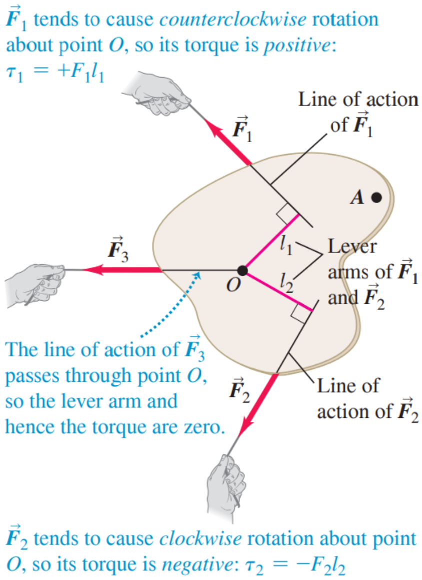


Figure 10.2 - The torque of a force about a point is the product of the force magnitude and the lever arm of the force

Figure 10.2 shows this choice for the sign of torque. We'll often use the symbol ⊕ to indicate our choice of the positive sense of rotation.

The SI unit of torque is the newton-meter. In our discussion of work and energy we called this combination the joule. But torque is *not* work or energy, and torque should be expressed in newton-meters, *not* joules.

Figure 10.3 shows a force $\vec{F}$ applied at point P, located at position $\vec{r}$ with respect to point O. There are three ways to calculate the torque of $\vec{F}$:

1. Find the lever arm l and use $\tau = Fl$.
2. Determine the angle ϕ between the vectors $\vec{r}$ and $\vec{F}$; the lever arm is $r\sin\phi$, so $\tau = rF\sin\phi$.
3. Represent $\vec{F}$ in terms of a radial component F_{rad} Frad along the direction of $\vec{r}$ and a tangential component $F_{\tan}$ at right angles, perpendicular to $\vec{r}$. (We call this component *tangential* because if the object rotates, the point where the force acts moves in a circle, and this component is tangent to that circle). Then $F_{\tan} = F\sin\phi$ and $\tau = r(F\sin\phi) = F_{\tan} r$. The component F_{rad} produces *no* torque with respect to O because its lever arm with respect to that point is zero (compare to forces $\vec{F}_c$ in Fig. 10.1 and $\vec{F}_3$ in Fig. 10.2).

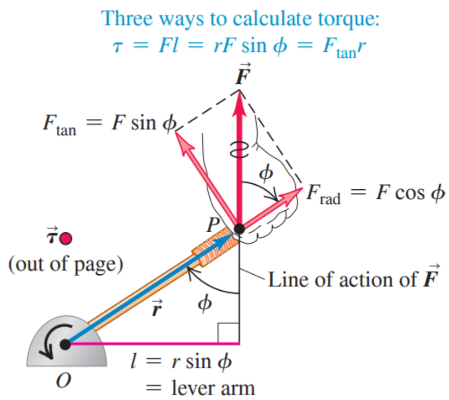


Figure 10.3 - Three ways to calculate the torque of force $\vec{F}$ about point O. In this figure, $\vec{r}$ and $\vec{F}$ are in the plane of the page and the torque vector $\vec{\tau}$ points out of the page toward you

Summarizing these three expressions for torque, we have

Magnitude of $\vec{r}$ (vector from O to where $\vec{F}$ acts)
Lever arm of $\vec{F}$
Magnitude of torque due to force $\vec{F}$ relative to point O

$$\tau = Fl = rF\sin\phi = F_{\tan} r \qquad (10.2)$$

Magnitude of $\vec{F}$
Angle between $\vec{r}$ and $\vec{F}$
Tangential component of $\vec{F}$

Torque as a Vector

We saw in Section 9.1 that angular velocity and angular acceleration can be represented as vectors; the same is true for torque. To see how to do this, note that the quantity $rF\sin\phi$ in Eq. (10.2) is the magnitude of the *vector product* $\vec{r}\times\vec{F}$ that we defined in Section 1.10. (Go back and review that definition). We generalize the definition of torque as follows: When a force $\vec{F}$ acts at a point having a position vector $\vec{r}$ with respect to an origin *O*, as in Fig. 10.3, the torque $\vec{\tau}$ of the force with respect to *O* is the *vector* quantity

$$\underbrace{\vec{\tau}}_{\text{Torque vector due to force } \vec{F} \text{ relative to point } O} = \underbrace{\vec{r}}_{\text{Vector from } O \text{ to where } \vec{F} \text{ acts}} \times \underbrace{\vec{F}}_{\text{Force } \vec{F}} \qquad (10.3)$$

The torque as defined in Eq. (10.2) is the magnitude of the torque vector $\vec{r}\times\vec{F}$. The direction of $\vec{\tau}$ is perpendicular to both $\vec{r}$ and $\vec{F}$. In particular, if both $\vec{r}$ and $\vec{F}$ lie in a plane perpendicular to the axis of rotation, as in Fig. 10.3, then the torque vector $\vec{\tau}=\vec{r}\times\vec{F}$ is directed along the axis of rotation, with a sense given by the right-hand rule (see Fig. 1.30 and **Fig. 10.4**).

Because $\vec{\tau}=\vec{r}\times\vec{F}$ is perpendicular to the plane of the vectors $\vec{r}$ and $\vec{F}$, it's common to have diagrams like Fig. 10.4, in which one of the vectors is perpendicular to the page. We use a dot (•) to represent a vector that points out of the page and a cross (×) to represent a vector that points into the page (see Figs. 10.3 and 10.4).

In the following sections we'll usually be concerned with rotation of an object about an axis oriented in a specified constant direction. In that case, only the component of torque along that axis will matter. We often call that component the torque with respect to the specified *axis*.

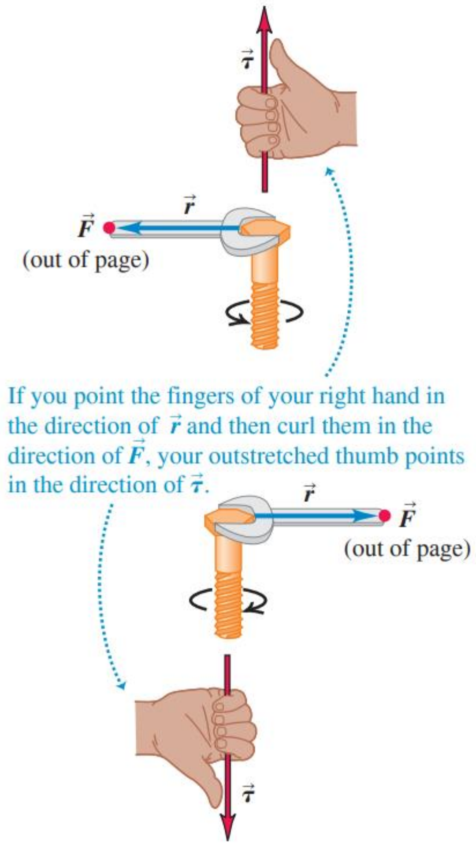


Figure 10.4 - The torque vector $\vec{\tau}=\vec{r}\times\vec{F}$ is directed along the axis of the bolt, perpendicular to both $\vec{r}$ and $\vec{F}$. The fingers of the right hand curl in the direction of the rotation that the torque tends to cause

10.2 Torque and Angular Acceleration for a Rigid Body

We're now ready to develop the fundamental relationship for the rotational dynamics of a rigid body (an object with a definite and unchanging shape and size). We'll show that the angular acceleration of a rotating rigid body is directly proportional to the sum of the torque components along the axis of rotation. The proportionality factor is the moment of inertia.

To develop this relationship, let's begin as we did in Section 9.4 by envisioning the rigid body as being made up of a large number of particles. We choose the axis of rotation to be the *z*-axis; the first particle has mass m_1 and distance r_1 from this axis (**Fig. 10.5**). The *net force* $\vec{F}_1$ acting on this particle has a component $F_{1,\text{rad}}$,rad along the radial direction, a component $F_{1,\tan}$,tan that is tangent to the circle of radius r_1 in which the particle moves as the body rotates, and a component F_{1z} along the axis of rotation. Newton's second law for the tangential component is

$$F_{1,\tan} = m_1 a_{1,\tan}\,. \qquad (10.4)$$

We can express the tangential acceleration of the first particle in terms of the angular acceleration α_z of the body by using Eq. (9.14): $a_{1,\tan} = r_1\alpha_z$. Using this relationship and multiplying both sides of Eq. (10.4) by r_1, we obtain

$$F_{1,\tan} r_1 = m_1 r_1^2 \alpha_z. \tag{10.5}$$

From Eq. (10.2), $F_{1,\tan} r_1$ is the *torque* of the net force with respect to the rotation axis, equal to the component τ_{1z} of the torque vector along the rotation axis. The subscript *z* is a reminder that the torque affects rotation around the *z*-axis, in the same way that the subscript on F_{1z} is a reminder that this force affects the motion of particle 1 along the *z*-axis.

Neither of the components $F_{1,\tan}$ or F_{1z} contributes to the torque about the *z*-axis, since neither tends to change the particle's rotation about that axis. So $\tau_{1z} = F_{1,\tan} r_1$ is the total torque acting on the particle with respect to the rotation axis. Also, $m_1 r_1^2$ is I_1, the moment of inertia of the particle about the rotation axis. Hence we can rewrite Eq. (10.5) as

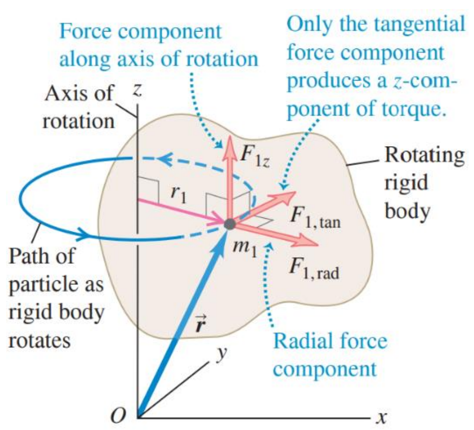


Figure 10.5 - As a rigid body rotates around the *z*-axis, a net force $\vec{F}_1$ acts on one particle of the body. Only the force component $F_{1,\tan}$ can affect the rotation, because only $F_{1,\tan}$ exerts a torque about *O* with a *z*-component (along the rotation axis)

$$\tau_{1z} = I_1\alpha_z = m_1 r_1^2 \alpha_z.$$

We write such an equation for every particle in the body, then add all these equations:

$$\tau_{1z} + \tau_{2z} + \ldots = I_1\alpha_z + I_2\alpha_z + \ldots = m_1 r_1^2\alpha_z + m_2 r_2^2 \alpha_z + \ldots$$

or

$$\sum \tau_{iz} = \left(\sum m_i r_1^2\right)\alpha_z. \tag{10.6}$$

The left side of Eq. (10.6) is the sum of all the torques about the rotation axis that act on all the particles. The right side is $I = \sum m_i r_i^2$, the total moment of inertia about the rotation axis, multiplied by the angular acceleration α_z. Note that α_z is the same for every particle because this is a *rigid* body. Thus Eq. (10.6) says that for the rigid body as a whole,

Rotational analog of Newton's second law for a rigid body:

Net torque on a rigid body about z-axis

$$\sum \tau_z = I\alpha_z \tag{10.7}$$

Moment of inertia of rigid body about z-axis

Angular acceleration of rigid body about z-axis

Just as Newton's second law says that a net *force* on a particle causes an *acceleration* in the direction of the net force, Eq. (10.7) says that a net *torque* on a rigid body about an axis causes an *angular acceleration* about that axis.

Our derivation assumed that the angular acceleration α_z is the same for all particles in the body. So Eq. (10.7) is valid *only* for *rigid* bodies. Hence this equation doesn't apply to a rotating tank of water or a swirling tornado of air, different parts of which have different angular accelerations. Note that since our derivation used Eq. (9.14), $a_{\tan} = r\alpha_z, \alpha_z$ must be measured in rad/s^2 .

The torque on each particle is due to the net force on that particle, which is the vector sum of external and *internal* forces (see Section 8.2). According to Newton's third law, the internal forces that any pair of particles in the rigid body exert on each other are equal in magnitude and opposite in direction (**Fig. 10.6**). If these forces act along the line joining the two particles, their lever arms with respect to any axis are also equal. So the torques for each such pair are equal and opposite, and add to zero. Hence *all* the internal torques add to zero, so the sum $\sum \tau_z$ in Eq. (10.7) includes only the torques of the *external* forces.

Often, an important external force acting on a rigid body is its *weight*. This force is not concentrated at a single point; it acts on every particle in the entire body. Nevertheless, if $\vec{g}$ has the same value at all points, we always get the correct torque (about any specified axis) if we assume that all the weight is concentrated at the *center of mass* of the body. We'll prove this statement in Chapter 11, but meanwhile we'll use it for some of the problems in this chapter.

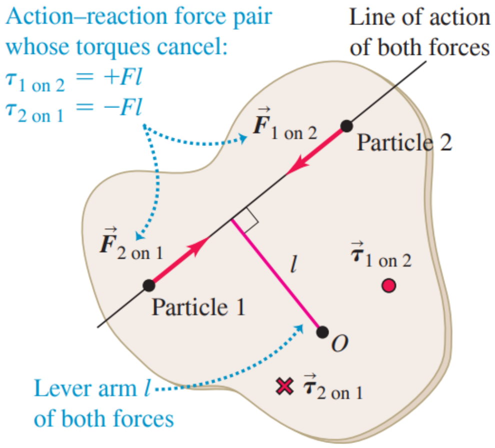


Figure 10.6 - Why only *external* torques affect a rigid body's rotation: Any two particles in the body exert equal and opposite forces on each other. If the forces act along the line joining the particles, the lever arms of the forces with respect to an axis through O are the same and the torques due to the two forces are equal and opposite

PROBLEM-SOLVING STRATEGY
10.1 Rotational Dynamics for Rigid Bodies

Our strategy for solving problems in rotational dynamics is very similar to Problem-Solving Strategy 5.2 for solving problems involving Newton's second law.

IDENTIFY *the relevant concepts:*

Equation (10.7), $\sum \tau_z = I\alpha_z$, is useful whenever torques act on a rigid body. Sometimes you can use an energy approach instead, as we did in Section 9.4. However, if the target variable is a force, a torque, an acceleration, an angular acceleration, or an elapsed time, using $\sum \tau_z = I\alpha_z$ is almost always best.

SET UP *the problem*:

- Sketch the situation and identify the body or bodies to be analyzed. Indicate the rotation axis.
- For each body, draw a free-body diagram that shows the body's *shape*, including all dimensions and angles. Label pertinent quantities with algebraic symbols.
- Choose coordinate axes for each body and indicate a positive sense of rotation (clockwise or counterclockwise) for each rotating body. If you know the sense of α_z, pick that as the positive sense of rotation.

EXECUTE *the solution*:

1. For each body, decide whether it undergoes translational motion, rotational motion, or both. Then apply $\sum \vec{F} = m\vec{a}$ (as in Section 5.2), $\sum \tau_z = I\alpha_z$, or both to the body.
2. Express in algebraic form any *geometrical* relationships between the motions of two or more bodies. An example is a string that unwinds, without slipping, from a pulley or a wheel that rolls without slipping (discussed in Section 10.3). These relationships usually appear as relationships between linear and/or angular accelerations.

3. Ensure that you have as many independent equations as there are unknowns. Solve the equations to find the target variables.

EVALUATE *your answer:*

Check that the algebraic signs of your results make sense. As an example, if you are unrolling thread from a spool, your answers should not tell you that the spool is turning in the direction that rolls the thread back onto the spool! Check that any algebraic results are correct for special cases or for extreme values of quantities.

10.3 Rigid-Body Rotation about a Moving Axis

We can extend our analysis of rigid-body rotational dynamics to some cases in which the axis of rotation moves. When that happens, the motion of the rigid body is **combined translation and rotation**. The key to understanding such situations is this: Every possible motion of a rigid body can be represented as a combination of *translational motion of the center of mass and rotation about an axis through the center of mass*. This is true even when the center of mass accelerates, so it is not at rest in any inertial frame. Figure 10.7 illustrates this for the motion of a tossed baton: The center of mass of the baton follows a parabolic curve, as though the baton were a particle located at the center of mass. A rolling ball is another example of combined translational and rotational motions.

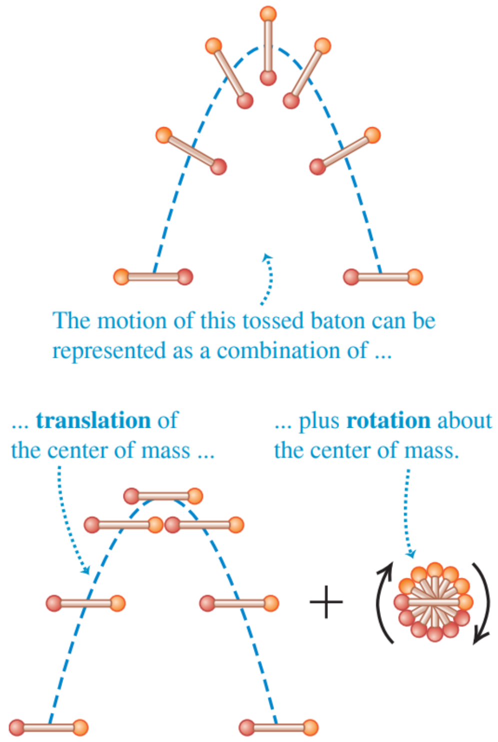


Figure 10.7 -The motion of a rigid body is a combination of translational motion of the center of mass and rotation around the center of mass

Combined Translation and Rotation: Energy Relationships

It's beyond our scope to prove that rigid-body motion can always be divided into translation of the center of mass and rotation about the center of mass. But we *can* prove this for the kinetic energy K of a rigid body that has both translational and rotational motions. For such a rigid body, K is the sum of two parts:

$$\underbrace{K}_{\text{Kinetic energy of a rigid body with both translation and rotation}} = \overbrace{\tfrac{1}{2}Mv_{\mathrm{cm}}^2}^{\text{Kinetic energy of translation of center of mass (cm)}} + \overbrace{\tfrac{1}{2}I_{\mathrm{cm}}\omega^2}^{\text{Kinetic energy of rotation around axis through cm}} \qquad (10.8)$$

M: Mass of rigid body; v_{cm}: Speed of cm; I_{cm}: Moment of inertia of rigid body about axis through cm; ω: Angular speed of rigid body

To prove this relationship, we again imagine the rigid body to be made up of particles. For a typical particle with mass m_i (**Fig. 10.8**), the velocity $\vec{v}_i$ of this particle relative to an inertial frame is the vector sum of the velocity $\vec{v}_{\mathrm{cm}}$ of the center of mass and the velocity $\vec{v}_i\,'$ of the particle *relative* to the center of mass:

$$\vec{v}_i = \vec{v}_{\mathrm{cm}} + \vec{v}_i\,'. \qquad (10.9)$$

The kinetic energy K_i of this particle in the inertial frame is $\frac{1}{2}m_i\upsilon_i^2$, which we can also express as $\frac{1}{2}m_i(\vec{\upsilon}_i \bullet \vec{\upsilon}_i)$. Substituting Eq. (10.9) into this, we get

$$\begin{aligned} K_i &= \tfrac{1}{2}m_i(\vec{\upsilon}_{\text{cm}} + \vec{\upsilon}_i\,')\bullet(\vec{\upsilon}_{\text{cm}} + \vec{\upsilon}_i\,') \\ &= \tfrac{1}{2}m_i(\vec{\upsilon}_{\text{cm}}\bullet\vec{\upsilon}_{\text{cm}} + 2\vec{\upsilon}_{\text{cm}}\bullet\vec{\upsilon}_i\,' + \vec{\upsilon}_i\,'\bullet\vec{\upsilon}_i\,') \\ &= \tfrac{1}{2}m_i(\upsilon_{\text{cm}}{}^2 + 2\vec{\upsilon}_{\text{cm}}\bullet\vec{\upsilon}_i\,' + \upsilon_i\,'^2) \end{aligned}$$

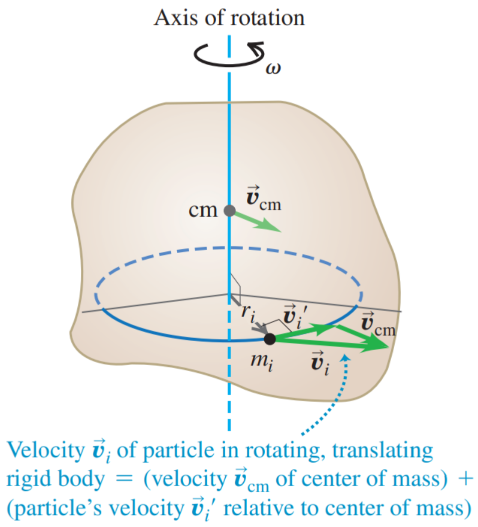


Figure 10.8 - A rigid body with both translational and rotational motions

The total kinetic energy is the sum $\sum K_i$ for all the particles making up the rigid body. Expressing the three terms in this equation as separate sums, we get

$$K = \sum K_i = \sum\left(\tfrac{1}{2}m_i\upsilon_{\text{cm}}{}^2\right) + \sum\left(m_i\vec{\upsilon}_{\text{cm}}\bullet\vec{\upsilon}_i\,'\right) + \sum\left(\tfrac{1}{2}m_i\upsilon_i\,'^2\right).$$

The first and second terms have common factors that we take outside the sum:

$$K = \tfrac{1}{2}\left(\sum m_i\right)\upsilon_{\text{cm}}{}^2 + \vec{\upsilon}_{\text{cm}}\bullet\left(\sum m_i\vec{\upsilon}_i\,'\right) + \sum\left(\tfrac{1}{2}m_i\upsilon_i\,'^2\right). \qquad (10.10)$$

Now comes the reward for our effort. In the first term, $\sum m_i$ is the total mass *M*. The second term is zero because $\sum m_i\vec{\upsilon}_i\,'$ is *M* times the velocity of the center of mass *relative to the center of mass*, and this is zero by definition. The last term is the sum of the kinetic energies of the particles computed by using their speeds with respect to the center of mass; this is just the kinetic energy of rotation around the center of mass. Using the same steps that led to Eq. (9.17) for the rotational kinetic energy of a rigid body, we can write this last term as $\frac{1}{2}I_{\text{cm}}\omega^2$, where I_{cm} is the moment of inertia with respect to the axis through the center of mass and ω is the angular speed. So Eq. (10.10) becomes Eq. (10.8):

$$K = \tfrac{1}{2}M\upsilon_{\text{cm}}{}^2 + \tfrac{1}{2}I_{\text{cm}}\omega^2.$$

Rolling Without Slipping

An important case of combined translation and rotation is **rolling without slipping**. The rolling wheel in **Fig. 10.9** is symmetrical, so its center of mass is at its geometric center. We view the motion in an inertial frame of reference in which the surface on which the wheel rolls is at rest.

In this frame, the point on the wheel that contacts the surface must be instantaneously *at rest* so that it does not slip. Hence the velocity $\vec{\upsilon}_1\,'$ of the point of contact relative to the center of mass must have the same magnitude but opposite direction as the center-of-mass velocity $\vec{\upsilon}_{\text{cm}}$. If the wheel's radius is R and its angular speed about the center of mass is v, then the magnitude of $\vec{\upsilon}_1\,'$ is $R\omega$; hence

Condition for rolling without slipping:

$$\underbrace{\upsilon_{\text{cm}}}_{\text{Speed of center of mass of rolling wheel}} = \underbrace{R}_{\text{Radius of wheel}}\underbrace{\omega}_{\text{Angular speed of wheel}} \qquad (10.11)$$

As Fig. 10.9 shows, the velocity of a point on the wheel is the vector sum of the velocity of the center of mass and the velocity of the point relative to the center of mass. Thus while point 1 the point of contact, is instantaneously at rest, point 3 at the top of the wheel is moving forward *twice as fast* as the center of mass, and points 2 and 4 at the sides have velocities at 45° to the horizontal.

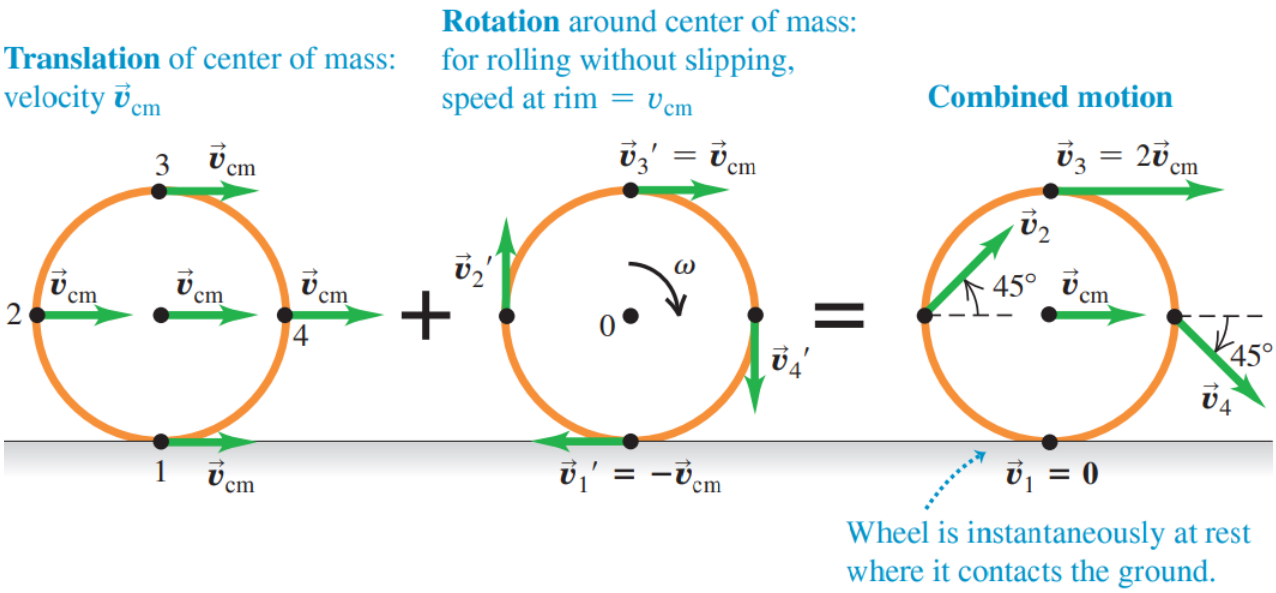


Figure 10.9 - The motion of a rolling wheel is the sum of the translational motion of the center of mass and the rotational motion of the wheel around the center of mass

At any instant we can think of the wheel as rotating about an "instantaneous axis" of rotation that passes through the point of contact with the ground. The angular velocity ω is the same for this axis as for an axis through the center of mass; an observer at the center of mass sees the rim make the same number of revolutions per second as does an observer at the rim watching the center of mass spin around him. If we think of the motion of the rolling wheel in Fig. 10.9 in this way, the kinetic energy of the wheel is $K = \frac{1}{2} I_i \omega^2$, where I_1 is the moment of inertia of the wheel about an axis through point 1. But by the parallel-axis theorem, Eq. (9.19), $I_1 = I_{cm} + MR^2$, where M is the total mass of the wheel and I_{cm} is the moment of inertia with respect to an axis through the center of mass. Using Eq. (10.11), we find that the wheel's kinetic energy is as given by Eq. (10.8):

$$K = \tfrac{1}{2} I_1 \omega^2 = \tfrac{1}{2} I_{cm} \omega^2 + \tfrac{1}{2} MR^2 \omega^2 + \tfrac{1}{2} M \upsilon_{cm}{}^2 .$$

CAUTION! **Rolling without slipping.** The relationship $\upsilon_{cm} = R\omega$ holds *only* if there is rolling without slipping. When a drag racer first starts to move, the rear tires are spinning very fast even though the racer is hardly moving, so $R\omega$ is greater than υ_{cm}. If a driver applies the brakes too heavily so that the car skids, the tires will spin hardly at all and $R\omega$ is less than υ_{cm}.

If a rigid body changes height as it moves, we must also consider gravitational potential energy. We saw in Section 9.4 that for any extended object of mass M, rigid or not, the gravitational potential energy U is the same as if we replaced the object by a particle of mass M located at the object's center of mass, so

$$U = Mgy_{cm} .$$

Combined Translation and Rotation: Dynamics

We can also analyze the combined translational and rotational motions of a rigid body from the standpoint of dynamics. We showed in Section 8.5 that for an extended object, the acceleration of the center of mass is the same as that of a particle of the same mass acted on by all the external forces on the actual object:

$$\underbrace{\sum \vec{F}_{ext}}_{\text{Net external force on an object}} = \underbrace{M}_{\text{Total mass of object}} \underbrace{\vec{a}_{cm}}_{\text{Acceleration of center of mass}} \tag{10.12}$$

The rotational motion about the center of mass is described by the rotational analog of Newton's second law, Eq. (10.7):

Net torque on a rigid body about z-axis through center of mass

Moment of inertia of rigid body about z-axis

$$\sum \tau_z = I_{\text{cm}}\alpha_z \tag{10.13}$$

Angular acceleration of rigid body about z-axis

It's not immediately obvious that Eq. (10.13) should apply to the motion of a translating rigid body; after all, our derivation of $\sum \tau_z = I\alpha_z$ in Section 10.2 assumed that the axis of rotation was stationary. But Eq. (10.13) is valid *even when the axis of rotation moves*, provided the following two conditions are met:

1. The axis through the center of mass must be an axis of symmetry.
2. The axis must not change direction.

These conditions are satisfied for many types of rotation. Note that in general this moving axis of rotation is *not* at rest in an inertial frame of reference.

We can now solve dynamics problems involving a rigid body that undergoes translational and rotational motions at the same time, provided that the rotation axis satisfies the two conditions just mentioned. Problem-Solving Strategy 10.1 (Section 10.2) is equally useful here, and you should review it now. Keep in mind that when a rigid body undergoes translational and rotational motions at the same time, we need two separate equations of motion *for the same body* Eq. (10.12) for the translation of the center of mass and Eq. (10.13) for rotation about an axis through the center of mass.

Rolling Friction

In Example 10.5 we said that we can ignore rolling friction if both the rolling body and the surface over which it rolls are perfectly rigid. In Fig. 10.10a a perfectly rigid sphere is rolling down a perfectly rigid incline. The line of action of the normal force passes through the center of the sphere, so its torque is zero; there is no sliding at the point of contact, so the friction force does no work. Figure 10.10b shows a more realistic situation, in which the surface "piles up" in front of the sphere and the sphere rides in a shallow trench. Because of these deformations, the contact forces on the sphere no longer act along a single point but over an area; the forces are concentrated on the front of the sphere as shown. As a result, the normal force now exerts a torque that opposes the rotation. In addition, there is some sliding of the sphere over the surface due to the deformation, causing mechanical energy to be lost. The combination of these two effects is the phenomenon of *rolling friction*. Rolling friction also occurs if the rolling body is deformable, such as an automobile tire. Often the rolling body and the surface are rigid enough that rolling friction can be ignored, as we have assumed in all the examples in this section.

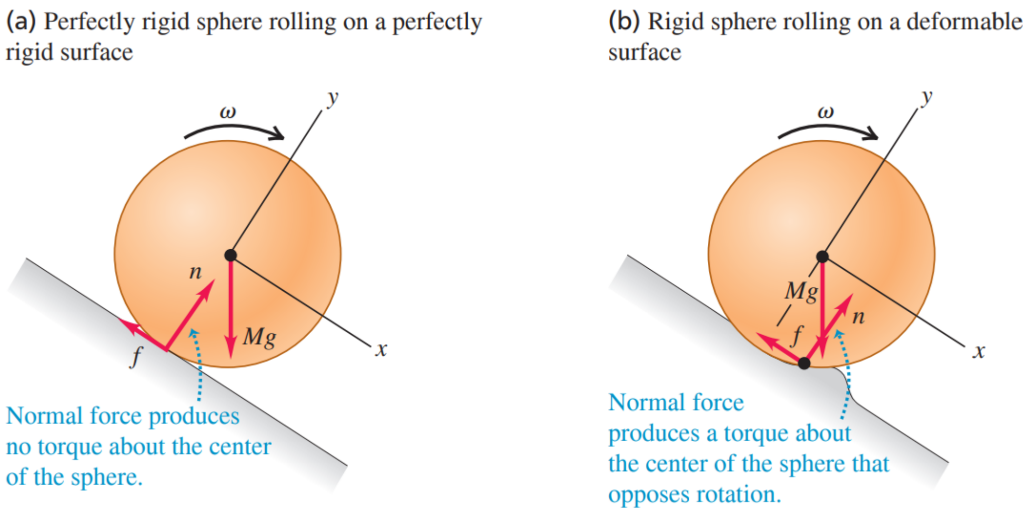


Figure 10.10 - Rolling down (a) a perfectly rigid surface and (b) a deformable surface. In (b) the deformation is greatly exaggerated, and the force n is the component of the contact force that points normal to the plane of the surface before it is deformed

10.4 Work and Power in Rotational Motion

When you pedal a bicycle, you apply forces to a rotating body and do work on it. Similar things happen in many other real-life situations, such as a rotating motor shaft driving a power tool or a car engine propelling the vehicle. Let's see how to apply our ideas about work from Chapter 6 to rotational motion.

Suppose a tangential force $\vec{F}_{\tan}$ acts at the rim of a pivoted disk - for example, a child running while pushing on a playground merry-go-round (Fig. 10.11a). The disk rotates through an infinitesimal angle $d\theta$ about a fixed axis during an infinitesimal time interval *dt* (Fig. 10.11b).

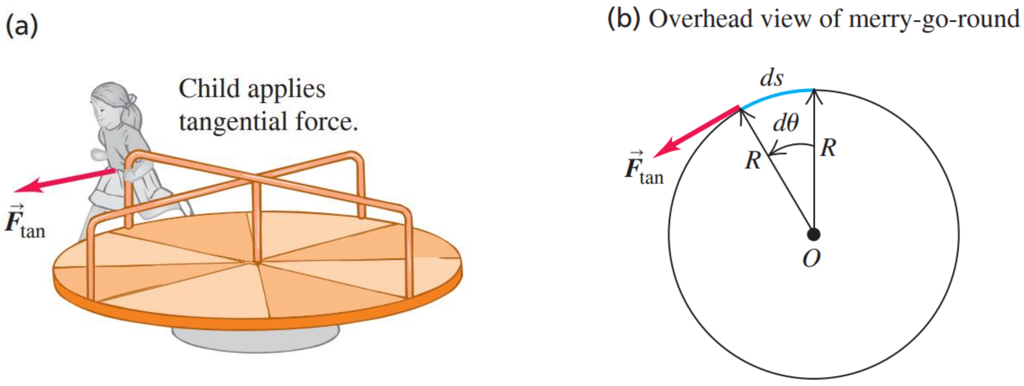


Figure 10.11 - A tangential force applied to a rotating body does work

The work *dW* done by the force $\vec{F}_{\tan}$ while a point on the rim moves a distance *ds* is $dW = \vec{F}_{\tan} ds$. If $d\theta$ is measured in radians, then $ds = R\ d\theta$ and

$$dW = F_{\tan} R\ d\theta\,.$$

Now $F_{\tan} R$ is the torque τ_z due to the force $\vec{F}_{\tan}$, so

$$dW = \tau_z\ d\theta\,. \tag{10.19}$$

As the disk rotates from θ_1 to θ_2, the total work done by the torque is

Work done by a torque τ_z

Upper limit = final angular position

$$W = \int_{\theta_1}^{\theta_2} \tau_z\, d\theta \tag{10.20}$$

Integral of the torque with respect to angle

Lower limit = initial angular position

If the torque remains *constant* while the angle changes, then the work is the product of torque and angular displacement:

Work done by a constant torque τ_z

Torque

$$W = \tau_z(\theta_2 - \theta_1) = \tau_z \Delta\theta \tag{10.21}$$

Final minus initial angular position = angular displacement

If torque is expressed in newton-meters (N • m) and angular displacement in radians, the work is in joules. Equation (10.21) is the rotational analog of Eq. (6.1), *W* = *Fs*, and Eq. (10.20) is the analog of Eq. (6.7), $W = \int F_x dx$, for the work done by a force in a straight-line displacement.

If the force in Fig. 10.21 had an axial component (parallel to the rotation axis) or a radial component (directed toward or away from the axis), that component would do no work because the displacement of the point of application has only a tangential component. An axial or radial component of force would also make no contribution to the torque about the axis of rotation. So Eqs. (10.20) and (10.21) are correct for *any* force, no matter what its components.

When a torque does work on a rotating rigid body, the kinetic energy changes by an amount equal to the work done. We can prove this by using exactly the same procedure that we used in Eqs. (6.11) through (6.13) for the translational kinetic energy of a particle. Let τ_z represent the *net* torque on the body so that $\tau_z = I\alpha_z$ from Eq. (10.7), and assume that the body is rigid so that the moment of inertia I is constant. We then transform the integrand in Eq. (10.20) into an integrand with respect to ω_z as follows:

$$\tau_z\, d\theta = \left(I\alpha_z\right)d\theta = I\frac{d\omega_z}{dt}d\theta = I\frac{d\theta}{dt}d\omega_z = I\omega_z d\omega_z .$$

Since τ_z is the net torque, the integral in Eq. (10.20) is the *total* work done on the rotating rigid body. This equation then becomes

Total work done on a rotating rigid body = work done by the net external torque

$$W_{\text{tot}} = \int_{\omega_1}^{\omega_2} I\omega_z\, d\omega_z = \overbrace{\tfrac{1}{2}I\omega_2^2}^{\text{Final rotational kinetic energy}} - \underbrace{\tfrac{1}{2}I\omega_1^2}_{\text{Initial rotational kinetic energy}} \qquad (10.22)$$

The change in the rotational kinetic energy of a *rigid* body equals the work done by forces exerted from outside the body. This equation is analogous to Eq. (6.13), the work–energy theorem for a particle.

How does *power* relate to torque? When we divide both sides of Eq. (10.19) by the time interval dt during which the angular displacement occurs, we find

$$\frac{dW}{dt} = \tau_z \frac{d\theta}{dt} .$$

But dW/dt is the rate of doing work, or *power*P, and $d\theta/dt$ is angular velocity ω_z:

Power due to a torque acting on a rigid body

$$P = \underbrace{\overbrace{\tau_z}^{\text{Torque with respect to rigid body's rotation axis}}\ \omega_z}_{\omega_z:\ \text{Angular velocity of rigid body about axis}} \qquad (10.23)$$

This is the analog of the relationship $P = \vec{F} \cdot \vec{v}$ that we developed in Section 6.4 for particle motion.

10.5 Angular Momentum

Every rotational quantity that we have encountered in Chapters 9 and 10 is the analog of some quantity in the translational motion of a particle. The analog of *momentum* of a particle is **angular momentum**, a vector quantity denoted as $\vec{L}$. Its relationship to momentum $\vec{p}$ (which we'll often call *linear momentum* for clarity) is exactly the same as the relationship of torque to force, $\vec{\tau} = \vec{r} \times \vec{F}$. For a particle with constant mass m and velocity $\vec{v}$, the angular momentum is

Angular momentum of a particle relative to origin O of an inertial frame of reference

Position vector of particle relative to O

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \qquad (10.24)$$

Linear momentum of particle = mass times velocity

The value of $\vec{L}$ depends on the choice of origin *O*, since it involves the particle's position vector $\vec{r}$ relative to *O*. The units of angular momentum are $\mathrm{kg \cdot m^2/s}$.

In Fig. 10.12 a particle moves in the *xy*-plane; its position vector $\vec{r}$ and momentum $\vec{p} = m\vec{\upsilon}$ are shown. The angular momentum vector $\vec{L}$ is perpendicular to the *xy*-plane. The right-hand rule for vector products shows that its direction is along the +*z*-axis, and its magnitude is

$$L = m\upsilon r \sin\phi = m\upsilon l \,, \tag{10.25}$$

where *l* is the perpendicular distance from the line of $\vec{\upsilon}$ to *O*. This distance plays the role of "lever arm" for the momentum vector.

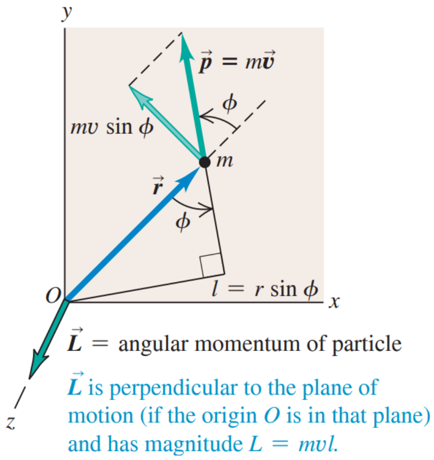


Figure 10.12 - Calculating the angular momentum $\vec{L} = \vec{r} \times m\vec{\upsilon} = \vec{r} \times \vec{p}$ of a particle with mass *m* moving in the *xy*-plane

When a net force $\vec{F}$ acts on a particle, its velocity and momentum change, so its angular momentum may also change. We can show that the *rate of change* of angular momentum is equal to the torque of the net force. We take the time derivative of Eq. (10.24), using the rule for the derivative of a product:

$$\frac{d\vec{L}}{dt} = \left(\frac{d\vec{r}}{dt} + m\vec{\upsilon}\right) + \left(\vec{r} \times m\frac{d\vec{\upsilon}}{dt}\right) = \left(\vec{\upsilon} \times m\vec{\upsilon}\right) + \left(\vec{r} \times m\vec{a}\right).$$

The first term is zero because it contains the vector product of the vector $\vec{\upsilon} = d\vec{r}/dt$ with itself. In the second term we replace $m\vec{a}$ with the net force $\vec{F}$:

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau} \ \text{(for a particle acted on by net force } \vec{F}\text{).} \tag{10.26}$$

The rate of change of angular momentum of a particle equals the torque of the net force acting on it. Compare this result to Eq. (8.4): The rate of change $d\vec{p}/dt$ of the *linear* momentum of a particle equals the net force that acts on it.

Angular Momentum of a Rigid Body

We can use Eq. (10.25) to find the total angular momentum of a *rigid body* rotating about the *z*-axis with angular speed ω . First consider a thin slice of the body lying in the *xy*-plane (**Fig. 10.13**). Each particle in the slice moves in a circle centered at the origin, and at each instant its velocity $\vec{\upsilon}_i$ is perpendicular to its position vector $\vec{r}_i$, as shown. Hence in Eq. (10.25), $\phi = 90°$ for every particle. A particle with mass m_i at a distance r_i from *O* has a speed υ_i equal to $r_i\omega$. From Eq. (10.25) the magnitude L_i of its angular momentum is

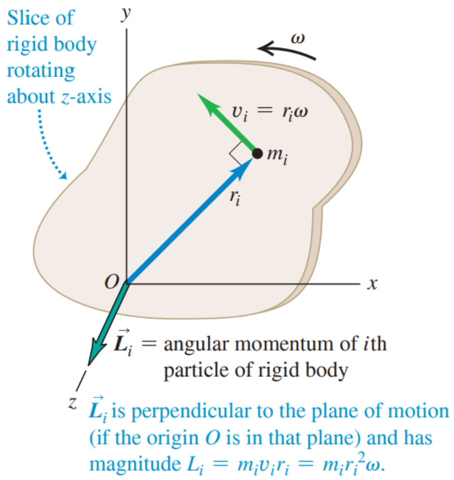


Figure 10.13 - Calculating the angular momentum of a particle of mass m_i in a rigid body rotating at angular speed ω . (Compare Fig. 10.23)

$$L_i = m_i(r_i\omega)r_i = m_i r_i^2 \omega \,. \tag{10.27}$$

The direction of each particle's angular momentum, as given by the right-hand rule for the vector product, is along the +z-axis.

The *total* angular momentum of the slice of the rigid body that lies in the xy-plane is the sum $\sum L_i$ of the angular momenta L_i of all of its particles. From Eq. (10.27),

$$L = \sum L_i = \left(\sum m_i r_i^2\right)\omega = I\omega,$$

where I is the moment of inertia of the slice about the z-axis.

We can do this same calculation for the other slices of the rigid body, all parallel to the xy-plane. For points that do not lie in the xy-plane, a complication arises because the $\vec{r}$ vectors have components in the z-direction as well as in the x- and y-directions; this gives the angular momentum of each particle a component perpendicular to the z-axis. But if the *z-axisis an axis of symmetry*, the perpendicular components for particles on opposite sides of this axis add up to zero (**Fig. 10.14**). So when a rigid body rotates about an axis of symmetry, its angular momentum vector $\vec{L}$ lies along the symmetry axis, and its magnitude is $L = I\omega$.

The angular velocity vector $\vec{\omega}$ also lies along the rotation axis, as we saw in Section 9.1. Hence for a rigid body rotating around an axis of symmetry, $\vec{L}$ and $\vec{\omega}$ are in the same direction (**Fig. 10.15**). So we have the *vector* relationship

Angular momentum of a rigid body rotating around a symmetry axis

Moment of inertia of rigid body about symmetry axis

$$\vec{L} = I\vec{\omega} \tag{10.28}$$

Angular velocity vector of rigid body

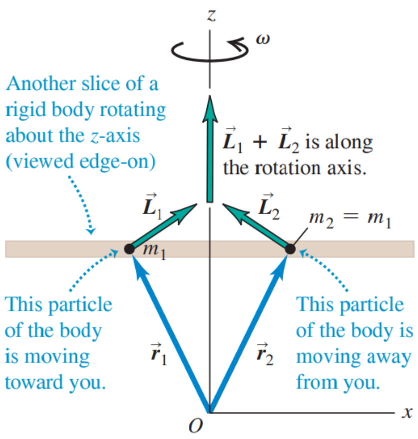


Figure 10.14 - Two particles of the same mass located symmetrically on either side of the rotation axis of a rigid body.
The angular momentum vectors $\vec{L}_1$ and $\vec{L}_2$ of the two particles do not lie along the rotation axis, but their vector sum $\vec{L}_1 + \vec{L}_2$ does

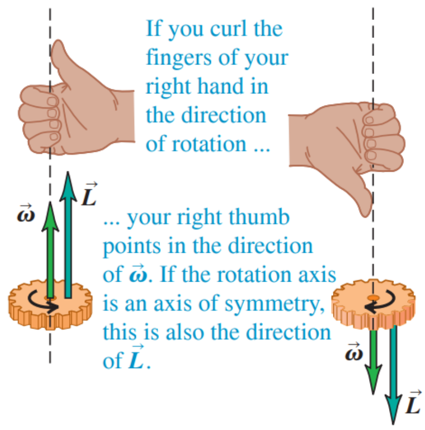


Figure 10.15 - For rotation about an axis of symmetry, $\vec{\omega}$ and $\vec{L}$ are parallel and along the axis. The directions of both vectors are given by the right-hand rule (compare Fig. 9.5)

From Eq. (10.26) the rate of change of angular momentum of a particle equals the torque of the net force acting on the particle. For any system of particles (including both rigid and nonrigid bodies), the rate of change of the *total* angular momentum equals the sum of the torques of all forces acting on all the particles. The torques of the *internal* forces add to zero if these forces act along the line from one particle to another, as in Fig. 10.8, and so the sum of the torques includes only the torques of the *external* forces. (We saw a similar cancellation in our discussion of center-of-mass motion in Section 8.5). So we conclude that

For a system of particles:

Sum of external torques on the system

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt} \tag{10.29}$$

Rate of change of total angular momentum $\vec{L}$ of system

Finally, if the system of particles is a rigid body rotating about a symmetry axis (the z-axis), then $L_z = I\omega_z$ and I is constant. If this axis has a fixed direction in space, then vectors $\vec{L}$ and $\vec{\omega}$ change only in magnitude, not in direction. In that case, $dL_z / dt = I \; d\omega_z / dt = I\alpha_z$, or

$$\sum \tau_z = I\alpha_z \; .$$

which is again our basic relationship for the dynamics of rigid-body rotation. If the body is *not* rigid, I may change; in that case, L changes even when ω is constant. For a nonrigid body, Eq. (10.29) is still valid, even though Eq. (10.7) is not.

When the axis of rotation is *not* a symmetry axis, the angular momentum is in general *not* parallel to the axis (**Fig. 10.16**). As the rigid body rotates, the angular momentum vector $\vec{L}$ traces out a cone around the rotation axis. Because $\vec{L}$ changes, there must be a net external torque acting on the body even though the angular velocity magnitude ω may be constant. If the body is an unbalanced wheel on a car, this torque is provided by friction in the bearings, which causes the bearings to wear out. "Balancing" a wheel means distributing the mass so that the rotation axis is an axis of symmetry; then $\vec{L}$ points along the rotation axis, and no net torque is required to keep the wheel turning.

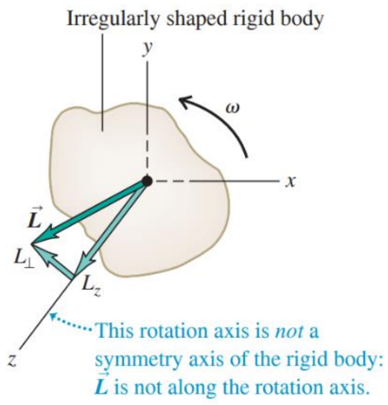


Figure 10.16 - If the rotation axis of a rigid body is not a symmetry axis, $\vec{L}$ does not in general lie along the rotation axis. Even if $\vec{\omega}$ is constant, the direction of $\vec{L}$ changes and a net torque is required to maintain rotation

In fixed-axis rotation we often use the term "angular momentum of the body" to refer to only the *component* of $\vec{L}$ along the rotation axis of the body (the z-axis in **Fig. 10.17**), with a positive or negative sign to indicate the sense of rotation just as with angular velocity.

10.6 Conservation of Angular Momentum

We have just seen that angular momentum can be used for an alternative statement of the basic dynamic principle for rotational motion. It also forms the basis for the principle of **conservation of angular momentum**. Like conservation of energy and of linear momentum, this principle is a universal conservation law, valid at all scales from atomic and nuclear systems to the motions of galaxies. This principle follows directly from Eq. (10.29): $\sum \vec{\tau} = d\vec{L} / dt$. If $\sum \vec{\tau} = 0$, then $d\vec{L} / dt = 0$, and $\vec{L}$ is constant

CONSERVATION OF ANGULAR MOMENTUM When the net external torque acting on a system is zero, the total angular momentum of the system is constant (conserved).

A circus acrobat, a diver, and an ice skater pirouetting on one skate all take advantage of this principle. Suppose an acrobat has just left a swing; she has her arms and legs extended and is rotating counterclockwise about her center of mass. When she pulls her arms and legs in, her moment of inertia I_{cm} with respect to her center of mass changes from a large value I_1 to a much smaller value I_2. The only external force acting on her is her weight, which has no torque with respect to an axis through her center of mass. So her angular momentum $L_z = I_{cm}\omega_z$ remains constant, and her angular velocity ω_z increases as I_{cm} decreases. That is,

$$I_1\omega_{1z} = I_2\omega_{2z} \text{ (zero net external torque).} \qquad (10.30)$$

When a skater or ballerina spins with arms outstretched and then pulls her arms in, her angular velocity increases as her moment of inertia decreases. In each case there is conservation of angular momentum in a system in which the net external torque is zero.

When a system has several parts, the internal forces that the parts exert on one another cause changes in the angular momenta of the parts, but the *total* angular momentum doesn't change.

Here's an example. Consider two objects A and B that interact with each other but not with anything else, such as the astronauts we discussed in Section 8.2 (see Fig. 8.9). Suppose object A exerts a force $\vec{F}_{A \text{ on } B}$ on B on object B; the corresponding torque (with respect to whatever point we choose) is $\vec{\tau}_{A \text{ on } B}$. According to Eq. (10.29), this torque is equal to the rate of change of angular momentum of B:

$$\vec{\tau}_{A \text{ on } B} = \frac{d\vec{F}_B}{dt}.$$

At the same time, object B exerts a force $\vec{F}_{A \text{ on } B}$ on object A, with a corresponding torque $\vec{\tau}_{A \text{ on } B}$, and

$$\vec{\tau}_{A \text{ on } B} = \frac{d\vec{L}_A}{dt}.$$

From Newton's third law, $\vec{F}_{A \text{ on } B} = -\vec{F}_{A \text{ on } B}$. Furthermore, if the forces act along the same line, as in Fig. 10.8, their lever arms with respect to the chosen axis are equal. Thus the *torques* of these two forces are equal and opposite, and $\vec{\tau}_{A \text{ on } B} = -\vec{\tau}_{A \text{ on } B}$. So if we add the two preceding equations, we find

$$\frac{d\vec{L}_A}{dt} + \frac{d\vec{L}_B}{dt} = 0$$

or, because $\vec{L}_A + \vec{L}_B$ is the total angular momentum $\vec{L}$ of the system,

$$\frac{d\vec{L}}{dt} = 0 \text{ (zero net external torque).} \qquad (10.31)$$

That is, the total angular momentum of the system is constant. The torques of the internal forces can transfer angular momentum from one object to the other, but they can't change the *total* angular momentum of the system (Fig. 10.17).

Figure 10.17 - A falling cat twists different parts of its body in different directions so that it lands feet first. At all times during this process the angular momentum of the cat as a whole remains zero

10.7 Gyroscopes and Precession

In all the situations we've looked at so far in this chapter, the axis of rotation either has stayed fixed or has moved and kept the same direction (such as rolling without slipping). But a variety of new physical phenomena, some quite unexpected, can occur when the axis of rotation changes direction. For example, consider a toy gyroscope that's supported at one end (**Fig. 10.18**). If we hold it with the flywheel axis horizontal and let go, the free end of the axis simply drops owing to gravity - *if* the flywheel isn't spinning. But if the flywheel *is* spinning, what happens is quite different. One possible motion is a steady circular motion of the axis in a horizontal plane, combined with the spin motion of the flywheel about the axis. This surprising, nonintuitive motion of the axis is called **precession**. Precession is found in nature as well as in rotating machines such as gyroscopes. As you read these words, the earth itself is precessing; its spin axis (through the north and south poles) slowly changes direction, going through a complete cycle of precession every 26,000 years.

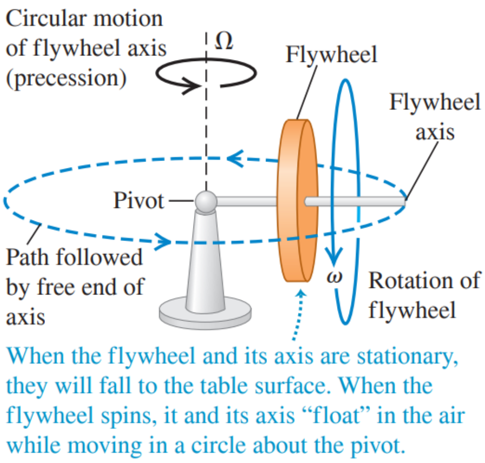


Figure 10.18 - A gyroscope supported at one end. The horizontal circular motion of the flywheel and axis is called precession. The angular speed of precession is Ω

To study this strange phenomenon of precession, we must remember that angular velocity, angular momentum, and torque are all *vector* quantities. In particular, we need the general relationship between the net torque $\sum\vec{\tau}$ that acts on an object and the rate of change of the object's angular momentum $\vec{L}$, given by Eq. (10.29), $\sum\vec{\tau} = d\vec{L}/dt$. Let's first apply this equation to the case in which the flywheel is *not* spinning (Fig. 10.19a). We take the origin *O* at the pivot and assume that the flywheel is symmetrical, with mass *M* and moment of inertia *I* about the flywheel axis. The flywheel axis is initially along the *x*-axis. The only external forces on the gyroscope are the normal force $\vec{n}$ acting at the pivot (assumed to be frictionless) and the weight $\vec{w}$ of the flywheel that acts at its center of mass, a distance *r* from the pivot. The normal force has zero torque with respect to the pivot, and the weight has a torque $\vec{\tau}$ in the *y*-direction, as shown in Fig. 10.19a. Initially, there is no rotation, and the initial angular momentum $\vec{L}_i$ is zero. From Eq. (10.29) the *change* $d\vec{L}$ in angular momentum in a short time interval *dt* following this is

$$d\vec{L} = \vec{\tau}\,dt\,. \tag{10.32}$$

This change is in the *y*-direction because T S is. As each additional time interval *dt* elapses, the angular momentum changes by additional increments $d\vec{L}$ in the *y*-direction because the direction of the torque is constant (Fig. 10.19b). The steadily increasing horizontal angular momentum means that the gyroscope rotates downward faster and faster around the *y*-axis until it hits either the stand or the table on which it sits.

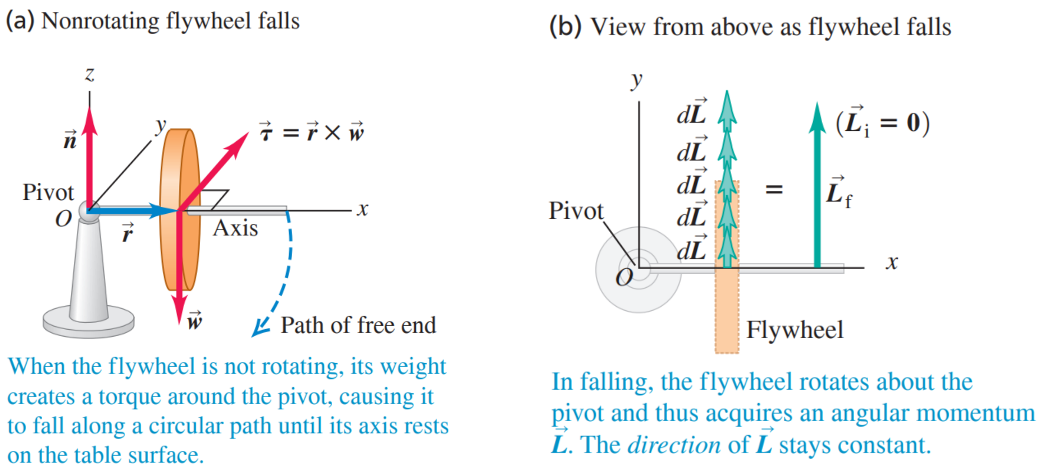


Figure 10.19 - (a) If the flywheel in Fig. 10.18 is initially not spinning, its initial angular momentum is zero. (b) In each successive time interval *dt*, the torque produces a change $d\vec{L} = \vec{\tau}\,dt$ in the angular momentum. The flywheel acquires an angular momentum $\vec{L}$ in the same direction as $\vec{\tau}$, and the flywheel axis falls

Now let's see what happens if the flywheel is spinning initially, so the initial angular momentum $\vec{L}_i$ is not zero (Fig. 10.20a). Since the flywheel rotates around its symmetry axis, $\vec{L}_i$ lies along this axis. But each change in angular momentum $d\vec{L}$ is perpendicular to the flywheel axis because the torque $\vec{\tau} = \vec{r} \times \vec{w}$ is perpendicular to that axis (Fig. 10.20b).

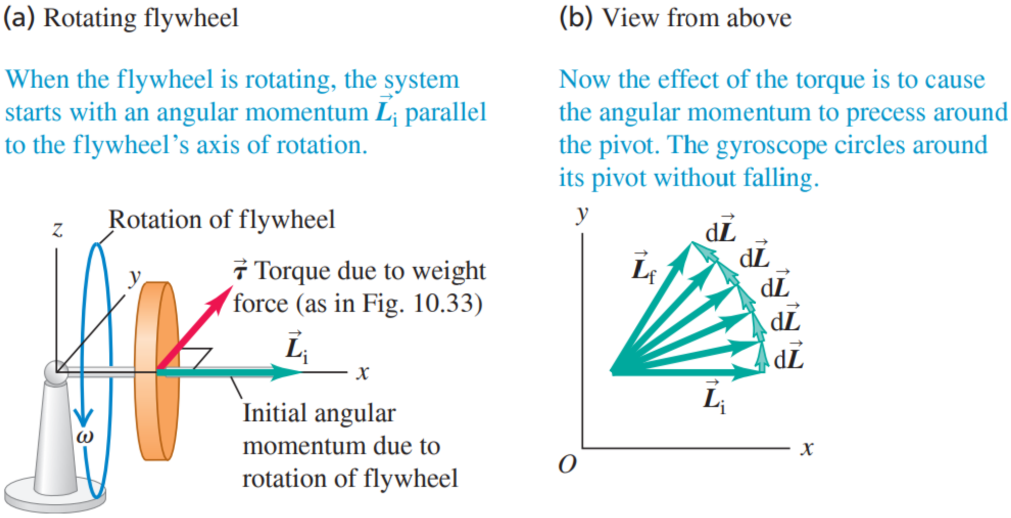


Figure 10.20 - (a) The flywheel is spinning initially with angular momentum $\vec{L}_i$. The forces (not shown) are the same as those in Fig. 10.19a. (b) Because the initial angular momentum is not zero, each change $d\vec{L} = \vec{\tau}dt$ in angular momentum is perpendicular to $\vec{L}$. As a result, the magnitude of $\vec{L}$ remains the same but its direction changes continuously

This causes the *direction* of $\vec{L}$ to change, but not its magnitude. The changes $d\vec{L}$ are always in the horizontal *xy*-plane, so the angular momentum vector and the flywheel axis with which it moves are always horizontal. That is, the axis doesn't fall - it precesses.

If this still seems mystifying to you, think about a ball attached to a string. If the ball is initially at rest and you pull the string toward you, the ball moves toward you also. But if the ball is initially moving and you continuously pull the string in a direction perpendicular to the ball's motion, the ball moves in a circle around your hand; it does not approach your hand at all. In the first case the ball has zero linear momentum $\vec{p}$ to start with; when you apply a force $\vec{F}$ toward you for a time *dt*, the ball acquires a momentum $d\vec{p} = \vec{F}\ dt$, which is also toward you. But if the ball already has linear momentum $\vec{p}$, a change in momentum $d\vec{p}$ that's perpendicular to $\vec{p}$ changes the direction of motion, not the speed. Replace $\vec{p}$ with $\vec{L}$ and $\vec{F}$ with $\vec{\tau}$ in this argument, and you'll see that precession is simply the rotational analog of uniform circular motion.

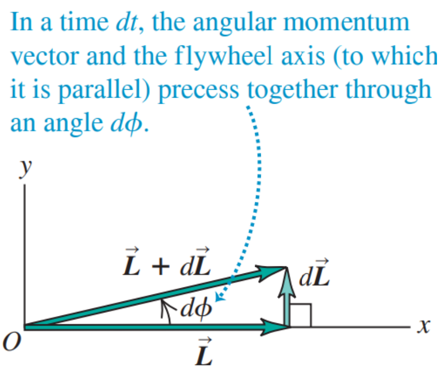


Figure 10.21 Detailed view of part of Fig. 10.20b

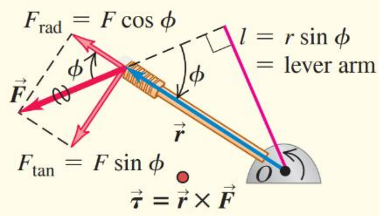
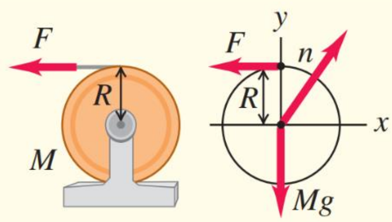
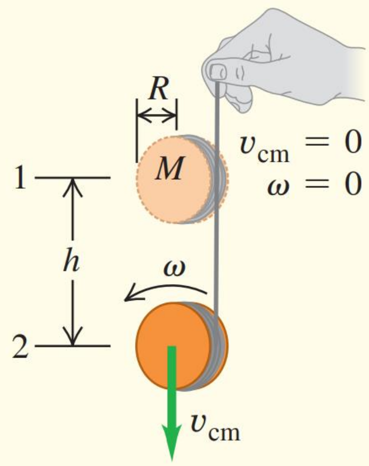
At the instant shown in Fig. 10.20a, the gyroscope has angular momentum $\vec{L}$. A short time interval *dt* later, the angular momentum is $\vec{L} + d\vec{L}$; the infinitesimal change in angular momentum is $d\vec{L} = \vec{\tau}dt$, which is perpendicular to $\vec{L}$. As the vector diagram in **Fig. 10.21** shows, this means that the flywheel axis of the gyroscope has turned through a small angle $d\phi$ given by $d\phi = |d\vec{L}|/|\vec{L}|$. The rate at which the axis moves, $d\phi/dt$, is called the **precession angular speed**; denoting this quantity by Ω, we find

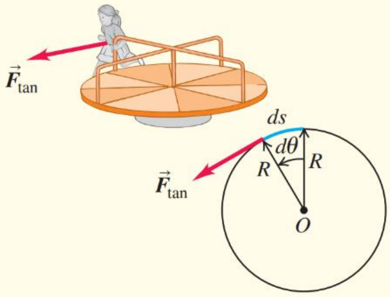
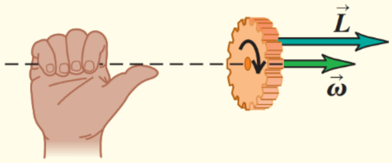
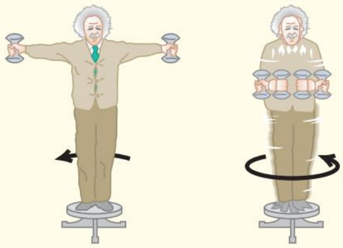
$$\Omega = \frac{d\phi}{dt} = \frac{|d\vec{L}|/|\vec{L}|}{dt} = \frac{\tau_z}{L_z} = \frac{wr}{I\omega}. \tag{10.33}$$

Thus the precession angular speed is *inversely* proportional to the angular speed of spin about the axis. A rapidly spinning gyroscope precesses slowly; if friction in its bearings causes the flywheel to slow down, the precession angular speed *increases*! The precession angular speed of the earth is very slow (1 rev/26,000 yr) because its spin angular momentum L_z is large and the torque τ_z, due to the gravitational influences of the moon and sun, is relatively small.

As a gyroscope precesses, its center of mass moves in a circle with radius r in a horizontal plane. Its vertical component of acceleration is zero, so the upward normal force $\vec{n}$ exerted by the pivot must be just equal in magnitude to the weight. The circular motion of the center of mass with angular speed Ω requires a force $\vec{F}$ directed toward the center of the circle, with magnitude $F = M\Omega^2 r$. This force must also be supplied by the pivot.

One key assumption that we made in our analysis of the gyroscope was that the angular momentum vector $\vec{L}$ is associated with only the spin of the flywheel and is purely horizontal. But there will also be a vertical component of angular momentum associated with the precessional motion of the gyroscope. By ignoring this, we've tacitly assumed that the precession is *slow*—that is, that the precession angular speed Ω is very much less than the spin angular speed ω. As Eq. (10.33) shows, a large value of ω automatically gives a small value of Ω, so this approximation is reasonable. When the precession is not slow, additional effects show up, including an up-and-down wobble or *nutation* of the flywheel axis that's superimposed on the precessional motion. You can see nutation occurring in a gyroscope as its spin slows down, so that Ω increases and the vertical component of $\vec{L}$ can no longer be ignored.

CHAPTER 10: SUMMARY		
Torque: When a force $\vec{F}$ acts on an object, the torque of that force with respect to a point O has a magnitude given by the product of the force magnitude F and the lever arm l. More generally, torque is a vector $\vec{\tau}$ equal to the vector product of $\vec{r}$ (the position vector of the point at which the force acts) and $\vec{F}$. (See Example 10.1)	$\tau = Fl = rF\sin\phi = F_{\tan} r$ $\vec{\tau} = \vec{r} \times \vec{F}$	
Rotational dynamics: The rotational analog of Newton's second law says that the net torque acting on an object equals the product of the object's moment of inertia and its angular acceleration. (See Examples 10.2 and 10.3)	$\sum \tau_z = I\alpha_z$	
Combined translation and rotation: If a rigid body is both moving through space and rotating, its motion can be regarded as translational motion of the center of mass plus rotational motion about an axis through the center of mass. Thus the kinetic energy is a sum of translational and rotational kinetic energies. For dynamics, Newton's second law describes the motion of the center of mass, and the rotational equivalent of Newton's second law describes rotation about the center of mass. In the case of rolling without slipping, there is a special relationship between the motion of the center of mass and the rotational motion. (See Examples 10.4–10.5)	$K = \frac{1}{2}Mv_{cm}^{\ 2} + \frac{1}{2}I_{cm}\omega^2$ $\sum \vec{F}_{ext} = M\vec{a}_{cm}$ $\sum \tau_z = I_{cm}\alpha_z$ $v_{cm} = R\omega$ (rolling without slipping)	

Work done by a torque: A torque that acts on a rigid body as it rotates does work on that body. The work can be expressed as an integral of the torque. The work– energy theorem says that the total rotational work done on a rigid body is equal to the change in rotational kinetic energy. The power, or rate at which the torque does work, is the product of the torque and the angular velocity (See Example 10.6)	$W=\int_{\theta_1}^{\theta_2}\tau_z d\theta$ $W=\tau_z(\theta_2-\theta_1)=\tau_z\Delta\theta$ (constant torque only) $W_{\text{tot}}=\frac{1}{2}I\omega_2^2-\frac{1}{2}\omega_1^2$ $P=\tau_z\omega_z$	
Angular momentum: The angular momentum of a particle with respect to point O is the vector product of the particle's position vector $\vec{r}$ relative to O and its momentum $\vec{p}=m\vec{v}$. When a symmetrical object rotates about a stationary axis of symmetry, its angular momentum is the product of its moment of inertia and its angular velocity vector $\vec{\omega}$. If the object is not symmetrical or the rotation (z) axis is not an axis of symmetry, the component of angular momentum along the rotation axis is $I\omega_z$	$L=\vec{r}\times\vec{p}=\vec{r}\times m\vec{v}$ (particle) $\vec{L}=I\vec{\omega}$ (rigid body rotating about axis of symmetry)	
Rotational dynamics and angular momentum: The net external torque on a system is equal to the rate of change of its angular momentum. If the net external torque on a system is zero, the total angular momentum of the system is constant (conserved). (See Examples 10.7–10.9)	$\sum\vec{\tau}=\frac{d\vec{L}}{dt}$ $\frac{d\vec{L}}{dt}=0$ (zero net external torque)	

11 EQUILIBRIUM AND ELASTICITY

We've devoted a good deal of effort to understanding why and how objects accelerate in response to the forces that act on them. But very often we're interested in making sure that objects *don't* accelerate. Any building, from a multistory skyscraper to the humblest shed, must be designed so that it won't topple over. Similar concerns arise with a suspension bridge, a ladder leaning against a wall, or a crane hoisting a bucket full of concrete.

An object that can be modeled as a *particle* is in equilibrium whenever the vector sum of the forces acting on it is zero. But for the situations we've just described, that condition isn't enough. If forces act at different points on an extended object, an additional requirement must be satisfied to ensure that the object has no tendency to *rotate*: The sum of the *torques* about any point must be zero. This requirement is based on the principles of rotational dynamics. We can compute the torque due to the weight of an object by using the concept of center of gravity, which we introduce in this chapter.

Idealized rigid bodies don't bend, stretch, or squash when forces act on them. But all real materials are *elastic* and do deform to some extent. Elastic properties of materials are tremendously important. You want the wings of an airplane to be able to bend a little, but you'd rather not have them break off. Tendons in your limbs need to stretch when you exercise, but they must return to their relaxed lengths when you stop. Many of the necessities of everyday life, from rubber bands to suspension bridges, depend on the elastic properties of materials. In this chapter we'll introduce the concepts of *stress*, *strain*, and *elastic modulus* and a simple principle called *Hooke's law*, which helps us predict what deformations will occur when forces are applied to a real (not perfectly rigid) object.

11.1 Conditions for Equilibrium

We know that a particle is in *equilibrium*—that is, the particle does not accelerate—in an inertial frame of reference if the vector sum of all the forces acting on the particle is zero, $\sum \vec{F} = 0$. For an *extended* object, the equivalent statement is that the center of mass of the object has zero acceleration if the vector sum of all external forces acting on the object is zero. This is often called the **first condition for equilibrium:**

First condition for equilibrium: For the center of mass of an object at rest to remain at rest ...

$$\sum \vec{F} = \mathbf{0} \tag{11.1}$$

... the *net external force* on the object must be *zero*.

A second condition for an extended object to be in equilibrium is that the object must have no tendency to *rotate*. A rigid body that, in an inertial frame, is not rotating about a certain point has zero angular momentum about that point. If it is not to start rotating about that point, the rate of change of angular momentum must *also* be zero. From the discussion in Section 10.5, particularly Eq. (10.29), this means that the sum of torques due to all the external forces acting on the object must be zero. A rigid body in equilibrium can't have *any* tendency to start rotating about any point,

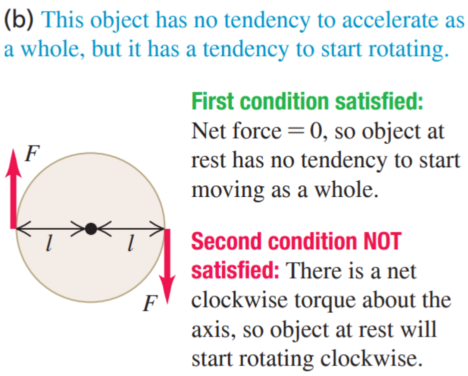
so the sum of external torques must be zero about any point. This is the **second condition for equilibrium:**

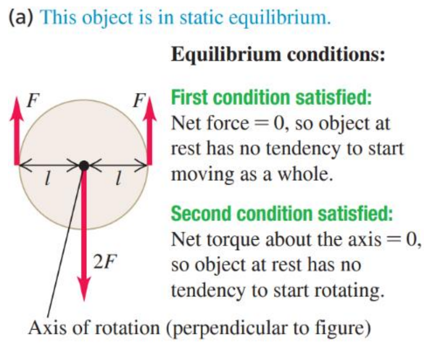
Second condition for equilibrium: For a nonrotating object to remain nonrotating ...

$$\sum \vec{\tau} = \mathbf{0} \tag{11.2}$$

...the *net external torque around any point* on the object must be *zero*.

In this chapter we'll apply the first and second conditions for equilibrium to situations in which a rigid body is at rest (no translation or rotation). Such a rigid body is said to be in **static equilibrium** (**Fig. 11.1**). But the same conditions apply to a rigid body in uniform *translational* motion (without rotation), such as an airplane in flight with constant speed, direction, and altitude. Such a rigid body is in equilibrium but is not static.





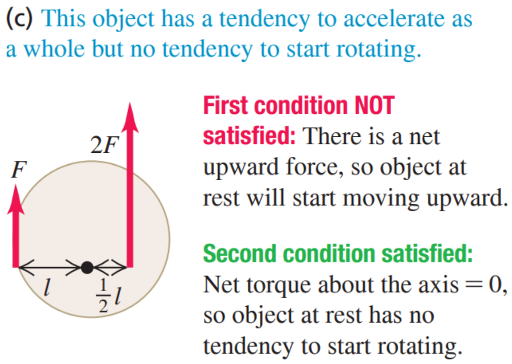


Figure 11.1 - To be in static equilibrium, an object at rest must satisfy both conditions for equilibrium: It can have no tendency to accelerate as a whole or to start rotating

11.2 Center of Gravity

In most equilibrium problems, one of the forces acting on the object is its weight. We need to be able to calculate the *torque* of this force. The weight doesn't act at a single point; it is distributed over the entire object. But we can always calculate the torque due to the object's weight by assuming that the entire force of gravity (weight) is concentrated at a point called the **center of gravity** (abbreviated "cg"). The acceleration due to gravity decreases with altitude; but if we can ignore this variation over the vertical dimension of the object, then the object's center of gravity is identical to its *center of mass* (abbreviated "cm"), which we defined in Section 8.5. We stated this result without proof in Section 10.2, and now we'll prove it.

First let's review the definition of the center of mass. For a collection of particles with masses m_1, m_2 …, and coordinates (x_1, y_1, z_1), (x_2, y_2, z_2),…, the coordinates x_{cm}, y_{cm}, and z_{cm}, of the center of mass of the collection are

$$x_{\text{cm}} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \dots}{m_1 + m_2 + m_3 + \dots} = \frac{\sum_i m_i x_i}{\sum_i m_i},$$

$$y_{\text{cm}} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \dots}{m_1 + m_2 + m_3 + \dots} = \frac{\sum_i m_i y_i}{\sum_i m_i} \text{(center of mass)}, \qquad (11.3)$$

$$z_{\text{cm}} = \frac{m_1z_1 + m_2z_2 + m_3z_3 + \dots}{m_1 + m_2 + m_3 + \dots} = \frac{\sum_i m_i z_i}{\sum_i m_i}.$$

Also, x_{cm}, y_{cm}, *and* z_{cm} are the components of the position vector $\vec{r}_{cm}$ of the center of mass, so Eqs. (11.3) are equivalent to the vector equation

Position vectors of individual particles

Position vector of center of mass of a system of particles

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} \quad (11.4)$$

Masses of individual particles

Now consider the gravitational torque on an object of arbitrary shape (**Fig. 11.2**). We assume that the acceleration due to gravity $\vec{g}$ is the same at every point in the object. Every particle in the object experiences a gravitational force, and the total weight of the object is the vector sum of a large number of parallel forces. A typical particle has mass m_i and weight $\vec{w}_i = m_i\vec{g}_i$. If $\vec{r}_i$ is the position vector of this particle with respect to an arbitrary origin O, then the torque vector $\vec{\tau}_i$ of the weight $\vec{w}_i$ with respect to O is, from Eq. (10.3),

$$\vec{\tau}_i = \vec{r}_i \times \vec{w}_i = \vec{r}_i \times m_i\vec{g} .$$

The *total* torque due to the gravitational forces on all the particles is

$$\begin{aligned}\vec{\tau} &= \sum_i \vec{\tau}_i = \vec{r}_1 \times m_1\vec{g} + \vec{r}_2 \times m_2\vec{g} + \ldots \\ &= (m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots) \times \vec{g} \\ &= (\sum_i m_i\vec{r}_i) \times \vec{g}\end{aligned}$$

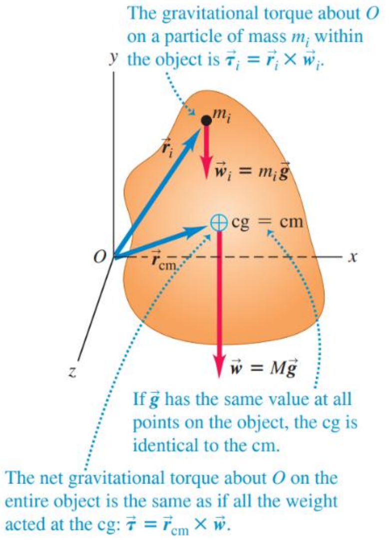


Figure 11.2 - The center of gravity (cg) and center of mass (cm) of an extended object

When we multiply and divide this result by the total mass of the object,

$$M = m_1 + m_2 + \ldots = \sum_i m_i ,$$

we get

$$\vec{\tau} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots}{m_1 + m_2 + \ldots} \times M\vec{g} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} \times M\vec{g} .$$

The fraction in this equation is just the position vector $\vec{r}_{cm}$ of the center of mass, with components x_{cm}, y_{cm}, and z_{cm}, as given by Eq. (11.4), and $M\vec{g}$ is equal to the total weight $\vec{w}$ of the object. Thus

$$\vec{\tau} = \vec{r}_{cm} \times M\vec{g} = \vec{r}_{cm} \times \vec{w} . \quad (11.5)$$

The total gravitational torque, given by Eq. (11.5), is the same as though the total weight $\vec{w}$ were acting at the position $\vec{r}_{cm}$ of the center of mass, which we also call the *center of gravity*. **If $\vec{g}$ has the same value at all points on an object, its center of gravity is identical to its center of mass.** Note, however, that the center of mass is defined independently of any gravitational effect. While the value of $\vec{g}$ varies somewhat with elevation, the variation is extremely slight. We'll assume throughout this chapter that the center of gravity and center of mass are identical unless explicitly stated otherwise.

Finding and Using the Center of Gravity

We can often use symmetry considerations to locate the center of gravity of an object, just as we did for the center of mass. The center of gravity of a homogeneous sphere, cube, or rectangular plate is at its geometric center. The center of gravity of a right circular cylinder or cone is on its axis of symmetry.

For an object with a more complex shape, we can sometimes locate the center of gravity by thinking of the object as being made of symmetrical pieces. For example, we could approximate the human body as a collection of solid cylinders, with a sphere for the head. Then we can locate the center of gravity of the combination with Eqs. (11.3), letting $m_1, m_2, \dots$ be the masses of the individual pieces and $(x_1, y_1, z_1), (x_2, y_2, z_2), \dots$ be the coordinates of their centers of gravity.

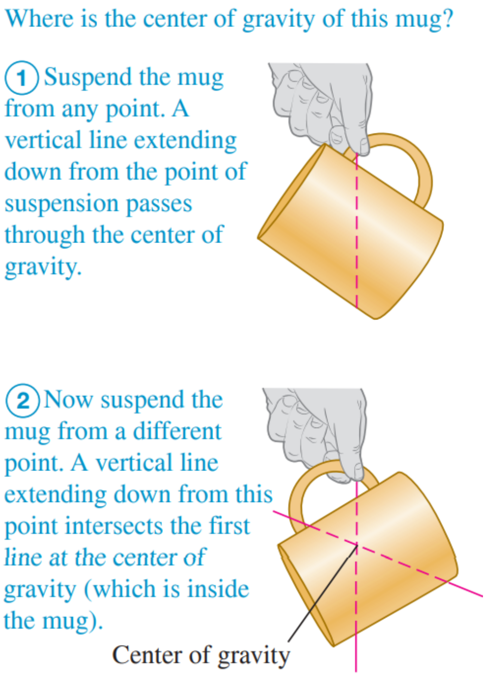


Figure 11.3 - Finding the center of gravity of an irregularly shaped object - in this case, a coffee mug.

When an object in rotational equilibrium and acted on by gravity is supported or suspended at a single point, the center of gravity is always at or directly above or below the point of suspension. If it were anywhere else, the weight would have a torque with respect to the point of suspension, and the object could not be in rotational equilibrium. **Figure 11.3** shows an application of this idea.

Using the same reasoning, we can see that an object supported at several points must have its center of gravity somewhere within the area bounded by the supports. This explains why a car can drive on a straight but slanted road if the slant angle is relatively small (Fig. 11.4a) but will tip over if the angle is too steep (Fig. 11.4b). The truck in Fig. 11.4c has a higher center of gravity than the car and will tip over on a shallower incline.

The lower the center of gravity and the larger the area of support, the harder it is to overturn an object. Four-legged animals such as deer and horses have a large area of support bounded by their legs; hence they are naturally stable and need only small feet or hooves. Animals that walk on two legs, such as humans and birds, need relatively large feet to give them a reasonable area of support. If a two-legged animal holds its body approximately horizontal, like a chicken or the dinosaur *Tyrannosaurus rex*, it must perform a balancing act as it walks to keep its center of gravity over the foot that is on the ground. A chicken does this by moving its head; *T. rex* probably did it by moving its massive tail.

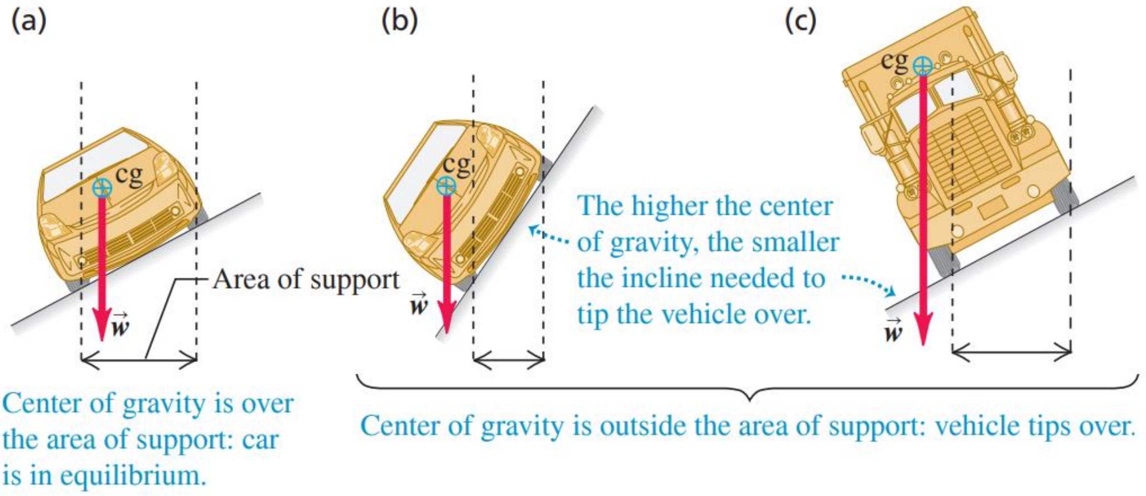


Figure 11.4 - In (a) the center of gravity is within the area bounded by the supports, and the car is in equilibrium. The car in (b) and the truck in (c) will tip over because their centers of gravity lie outside the area of support

EXAMPLE 11.1 Walking the plank

A uniform plank of length L = 6.0 m and mass M = 90 kg rests on sawhorses separated by D = 1.5m and equidistant from the center of the plank. Cousin Throckmorton wants to stand on the right-hand end of the plank. If the plank is to remain at rest, how massive can Throckmorton be?

IDENTIFY and SET UP

To just balance, Throckmorton's mass m must be such that the center of gravity of the plank–Throcky system is directly over the right-hand sawhorse (Fig. 11.6). We take the origin at C, the geometric center and center of gravity of the plank, and take the positive x-axis horizontally to the right. Then the centers of gravity of the plank and Throcky are at $x_P = 0$ and $x_T = L/2 = 3.0$ m, respectively, and the right-hand sawhorse is at $x_S = D/2$. We'll use Eqs. (11.3) to locate the center of gravity x_{cg} of the plank–Throcky system.

EXECUTE

From the first of Eqs. (11.3),

$$x_{cg} = \frac{M(0) + m(L/2)}{M+m} = \frac{m}{M+m}\frac{L}{2}.$$

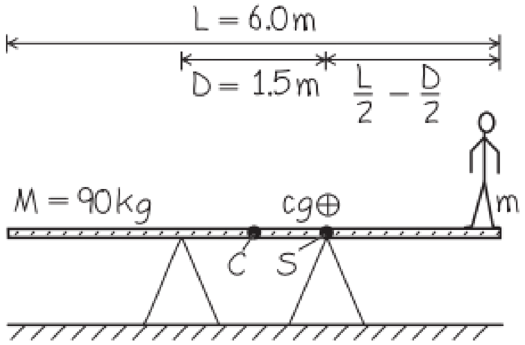


Figure 11.6 - Our sketch for this problem

We set $x_{cg} = x_S$ and solve for m:

$$\frac{m}{M+m}\frac{L}{2} = \frac{D}{2},$$

$$mL = (M+m)D,$$

$$m = M\frac{D}{L-D} = (90\text{ kg})\frac{1.5m}{6.0m - 1.5m} = 30\text{ kg}.$$

EVALUATE

As a check, let's repeat the calculation with the origin at the right-hand sawhorse. Now $x_S = 0$, $x_P = -D/2$, and $x_T = (L/2) - (D/2)$, and we require $x_{cg} = x_S = 0$:

$$x_{cg} = \frac{M(-D/2) + m[(L/2) - (D/2)]}{M+m} = 0,$$

$$m = \frac{MD/2}{(L/2)-(D/2)} = M\frac{D}{L-D} = 30\text{ kg}.$$

The result doesn't depend on our choice of origin.

A 60 kg adult could stand only halfway between the right-hand sawhorse and the end of the plank. Can you see why?

KEYCONCEPT

If an extended object supported at two or more points is to be in equilibrium, its center of gravity must be somewhere within the area bounded by the supports. If the object is supported at only one point, its center of gravity must be above that point.

11.3 Solving Rigid-body Equilibrium problems

There are just two key conditions for rigid-body equilibrium: The vector sum of the forces on the object must be zero, and the sum of the torques about any point must be zero. To keep things simple, we'll restrict our attention to situations in which we can treat all forces as acting in a single plane, which we'll call the *xy*-plane. Then we need consider only the *x*- and *y*-components of force in Eq. (11.1), and in Eq. (11.2) we need consider only the *z*-components of torque (perpendicular to the plane). The first and second conditions for equilibrium are then

$\sum F_x = 0$ and $\sum F_y = 0$ (first condition for equilibrium, forces in *xy*-plane),

$\sum \tau_z = 0$ (second condition for equilibrium, forces in *xy*-plane).

CAUTION! **Choosing the reference point for calculating torques.** In equilibrium problems, the choice of reference point for calculating torques in $\sum \tau_z$ is completely arbitrary. But once you make your choice, you must use the *same* point to calculate *all* the torques on an object. Choose the point so as to simplify the calculations as much as possible.

PROBLEM-SOLVING STRATEGY

11.1 Equilibrium of a Rigid Body

IDENTIFY *the relevant concepts:*

The first and second conditions for equilibrium ($\sum F_x = 0, \sum F_y = 0$, and $\sum \tau_z = 0$) are applicable to any rigid body that is not accelerating in space and not rotating.

SET UP *the problem* :

- Sketch the physical situation and identify the object in equilibrium to be analyzed. Sketch the object accurately; do *not* represent it as a point. Include dimensions.
- Draw a free-body diagram showing all forces acting *on* the object. Show the point on the object at which each force acts.
- Choose coordinate axes and specify their direction. Specify a positive direction of rotation for torques. Represent forces in terms of their components with respect to the chosen axes.
- Choose a reference point about which to compute torques. Choose wisely; you can eliminate from your torque equation any force whose line of action goes through the point you choose. The object doesn't actually have to be pivoted about an axis through the reference point.

EXECUTE *the solution*:

1 Write equations expressing the equilibrium conditions. Remember that $\sum F_x = 0$, $\sum F_y = 0$, and $\sum \tau_z = 0$ are *separate* equations. You can compute the torque of a force by finding the torque of each of its components separately, each with its appropriate lever arm and sign, and adding the results.

2 To obtain as many equations as you have unknowns, you may need to compute torques with respect to two or more reference points; choose them wisely, too.

EVALUATE *your answer:*

Check your results by writing $\sum \tau_z = 0$ with respect to a different reference point. You should get the same answers.

11.4 Stress, strain, and Elastic Moduli

The rigid body is a useful idealized model, but the stretching, squeezing, and twisting of real objects when forces are applied are often too important to ignore. **Figure 11.5** shows three examples. We want to study the relationship between the forces and deformations for each case.

(a) (b) (c)

Figure 11.5 - Three types of stress. (a) Guitar strings under *tensile stress*, being stretched by forces acting at their ends. (b) A diver under *bulk stress*, being squeezed from all sides by forces due to water pressure. (c) A ribbon under *shear stress*, being deformed and eventually cut by forces exerted by the *scissors*

You don't have to look far to find a deformable object; it's as plain as the nose on your face (**Fig. 11.6**). If you grasp the tip of your nose between your index finger and thumb, you'll find that the harder you pull your nose outward or push it inward, the more it stretches or compresses. Likewise, the harder you squeeze your index finger and thumb together, the more the tip of your nose compresses. If you try to twist the tip of your nose, you'll get a greater amount of twist if you apply stronger forces.

Figure 11.6 - When you pinch your nose, the force per area that you apply to your nose is called *stress*. The fractional change in the size of your nose (the change in size divided by the initial size) is called *strain*. The deformation is *elastic* because your nose springs back to its initial size when you stop pinching

These observations illustrate a general rule. In each case you apply a **stress** to your nose; the amount of stress is a measure of the forces causing the deformation, on a "force per unit area" basis. And in each case the stress causes a deformation, or **strain**. More careful versions of the experiments with your nose suggest that for relatively small stresses, the resulting strain is proportional to the stress: The greater the deforming forces, the greater the resulting deformation. This proportionality is called **Hooke's law**, and the ratio of stress to strain is called the **elastic modulus:**

Measure of forces applied to deform an object

$$\text{Hooke's law:} \quad \frac{\text{Stress}}{\text{Strain}} = \text{Elastic modulus} \tag{11.7}$$

Property of material of which object is made

Measure of how much deformation results from stress

The value of the elastic modulus depends on what the object is made of but not its shape or size. If a material returns to its original state after the stress is removed, it is called **elastic**; Hooke's law is a special case of elastic behavior. If a material instead remains deformed after the stress is removed, it is called **plastic**. Here we'll consider elastic behavior only; we'll return to plastic behavior in Section 11.5.

We used one form of Hooke's law in Section 6.3: The elongation of an ideal spring is proportional to the stretching force. Remember that Hooke's "law" is not really a general law; it is valid over only a limited range of stresses. In Section 11.5 we'll see what happens beyond that limited range.

Tensile and Compressive Stress and Strain

The simplest elastic behavior to understand is the stretching of a bar, rod, or wire when its ends are pulled (Fig. 11.5a). **Figure 11.7** shows an object that initially has uniform cross-sectional area A and length l_0. We then apply forces of equal magnitude $F_\perp$ but opposite directions at the ends (this ensures that the object has no tendency to move left or right). We say that the object is in **tension.** We've already talked a lot about tension in ropes and strings; it's the same concept here. The subscript $\perp$ is a reminder that the forces act perpendicular to the cross-section.

We define the **tensile stress** at the cross section as the ratio of the force $F_\perp$ to the cross-sectional area A

$$\text{Tensile stress} = \frac{F_\perp}{A}. \tag{11.8}$$

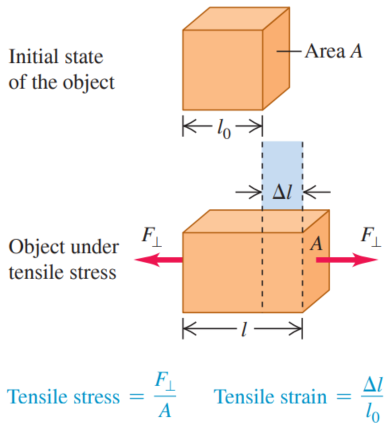


Figure 11.7 - An object in tension. The net force on the object is zero, but the object deforms. The tensile stress (the ratio of the force to the cross-sectional area) produces a tensile strain (the elongation divided by the initial length). The elongation Δl is exaggerated for clarity

This is a *scalar* quantity because $F_\perp$ is the *magnitude* of the force. The SI unit of stress is the **pascal** (abbreviated Pa and named for the 17th-century French scientist and philosopher Blaise Pascal). Equation (11.8) shows that 1 pascal equals 1 newton per square meter (N/m^2):

$$1 \text{ pascal} = 1 \text{ Pa} = 1 \text{ N/m}^2.$$

The units of stress are the same as those of *pressure*, which we'll encounter often in later chapters.

Under tension the object in Fig. 11.7 stretches to a length $l = l_0 + \Delta l$. The elongation Δl does not occur only at the ends; every part of the object stretches in the same proportion. The **tensile strain** of the object equals the fractional change in length, which is the ratio of the elongation Δl to the original length l_0:

$$\text{Tensile strain} = \frac{l - l_0}{l_0} = \frac{\Delta l}{l_0}. \tag{11.9}$$

Tensile strain is stretch per unit length. It is a ratio of two lengths, always measured in the same units, and so is a pure (dimensionless) number with no units.

Experiment shows that for a sufficiently small tensile stress, stress and strain are proportional, as in Eq. (11.7). The corresponding elastic modulus is called **Young's modulus**, denoted by Y:

Young's modulus for tension

Force applied perpendicular to cross section

Original length (see Fig. 11.14)

$$Y = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{F_\perp / A}{\Delta l / l_0} = \frac{F_\perp}{A}\frac{l_0}{\Delta l} \qquad (11.10)$$

Cross-sectional area of object

Elongation (see Fig. 11.14)

Table 11.1 - Approximate Elastic Moduli

Material	Young's modulus Y(Pa)	Bulk Modulus B(Pa)	Shear Modulus S(Pa)
Aluminum	7.0×10^{10}	7.5×10^{10}	2.5×10^{10}
Brass	9.0×10^{10}	6.0×10^{10}	3.5×10^{10}
Copper	11×10^{10}	14×10^{10}	4.4×10^{10}
Iron	21×10^{10}	16×10^{10}	7.7×10^{10}
Lead	1.6×10^{10}	4.1×10^{10}	0.6×10^{10}
Nickel	21×10^{10}	17×10^{10}	7.8×10^{10}
Silicone rubber	0.001×10^{10}	0.2×10^{10}	0.0002×10^{10}
Steel	20×10^{10}	16×10^{10}	7.5×10^{10}
Tendon (typical)	0.12×10^{10}	–	–

Since strain is a pure number, the units of Young's modulus are the same as those of stress: force per unit area. Table 11.1 lists some typical values. (This table also gives values of two other elastic moduli that we'll discuss later in this chapter). A material with a large value of Y is relatively unstretchable; a large stress is required for a given strain. For example, the value of Y for cast steel (1.2×10^{11} Pa) is much larger than that for a tendon (1.2×10^{9} Pa).

When the forces on the ends of a bar are pushes rather than pulls (**Fig. 11.8**), the bar is in **compression** and the stress is a **compressive stress**. The **compressive strain** of an object in compression is defined in the same way as the tensile strain, but Δl has the opposite direction. Hooke's law and Eq. (11.10) are valid for compression as well as tension if the compressive stress is not too great. For many materials, Young's modulus has the same value for both tensile and compressive stresses. Composite materials such as concrete and stone are an exception; they can withstand compressive stresses but fail under comparable tensile stresses. Stone was the primary building material used by ancient civilizations such as the Babylonians, Assyrians, and Romans, so their structures had to be designed to avoid tensile stresses. Hence they used arches in doorways and bridges, where the weight of the overlying material compresses the stones of the arch together and does not place them under tension.

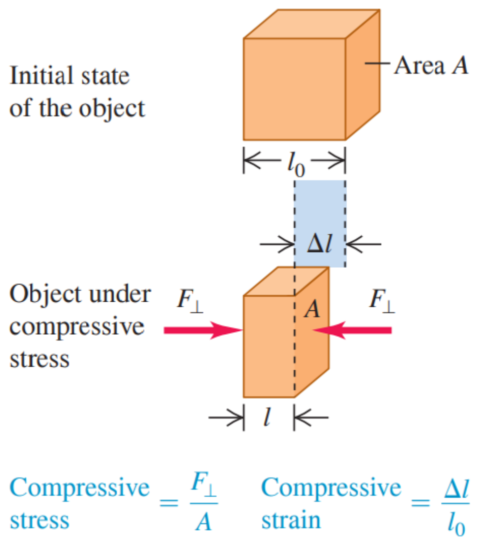


Figure 11.8 - An object in compression. The compressive stress and compressive strain are defined in the same way as tensile stress and strain (see Fig.11.7), except that Δl now denotes the distance that the object contracts

In many situations, objects can experience both tensile and compressive stresses at the same time. For example, a horizontal beam supported at each end sags under its own weight. As a result, the top of the beam is under compression while the bottom of the beam is under tension (Fig. 11.9a). To minimize the stress and hence the bending strain, the top and bottom of the beam are given a large cross-sectional area. There is neither compression nor tension along the centerline of the beam, so this part can have a small cross section; this helps keep the weight of the beam to a minimum and further helps reduce the stress. The result is an I-beam of the familiar shape used in building construction (Fig. 11.9b).

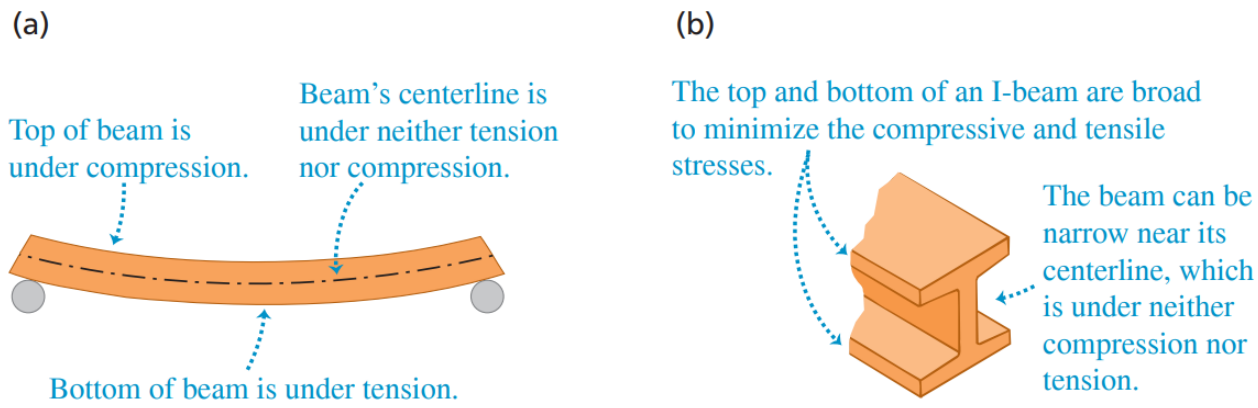


Figure 11.9 - (a) A beam supported at both ends is under both compression and tension.
(b) The cross-sectional shape of an I-beam minimizes both stress and weight

Bulk Stress and Strain

When a scuba diver plunges deep into the ocean, the water exerts nearly uniform pressure everywhere on his surface and squeezes him to a slightly smaller volume (see Fig. 11.5b). This is a different situation from the tensile and compressive stresses and strains we have discussed. The uniform pressure on all sides of the diver is a **bulk stress** (or **volume stress**), and the resulting deformation—a **bulk strain** (or **volume strain**)—is a change in his volume.

CAUTION! **Pressure vs. force.** Unlike force, pressure has no intrinsic direction: The pressure on the surface of an immersed object is the same no matter how the surface is oriented. Hence pressure is a *scalar*quantity, not a vector quantity.

If an object is immersed in a fluid (liquid or gas) at rest, the fluid exerts a force on any part of the object's surface; this force is *perpendicular* to the surface. (If we tried to make the fluid exert a force parallel to the surface, the fluid would slip sideways to counteract the effort). The force $F_{\perp}$ per unit area that the fluid exerts on an immersed object is called the **pressure***p* in the fluid:

Pressure in a fluid $\quad p = \dfrac{F_{\perp}}{A} \qquad$ (11.11)

Force that fluid applies to surface of an immersed object
Area over which force is exerted

Pressure has the same units as stress; commonly used units include 1 Pa (= 1 N/m^2), and 1 **atmosphere** (1atm). One atmosphere is the approximate average pressure of the earth's atmosphere at sea level:

$$1 \text{ atmosphere} = 1 \text{ atm} = 1.013 \times 10^5 \text{ Pa} .$$

The pressure in a fluid increases with depth. For example, the pressure in the ocean increases by about 1 atm every 10 m. If an immersed object is relatively small, however, we can ignore these pressure differences for purposes of calculating bulk stress. We'll then treat the pressure as having the same value at all points on an immersed object's surface.

Pressure plays the role of stress in a volume deformation. The corresponding strain is the fractional change in volume — that is, the ratio of the volume change ΔV to the original volume V_0:

$$\text{Bulk (volume) strain} = \frac{\Delta V}{V_0} . \qquad (11.12)$$

Volume strain is the change in volume per unit volume. Like tensile or compressive strain, it is a pure number, without units.

When Hooke's law is obeyed, an increase in pressure (bulk stress) produces a *proportional* bulk strain (fractional change in volume). The corresponding elastic modulus (ratio of stress to strain) is called the **bulk modulus**, denoted by B. When the pressure on an object changes by a small amount Δp , from p_0 to $p_0 + \Delta p$, and the resulting bulk strain is $\Delta V > V_0$, Hooke's law takes the form

Bulk modulus for compression; Additional pressure on object; Original volume (see Fig. 11.17); Change in volume (see Fig. 11.17)

$$B = \frac{\text{Bulk stress}}{\text{Bulk strain}} = -\frac{\Delta p}{\Delta V/V_0} \qquad (11.13)$$

We include a minus sign in this equation because an *increase* of pressure always causes a *decrease* in volume. In other words, if Δp is positive, ΔV is negative. The bulk modulus B itself is a positive quantity.

For small pressure changes in a solid or a liquid, we consider B to be constant. The bulk modulus of a *gas*, however, depends on the initial pressure p_0. Table 11.1 includes values of B for several solid materials. Its units, force per unit area, are the same as those of pressure (and of tensile or compressive stress).

The reciprocal of the bulk modulus is called the **compressibility** and is denoted by k. From Eq. (11.13),

$$k = \frac{1}{B} = -\frac{\Delta V/V_0}{\Delta p} = -\frac{1}{V_0}\frac{\Delta V}{\Delta p} \quad \text{(compressibility).} \qquad (11.14)$$

Compressibility is the fractional decrease in volume $-\Delta V/V_0$, per unit increase Δp incpressure. The units of compressibility are those of *reciprocal pressure*, Pa^{-1} or atm^{-1} .

For example, the compressibility of water is $46.4 \times 10^{-6}\ \mathrm{atm}^{-1}$, which means that the volume of water decreases by 46.4 parts per million for each 1 atmosphere increase in pressure. Materials with small bulk modulus B and large compressibility k are easiest to compress.

Shear Stress and Strain

The third kind of stress-strain situation is called *shear*. The ribbon in Fig. 11.5c is under **shear stress**: One part of the ribbon is being pushed up while an adjacent part is being pushed down, producing a deformation of the ribbon. Figure 11.10 shows an object being deformed by a shear stress. In the figure, forces of equal magnitude but opposite direction act *tangent* to the surfaces of opposite ends of the object. We define the shear stress as the force $F_{\parallel}$ acting tangent to the surface divided by the area A on which it acts:

$$\text{Shear stress} = \frac{F_{\parallel}}{A}. \qquad (11.15)$$

Shear stress, like the other two types of stress, is a force per unit area.

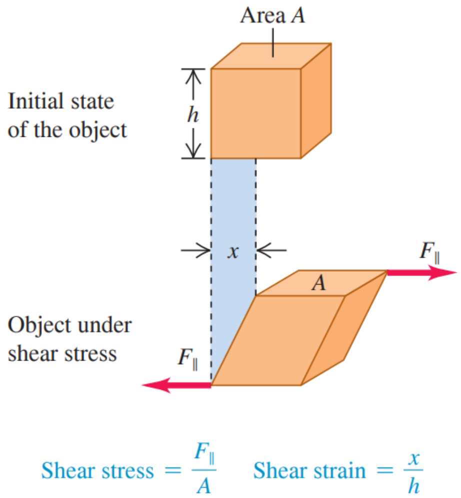


Figure 11.10 - An object under shear stress. Forces are applied tangent to opposite surfaces of the object (in contrast to the situation in Fig. 11.7, in which the forces act perpendicular to the surfaces). The deformation x is exaggerated for clarity

Figure 11.10 shows that one face of the object under shear stress is displaced by a distance x relative to the opposite face.

We define **shear strain** as the ratio of the displacement x to the transverse dimension h:

$$\text{Shear strain} = \frac{x}{h}. \qquad (11.16)$$

In real-life situations, x is typically much smaller than h. Like all strains, shear strain is a dimensionless number; it is a ratio of two lengths.

If the forces are small enough that Hooke's law is obeyed, the shear strain is *proportional* to the shear stress. The corresponding elastic modulus (ratio of shear stress to shear strain) is called the **shear modulus**, denoted by S:

Force applied tangent to surface of object; Transverse dimension (see Fig. 11.18)

Shear modulus for shear

$$S = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{F_{\parallel}/A}{x/h} = \frac{F_{\parallel}}{A}\frac{h}{x} \qquad (11.17)$$

Area over which force is exerted; Deformation (see Fig. 11.18)

For a given material, S is usually onethird to one-half as large as Young's modulus Y for tensile stress. Keep in mind that the concepts of shear stress, shear strain, and shear modulus apply to *solid* materials only. The reason is that *shear* refers to deforming an object that has a definite shape (see Fig. 11.10). This concept doesn't apply to gases and liquids, which do not have definite shapes.

11.5 Elasticity and plasticity

Hooke's law - the proportionality of stress and strain in elastic deformations - has a limited range of validity. In the preceding section we used phrases such as "if the forces are small enough that Hooke's law is obeyed." Just what *are* the limitations of Hooke's law? What's more, if you pull, squeeze, or twist *anything* hard enough, it will bend or break. Can we be more precise than that?

To address these questions, let's look at a graph of tensile stress as a function of tensile strain. **Figure 11.11** shows a typical graph of this kind for a metal such as copper or soft iron. The strain is shown as the *percent* elongation; the horizontal scale is not uniform beyond the first portion of the curve, up to a strain of less than 1%. The first portion is a straight line, indicating Hooke's law behavior with stress directly proportional to strain. This straight-line portion ends at point a; the stress at this point is called the *proportional limit*.

From a to b, stress and strain are no longer proportional, and Hooke's law is *not* obeyed. However, from a to b (and O to a), the behavior of the material is *elastic*: If the load is gradually removed starting at any point between O and b, the curve is retraced until the material returns to its original length. This elastic deformation is *reversible*.

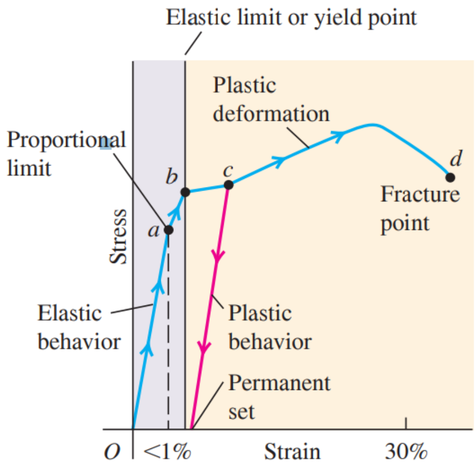


Figure 11.11 - Typical stress-strain diagram for a ductile metal under tension

Point b, the end of the elastic region, is called the *yield point*; the stress at the yield point is called the *elastic limit*. When we increase the stress beyond point b, the strain continues to increase. But if we remove the load at a point like c beyond the elastic limit, the material does *not* return to its original length. Instead, it follows the red line in Fig. 11.11. The material has deformed *irreversibly* and acquired a *permanent set*. This is the *plastic* behavior.

Once the material has become plastic, a small additional stress produces a relatively large increase in strain, until a point d is reached at which *fracture* takes place. That's what happens if a steel guitar string in Fig. 11.5a is tightened too much: The string breaks at the fracture point. Steel is *brittle* because it breaks soon after reaching its elastic limit; other materials, such as soft iron, are *ductile* - they can be given a large permanent stretch without breaking. (The material depicted in Fig. 11.11 is ductile, since it can stretch by more than 30 % before breaking).

Unlike uniform materials such as metals, stretchable biological materials such as tendons and ligaments have no true plastic region. That's because these materials are made of a collection of microscopic fibers; when stressed beyond the elastic limit, the fibers tear apart from each other. (A torn ligament or tendon is one that has fractured in this way).

If a material is still within its elastic region, something very curious can happen when it is stretched and then allowed to relax. **Figure 11.12** is a stress-strain curve for vulcanized rubber that has been stretched by more than seven times its original length. The stress is not proportional to the strain, but the behavior is elastic because when the load is removed, the material returns to its original length. However, the material follows *different* curves for increasing and decreasing stress. This is called *elastic hysteresis*. The work done by the material when it returns to its original shape is less than the work required to deform it; that's due to internal friction. Rubber with large elastic hysteresis is very useful for absorbing vibrations, such as in engine mounts and shock-absorber bushings for cars. Tendons display similar behavior.

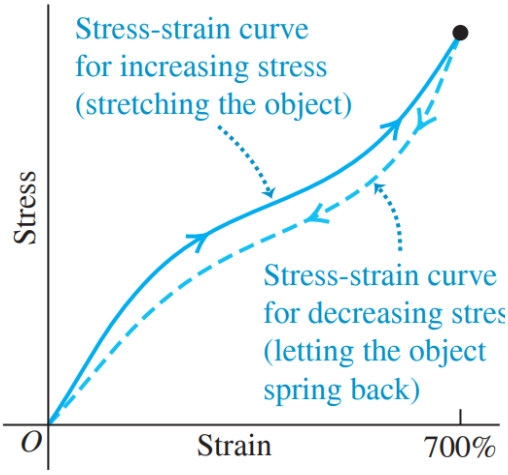
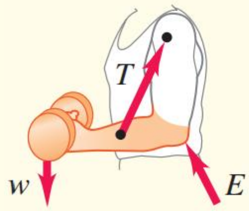
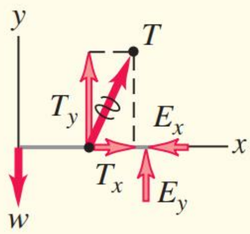
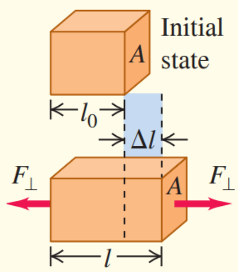
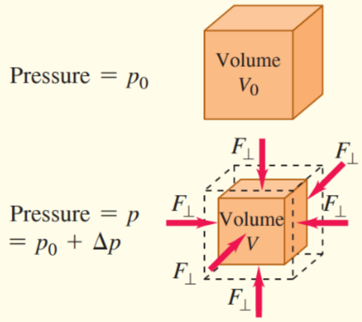
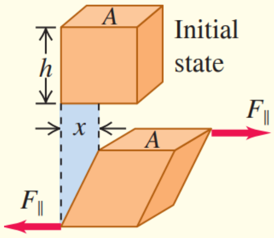


Figure 11.12 -Typical stress-strain diagram for vulcanized rubber. The curves are different for increasing and decreasing stress, a phenomenon called elastic hysteresis

The stress required to cause actual fracture of a material is called the *breaking stress*, the *ultimate strength*, or (for tensile stress) the *tensile strength*. Two materials, such as two types of steel, may have very similar elastic constants but vastly different breaking stresses. Iron and steel are comparably *stiff* (they have almost the same value of Young's modulus), but steel is *stronger* (it has a larger breaking stress than does iron).

CHAPTER 11: SUMMARY

Conditions for equilibrium: For a rigid body to be in equilibrium, two conditions must be satisfied. First, the vector sum of forces must be zero. Second, the sum of torques about any point must be zero. The torque due to the weight of an object can be found by assuming the entire weight is concentrated at the center of gravity, which is at the same point as the center of mass if $\vec{g}$ has the same value at all points	$\sum \vec{F} = 0$ $\sum \vec{\tau} = 0$ about *any* point $\vec{r}_{cm} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}$	 
Stress, strain, and Hooke's law: Hooke's law states that in elastic deformations, stress (force per unit area) is proportional to strain (fractional deformation). The proportionality constant is called the elastic modulus	$\dfrac{\text{Stress}}{\text{Strain}} = \text{Elastic modulus}$	
Tensile and compressive stress: Tensile stress is tensile force per unit area, $F_\perp / A$. Tensile strain is fractional change in length, $\Delta l / l_0$. The elastic modulus for tension is called Young's modulus Y. Compressive stress and strain are defined in the same way	$Y = \dfrac{\text{Tensile stress}}{\text{Tensile strail}} = \dfrac{F_\perp / A}{\Delta l / l_0} = \dfrac{F_\perp}{A}\dfrac{l_0}{\Delta l}$	

Bulk stress: Pressure in a fluid is force per unit area. Bulk stress is pressure change, Δp, and bulk strain is fractional volume change, $\Delta V/V_0$. The elastic modulus for compression is called the bulk modulus, B. Compressibility, k, is the reciprocal of bulk modulus: $k = 1/B$	$p = \frac{F_\perp}{A}$ $B = \frac{\text{Bulk stress}}{\text{Bulk strain}} = -\frac{\Delta p}{\Delta V / V_0}$	
Shear stress: Shear stress is force per unit area, $F_{\parallel} / A$, for a force applied tangent to a surface. Shear strain is the displacement x of one side divided by the transverse dimension h. The elastic modulus for shear is called the shear modulus, S	$S = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{F_{\parallel} / A}{x / h} = \frac{F_{\parallel}}{A}\frac{h}{x}$	
The limits of Hooke's law: The proportional limit is the maximum stress for which stress and strain are proportional. Beyond the proportional limit, Hooke's law is not valid. The elastic limit is the stress beyond which irreversible deformation occurs. The breaking stress, or ultimate strength, is the stress at which the material breaks		

12 FLUID MECHANICS

Fluids play a vital role in many aspects of everyday life. We drink them, breathe them, swim in them. They circulate through our bodies and control our weather. The physics of fluids is therefore crucial to our understanding of both nature and technology.

We begin our study with **fluid statics**, the study of fluids at rest in equilibrium situations. Like other equilibrium situations, it is based on Newton's first and third laws. We'll explore the KEYCONCEPTs of density, pressure, and buoyancy. **Fluid dynamics**, the study of fluids in motion, is much more complex; indeed, it is one of the most complex branches of mechanics. Fortunately, we can analyze many important situations by using simple idealized models and familiar principles such as Newton's laws and conservation of energy. Even so, we'll barely scratch the surface of this broad and interesting topic.

12.1 Elasticity and Plasticity

A **fluid** is any substance that can flow and change the shape of the volume that it occupies. (By contrast, a solid tends to maintain its shape). We use the term "fluid" for both gases and liquids. The key difference between them is that a liquid has *cohesion*, while a gas does not. The molecules in a liquid are close to one another, so they can exert attractive forces on each other and thus tend to stay together (that is, to cohere). That's why a quantity of liquid maintains the same volume as it flows: If you pour 500 mL of water into a pan, the water will still occupy a volume of 500 mL. The molecules of a gas, by contrast, are separated on average by distances far larger than the size of a molecule. Hence the forces between molecules are weak, there is little or no cohesion, and a gas can easily change in volume. If you open the valve on a tank of compressed oxygen that has a volume of 500 mL, the oxygen will expand to a far greater volume.

An important property of *any* material, fluid or solid, is its **density**, defined as its mass per unit volume. A homogeneous material such as ice or iron has the same density throughout. We use ρ(the Greek letter rho) for density. For a homogeneous material,

$$\underbrace{\rho}_{\text{Density of a homogeneous material}} = \frac{\overbrace{m}^{\text{Mass of material}}}{\underbrace{V}_{\text{Volume occupied by material}}} \tag{12.1}$$

Two objects made of the same material have the same density even though they may have different masses and different volumes. That's because the *ratio* of mass to volume is the same for both objects (**Fig. 12.1**).

The SI unit of density is the kilogram per cubic meter (1 kg/m^3). The cgs unit, the gram per cubic centimeter (1 g/cm^3), is also widely used:

$$1\ \text{g/cm}^3 = 1000\ \text{kg/m}^3.$$

The densities of some common substances at ordinary temperatures are given in Table 12.1. Note the wide range of magnitudes. The densest material found on earth is the metal osmium (ρ = 22,500 kg/m^3), but its density pales by comparison to the densities of exotic astronomical objects, such as white dwarf stars and neutron stars.

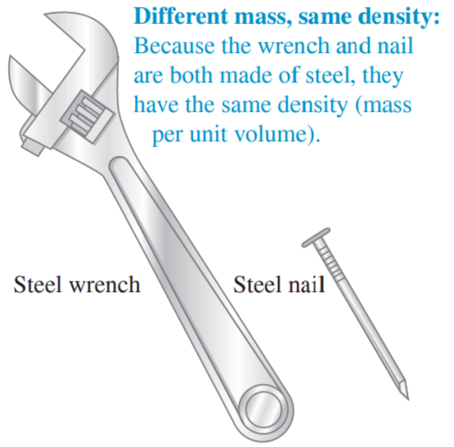


Figure 12.1 - Two objects with different masses and different volumes but the same density

The **specific gravity** of a material is the ratio of its density to the density of water at 4.0°C, 1000 kg/m^3; it is a pure number without units. For example, the specific gravity of aluminum is 2.7. "Specific gravity" is a poor term, since it has nothing to do with gravity; "relative density" would have been a better choice.

The density of some materials varies from point to point within the material. One example is the material of the human body, which includes low-density fat (about 940 kg/m^3) and high-density bone (from 1700 to 2500 kg/m^3). Two others are the earth's atmosphere (which is less dense at high altitudes) and oceans (which are denser at greater depths). For these materials, Eq. (12.1) describes the **average density**. In general, the density of a material depends on environmental factors such as temperature and pressure.

Table 12.1 - Densities of Some Common Substances

Material	**Density(kg/m^3)***	**Material**	**Density(kg/m^3)***
Air (1 atm, 20°C)	1.20	Iron steel	7.8×10^3
Ethanol	0.81×10^3	Brass	8.6×10^3
Benzene	0.90×10^3	Copper	8.9×10^3
Ice	0.92×10^3	Silver	10.5×10^3
Water	1.00×10^3	Lead	11.3×10^3
Seawater	1.03×10^3	Mercury	13.6×10^3
Blood	1.06×10^3	Gold	19.3×10^3
Glycerin	1.26×10^3	Platinum	21.4×10^3
Concrete	2×10^3	White dwarf star	10^{10}
Aluminum	2.7×10^3	Neutron star	10^{18}

*To obtain the densities in grams per cubic centimeter, simply divide by 10^3

EXAMPLE 12.1 The weight of a roomful of air

Find the mass and weight of the air at 20°C in a living room with a 4.0 m × 5.0 m floor and a ceiling 3.0 m high, and the mass and weight of an equal volume of water.

IDENTIFY and SET UP

We assume that the air density is the same throughout the room. (Air is less dense at high elevations than near sea level, but the density varies negligibly over the room's 3.0 m height). We use Eq. (12.1) to relate the mass m_{air} to the room's volume V (which we'll calculate) and the air density ρ_{air} (given in Table 12.1).

EXECUTE

We have V = (4.0 m)(5.0 m)(3.0 m) = 60 m^3 , so from Eq. (12.1),

$$m_{\text{air}} = \rho_{\text{air}} V = (1.20\ \text{kg/m}^3)(60\ \text{m}^3) = 72\ \text{kg},$$

$$w_{\text{air}} = m_{\text{air}} g = (72\ \text{kg})(9.8\ \text{m/s}^2) = 700\ \text{N}.$$

The mass and weight of an equal volume of water are

$$m_{\text{water}} = \rho_{\text{water}} V = (1000\ \text{kg/m}^3)(60\ \text{m}^3) = 6.0\times10^4\ \text{kg},$$

$$w_{\text{water}} = m_{\text{water}} g = (6.0\times10^4\ \text{kg})(9.8\ \text{m/s}^2) = 5.9\times10^5\ \text{N}.$$

EVALUATE

A roomful of air weighs about the same as an average adult. Water is nearly a thousand times denser than air, so its mass and weight are larger by the same factor. The weight of a roomful of water would collapse the floor of an ordinary house.

KEYCONCEPT
To find the density of a uniform substance, divide the mass of the substance by the volume that it occupies.

12.2 Pressure in a Fluid

A fluid exerts a force perpendicular to any surface in contact with it, such as a container wall or an object immersed in the fluid. This is the force that you feel pressing on your legs when you dangle them in a swimming pool. Even when a fluid as a whole is at rest, the molecules that make up the fluid are in motion; the force exerted by the fluid is due to molecules colliding with their surroundings.

Imagine a surface *within* a fluid at rest. For this surface and the fluid to remain at rest, the fluid must exert forces of equal magnitude but opposite direction on the surface's two sides. Consider a small surface of area dA centered on a point in the fluid; the normal force exerted by the fluid on each side is $dF_{\perp}$ (**Fig. 12.2**). We define the **pressure** p at that point as the normal force per unit area - that is, the ratio of $dF_{\perp}$ to dA (**Fig. 12.3**):

Pressure at a point in a fluid

$$p = \frac{dF_{\perp}}{dA} \qquad (12.2)$$

$dF_{\perp}$: Normal force exerted by fluid on a small surface at that point

dA: Area of surface

If the pressure is the same at all points of a finite plane surface with area A, then

$$p = \frac{F_{\perp}}{A}, \qquad (12.3)$$

where $F_{\perp}$ is the net normal force on one side of the surface. The SI unit of pressure is the **pascal**, where

$$1 \text{ pascal} = 1 \text{ Pa} = 1 \text{ N/m}^2.$$

We introduced the pascal in Chapter 11. Two related units, used principally in meteorology, are the *bar*, equal to 10^5 Pa, and the *millibar*, equal to 100 Pa.

Atmospheric pressure p_a is the pressure of the earth's atmosphere, the pressure at the bottom of this sea of air in which we live. This pressure varies with weather changes and with elevation. Normal atmospheric pressure at sea level (an average value) is 1 *atmosphere* (atm), defined to be exactly 101,325 Pa. To four significant figures,

$$(p_a)_{av} = 1 \text{ atm} = 1.013 \times 10^5 \text{ Pa} =$$
$$= 1.013 \text{ bar} = 1013 \text{ milibar}.$$

CAUTION! **Don't confuse pressure and force.** In everyday language "pressure" and "force" mean pretty much the same thing. In fluid mechanics, however, these words describe very different quantities. Pressure acts perpendicular to any surface in a fluid, no matter how that surface is oriented (Fig. 12.3). Hence pressure has no direction of its own; it's a scalar. By contrast, force is a vector with a definite direction. Remember, too, that pressure is force per unit area. As Fig. 12.3 shows, a surface with twice the area has twice as much force exerted on it by the fluid, so the

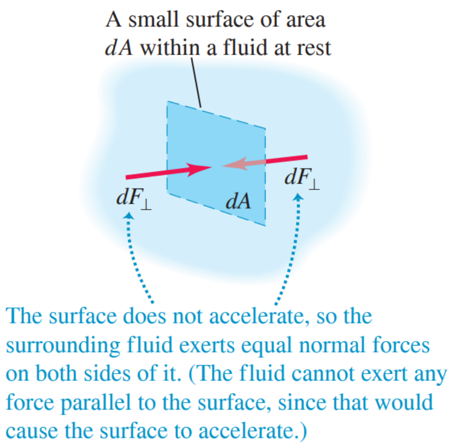


Figure 12.2 - Forces acting on a small surface within a fluid at rest

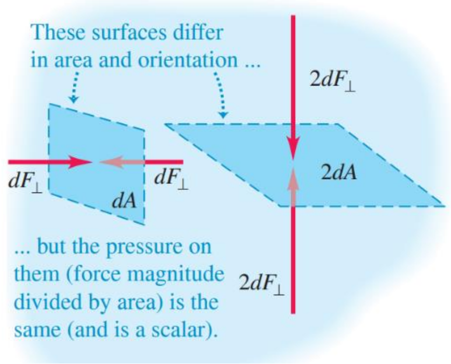


Figure 12.3 - Pressure is a scalar with units of newtons per square meter. By contrast, force is a vector with units of newtons

pressure is the same.

EXAMPLE 12.2 The force of air

In the room described in Example 12.1, what is the total downward force on the floor due to an air pressure of 1.00 atm?

IDENTIFY and SET UP

This example uses the relationship among the pressure p of a fluid (air), the area A subjected to that pressure, and the resulting normal force $F_{\perp}$ the fluid exerts. The pressure is uniform, so we use Eq. (12.3), $F_{\perp} = pA$, to determine $F_{\perp}$. The floor is horizontal, so $F_{\perp}$ is vertical (downward).

EXECUTE

We have $A = (4.0\text{ m})(5.0\text{ m}) = 20\text{ m}^3$, so from Eq. (12.3),

$$F_{\perp} = pA = (1.013\times10^5\text{ N/m}^2)(20\text{ m}^2) = 2.0\times10^6\text{ N}.$$

EVALUATE

Unlike the water in Example 12.1, $F_{\perp}$ will not collapse the floor here, because there is an *upward* force of equal magnitude on the floor's underside. If the house has a basement, this upward force is exerted by the air underneath the floor. In this case, if we ignore the thickness of the floor, the *net* force due to air pressure is zero.

KEYCONCEPT

To find the force exerted by a fluid perpendicular to a surface, multiply the pressure of the fluid by the surface's area. This relationship comes from the definition of pressure as the normal force per unit area within the fluid.

Pressure, Depth, and Pascal's Law

If the weight of the fluid can be ignored, the pressure in a fluid is the same throughout its volume. We used that approximation in our discussion of bulk stress and strain in Section 11.4. But often the fluid's weight is *not* negligible, and pressure variations are important. Atmospheric pressure is less at high altitude than at sea level, which is why airliner cabins have to be pressurized. When you dive into deep water, you can feel the increased pressure on your ears.

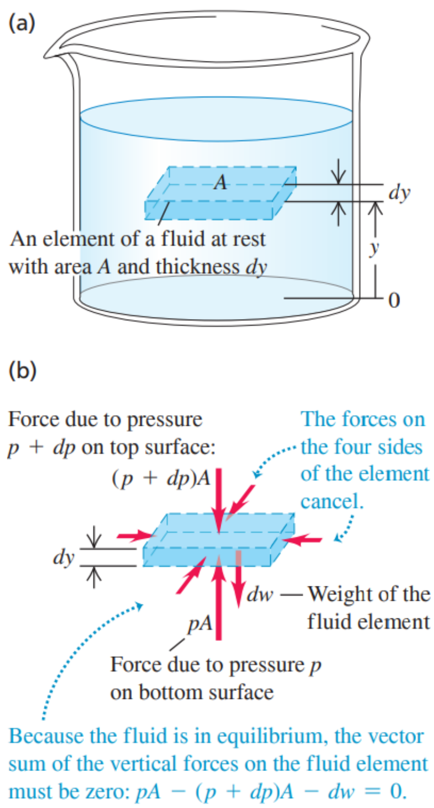


Figure 12.4 - The forces on an element of fluid in equilibrium

We can derive a relationship between the pressure p at any point in a fluid at rest and the elevation y of the point. We'll assume that the density ρ has the same value throughout the fluid (that is, the density is *uniform*), as does the acceleration due to gravity g. If the fluid is in equilibrium, any thin element of the fluid with thickness dy is also in equilibrium (Fig. 12.4a). The bottom and top surfaces each have area A, and they are at elevations y and $y + dy$ above some reference level where $y = 0$. The fluid element has volume $dV = Ady$, mass $dm = \rho dV = \rho Ady$, and weight $dw = dmg = \rho gAdy$.

What are the other forces on this fluid element (Fig 12.4b)? Let's call the pressure at the bottom surface p; then the total y-component of upward force on this surface is pA. The pressure at the top surface is $p + dp$, and the total y-component of (downward) force on the top surface is $-(p + dp)A$. The fluid element is in equilibrium, so the total y-component of force, including the weight and the forces at the bottom and top surfaces, must be zero:

$$\sum F_y = 0,$$

so

$$pA - (p + dp)A - \rho g A dy = 0.$$

When we divide out the area A and rearrange, we get

$$\frac{dp}{dy} = -\rho g. \tag{12.4}$$

This equation shows that when y increases, p decreases; that is, as we move upward in the fluid, pressure decreases, as we expect. If p^1 and p^2 are the pressures at elevations y^1 and y^2, respectively, and if ρ and g are constant, then

Pressure difference between two points in a **fluid of uniform density**
Uniform density of fluid

$$p_2 - p_1 = -\rho g(y_2 - y_1) \tag{12.5}$$

Heights of the two points
Acceleration due to gravity ($g > 0$)

It's often convenient to express Eq. (12.5) in terms of the *depth* below the surface of a fluid (Fig. 12.5). Take point 1 at any level in the fluid and let p represent the pressure at this point. Take point 2 at the *surface* of the fluid, where the pressure is p_0 (subscript zero for zero depth). The depth of point 1 below the surface is $h = y_2 - y_1$, and Eq. (12.5) becomes

$$p_0 - p = -\rho g(y_2 - y_1) = -\rho g h \quad \text{or}$$

Pressure at depth h in a **fluid of uniform density**
Uniform density of fluid

$$p = p_0 + \rho g h \tag{12.6}$$

Depth below surface
Pressure at surface of fluid
Acceleration due to gravity ($g > 0$)

The pressure p at a depth h is greater than the pressure p_0 at the surface by an amount $\rho g h$. Note that the pressure is the same at any two points at the same level in the fluid. The *shape* of the container does not matter (**Fig. 12.6**)

Equation (12.6) shows that if we increase the pressure p_0 at the top surface, possibly by using a piston that fits tightly inside the container to push down on the fluid surface, the pressure p at any depth increases by exactly the same amount. This observation is called *Pascal's law*.

PASCAL'S LAW: Pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and the walls of the containing vessel.

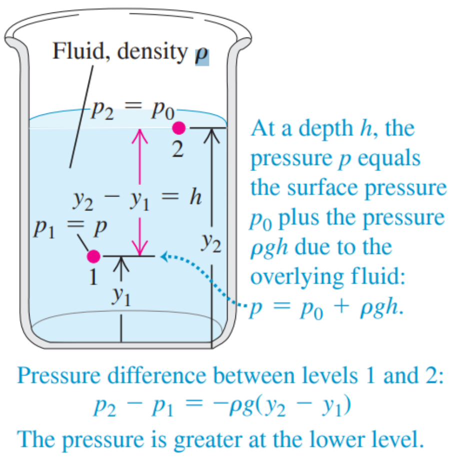


Figure 12.5 - How pressure varies with depth in a fluid with uniform density

The hydraulic lift (**Fig. 12.7**) illustrates Pascal's law. A piston with small cross-sectional area A_1 exerts a force F_1 on the surface of a liquid such as oil. The applied pressure $p = F_1/A_1$ is transmitted through the connecting pipe to a larger piston of area A_2. The applied pressure is the same in both cylinders, so

$$p = \frac{F_1}{A_1} = \frac{F_2}{A_2} \quad \text{and} \quad F_2 = \frac{A_2}{A_1} F_1 \tag{12.7}$$

The hydraulic lift is a force-multiplying device with a multiplication factor equal to the ratio of the areas of the two pistons. Dentist's chairs, car lifts and jacks, many lifts, and hydraulic brakes all use this principle.

For gases the assumption that the density ρ is uniform is realistic over only short vertical distances. In a room with a ceiling height of 3.0 m filled with air of uniform density 1.2 kg/m^3, the difference in pressure between floor and ceiling, given by Eq. (12.6), is

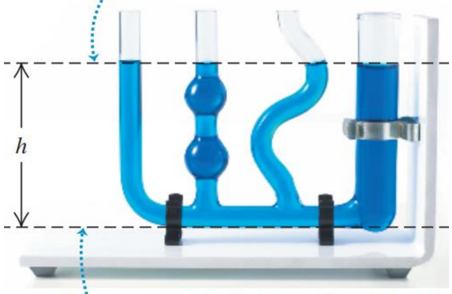


Figure 12.6 - Each fluid column has the same height, no matter what its shape

$$\rho gh = \left(1.2\ \text{kg/m}^3\right)\left(9.8\ \text{m/s}^2\right)\left(3.0\ \text{m}\right) = 35\ \text{Pa}$$

or about 0.00035 atm, a very small difference. But between sea level and the summit of Mount Everest (8882 m) the density of air changes by nearly a factor of 3, and in this case we cannot use Eq. (12.6). Liquids, by contrast, are nearly incompressible, and it is usually a very good approximation to regard their density as independent of pressure.

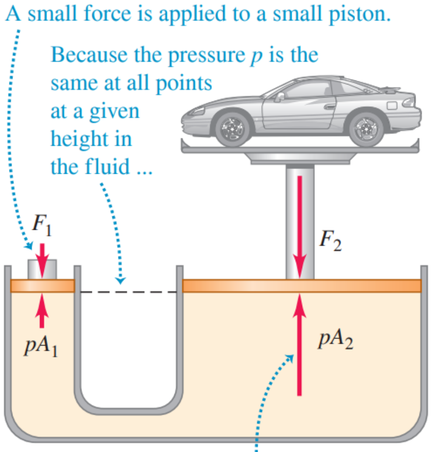


Figure 12.7 - The hydraulic lift is an application of Pascal's law. The size of the fluid-filled container is exaggerated for clarity

Absolute Pressure and Gauge Pressure

If the pressure inside a car tire is equal to atmospheric pressure, the tire is flat. The pressure has to be *greater* than atmospheric to support the car, so the significant quantity is the *difference* between the inside and outside pressures. When we say that the pressure in a car tire is "3 bar" (equal to 3.0×10^5 Pa), we mean that it is *greater* than atmospheric pressure (1.01×10^5 Pa) by this amount. The *total* pressure in the tire is then 4.01 bar or 4.01×10^5 Pa. The excess pressure above atmospheric pressure is usually called gauge **pressure**, and the total pressure is **called absolute** pressure. If the pressure is *less* than atmospheric, as in a partial vacuum, the gauge pressure is negative.

EXAMPLE 12.3 Finding absolute and gauge pressures

Water stands 12.0 m deep in a storage tank whose top is open to the atmosphere. What are the absolute and gauge pressures at the bottom of the tank?

IDENTIFY and SET UP The water is nearly incompressible, so we can treat it as having uniform density. The level of the top of the tank corresponds to point 2 in Fig. 12.5, and the level of the bottom of the tank corresponds to point 1. Our target variable is p in Eq. (12.6). We have h = 12.0 m and $p_0 = 1\ \text{atm} = 1.01 \times 10^5\ \text{Pa}$.

EXECUTE

From Eq. (12.6), the pressures are absolute:

$$p = p_0 + \rho g h = \left(1.01 \times 10^5 \text{ Pa}\right) + \left(1000 \text{ kg/m}^3\right)\left(9.80 \text{ m/s}^2\right)\left(12.0 \text{ m}\right) =$$
$$= 2.19 \times 10^5 \text{ Pa} = 2.16 \text{ atm}$$

gauge:

$$p - p_0 = \left(2.19 - 1.01\right) \times 10^5 \text{ Pa} = 1.18 \times 10^5 \text{ Pa} = 1.16.$$

EVALUATE

A pressure gauge at the bottom of such a tank would probably be calibrated to read gauge pressure rather than absolute pressure.

KEYCONCEPT

Absolute pressure is the total pressure at a given point in a fluid. Gauge pressure is the difference between absolute pressure and atmospheric pressure.

Pressure Gauges

The simplest pressure gauge is the open-tube *manometer* (Fig. 12.8a). The U-shaped tube contains a liquid of density ρ, often mercury or water. The left end of the tube is connected to the container where the pressure p is to be measured, and the right end is open to the atmosphere at pressure $p_0 = p_{\text{atm}}$. The pressure at the bottom of the tube due to the fluid in the left column is $p_{\text{atm}} + \rho g y_1$, and the pressure at the bottom due to the fluid in the right column is $p_{\text{atm}} + \rho g y_2$. These pressures are measured at the same level, so they must be equal:

$$\begin{aligned} p + \rho g y_1 &= p_{\text{atm}} + \rho g y_2 \\ p - p_{\text{atm}} &= \rho g (y_2 - y_1) = \rho g h \end{aligned} \tag{12.8}$$

In Eq. (12.8), p is the *absolute pressure*, and the difference $p - p_{\text{atm}}$ between absolute and atmospheric pressure is the gauge pressure. Thus the gauge pressure is proportional to the difference in height $h = y_2 - y_1$ of the liquid columns.

Another common pressure gauge is the **mercury barometer**. It consists of a long glass tube, closed at one end, that has been filled with mercury and then inverted in a dish of mercury (Fig. 12.8b). The space above the mercury column contains only mercury vapor; its pressure is negligibly small, so the pressure p_0 at the top of the mercury column is practically zero. From Eq. (12.6),

$$p_{\text{atm}} = p = 0 + \rho g (y_2 - y_1) = \rho g h \tag{12.9}$$

So the height h of the mercury column indicates the atmospheric pressure p_{atm}.

Pressures are often described in terms of the height of the corresponding mercury column, as so many "millimeters of mercury" (abbreviated mm Hg). A pressure of 1 mm Hg is called *1torr*, after Evangelista Torricelli, inventor of the mercury barometer. But these units depend on the density of mercury, which varies with temperature, and on the value of *g*, which varies with location, so the pascal is the preferred unit of pressure.

Many types of pressure gauges use a flexible sealed tube (**Fig. 12.9**). A change in the pressure either inside or outside the tube causes a change in its dimensions. This change is detected optically, electrically, or mechanically.

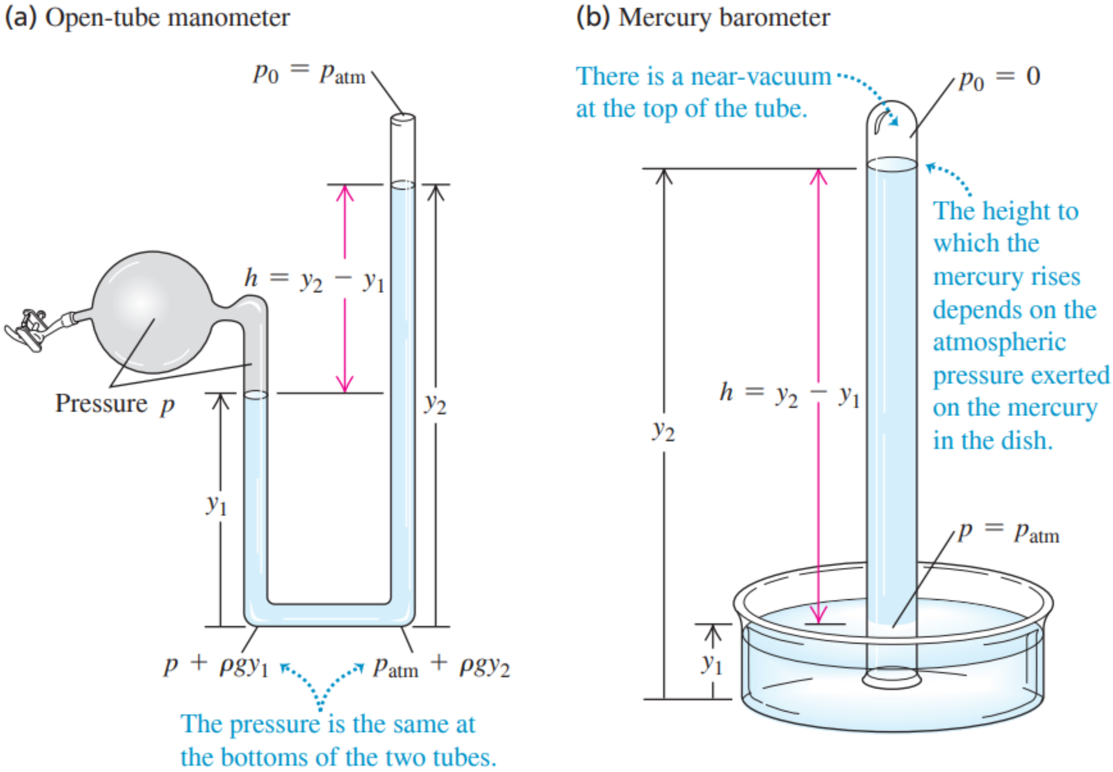


Figure 12.8 - Two types of pressure gauge

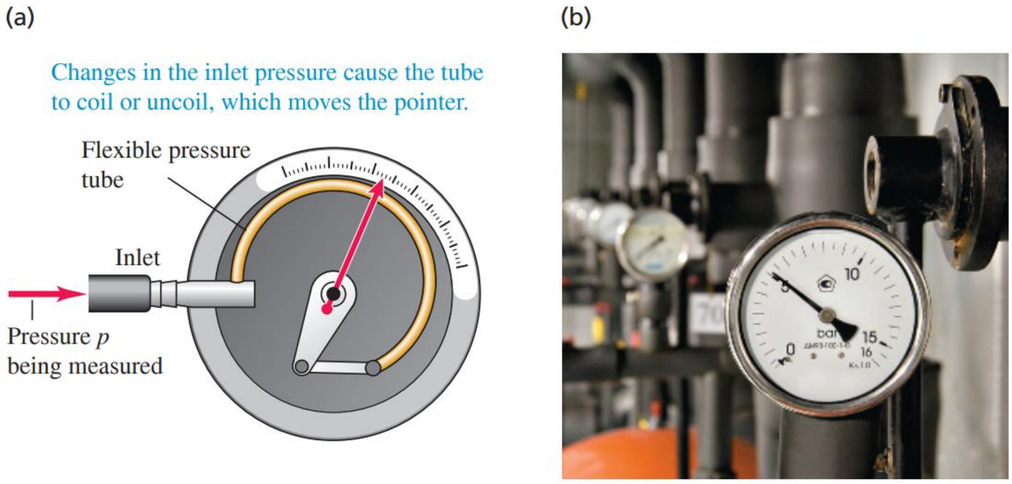


Figure 12.9 - (a) A Bourdon pressure gauge. When the pressure inside the flexible tube increases, the tube straightens out a little, deflecting the attached pointer. (b) This Bourdon-type pressure gauge is connected to a high-pressure gas line. The gauge pressure shown is just over 5 bars (1 bar = 10^5 Pa)

12.3 Buoyancy

An object immersed in water seems to weigh less than when it is in air. When the object is less dense than the fluid, it floats. The human body usually floats in water, and a heliumfilled balloon floats in air. These are examples of **buoyancy**, a phenomenon described by *Archimedes's principle*.

ARCHIMEDES'S PRINCIPLE: When an object is completely or partially immersed in a fluid, the fluid exerts an upward force on the object equal to the weight of the fluid displaced by the object.

To prove this principle, we consider an arbitrary element of fluid at rest. The dashed curve in Fig. 12.10a outlines such an element. The arrows labeled $dF_{\perp}$ represent the forces exerted on the element's surface by the surrounding fluid.

The entire fluid is in equilibrium, so the sum of all the y-components of force on this element of fluid is zero. Hence the sum of the y-components of the *surface* forces must be an upward force equal in magnitude to the weight mg of the fluid inside the surface. Also, the sum of the torques on the element of fluid must be zero, so the line of action of the resultant y-component of surface force must pass through the center of gravity of this element of fluid.

Now we replace the fluid inside the surface with a solid object that has exactly the same shape (Fig. 12.10b). The pressure at every point is the same as before. So the total upward force exerted on the object by the fluid is also the same, again equal in magnitude to the weight mg of the fluid displaced to make way for the object. We call this upward force the **buoyant force** on the solid object. The line of action of the buoyant force again passes through the center of gravity of the displaced fluid (which doesn't necessarily coincide with the center of gravity of the object).

When a balloon floats in equilibrium in air, its weight (including the gas inside it) must be the same as the weight of the air displaced by the balloon. A fish's flesh is denser than water, yet many fish can float while submerged. These fish have a gas-filled cavity within their bodies, which makes the fish's *average* density the same as water's.

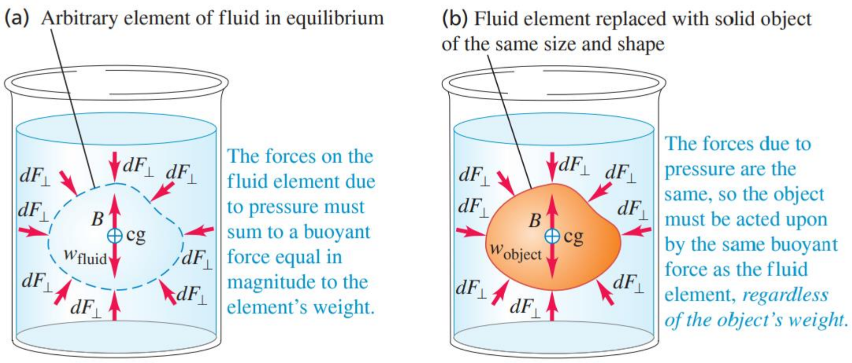


Figure 12.10 - Archimedes's principle

So the net weight of the fish is the same as the weight of the water it displaces. An object whose average density is *less* than that of a liquid can float partially submerged at the free upper surface of the liquid. A ship made of steel (which is much denser than water) can float because the ship is hollow, with air occupying much of its interior volume, so its average density is less than that of water. The greater the density of the liquid, the less of the object is submerged. When you swim in seawater (density 1030 kg/m^3), your body floats higher than in freshwater (1000 kg/m^3).

A practical example of buoyancy is the hydrometer, used to measure the density of liquids (Fig. 12.11a). The calibrated float sinks into the fluid until the weight of the fluid it displaces is exactly equal to its own weight. The hydrometer floats *higher* in denser liquids than in less dense liquids, and a scale in the top stem permits direct density readings. Hydrometers like this are used in medical diagnosis to measure the density of urine (which depends on a patient's level of hydration). Figure 12.11b shows a type of hydrometer used to measure the density of battery acid or antifreeze. The bottom of the large tube is immersed in the liquid; the bulb is squeezed to expel air and is then released, like a giant medicine dropper. The liquid rises into the outer tube, and the hydrometer floats in this liquid.

CAUTION! **The buoyant force depends on the fluid density.** The buoyant force on an object is proportional to the density of the *fluid* in which the object is immersed, *not* the density of the object. If a wooden block and an iron block have the same volume and both are submerged in water, both experience the same buoyant force. The wooden block rises and the iron block sinks because this buoyant force is greater than the weight of the wooden block but less than the weight of the iron block.

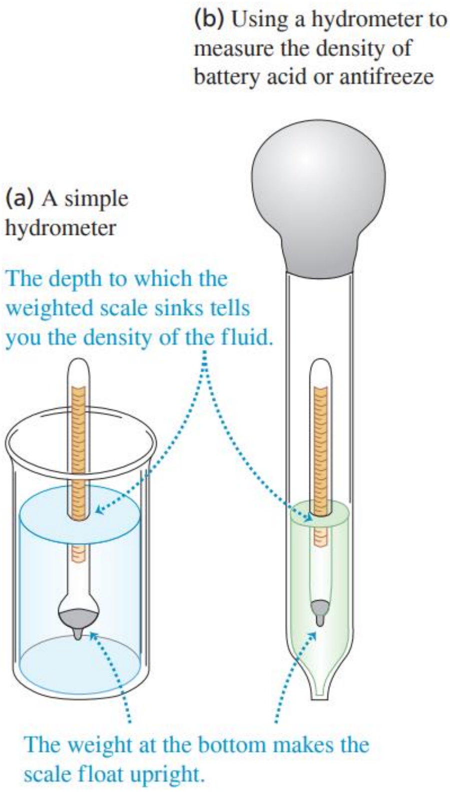


Figure 12.11 - Measuring the density of a fluid

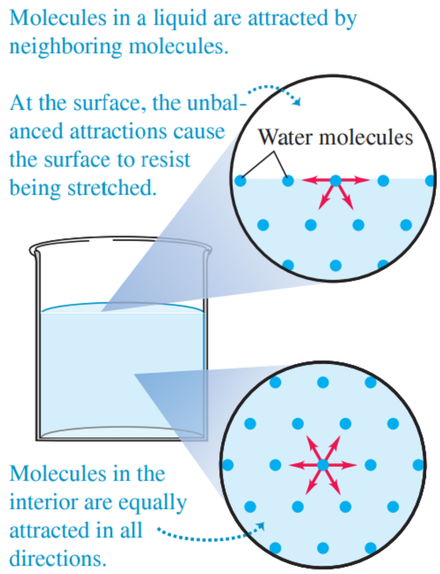


Figure 12.12 - A molecule at the surface of a liquid is attracted into the bulk liquid, which tends to reduce the liquid's surface area

Surface Tension

We've seen that if an object is less dense than water, it will float partially submerged. But a paper clip can rest *atop* a water surface even though its density is several times that of water. This is an example of **surface tension**: Surface tension arises because the molecules of the liquid exert attractive forces on each other. There is zero net force on a molecule within the interior of the liquid, but a surface molecule is drawn into the interior (**Fig. 12.12**). Thus the liquid tends to minimize its surface area, just as a stretched membrane does.

Surface tension explains why raindrops are spherical (*not* teardrop-shaped): A sphere has a smaller surface area for its volume than any other shape. It also explains why hot, soapy water is used for washing. To wash clothing thoroughly, water must be forced through the tiny spaces between the fibers (**Fig. 12.13**). This requires increasing the surface area of the water, which is difficult to achieve because of surface tension. The job is made easier by increasing the temperature of the water and adding soap, both of which decrease the surface tension.

Surface tension is important for a millimeter-sized water drop, which has a relatively large surface area for its volume. (A sphere of radius r has surface area $4\pi r^2$ and volume $(4\pi/3)r^3$. The ratio of surface area to volume is $3/r$, which increases with decreasing radius). But for large quantities of liquid, the ratio of surface area to volume is relatively small, and surface tension is negligible compared to pressure forces. For the remainder of this chapter, we'll consider only fluids in bulk and ignore the effects of surface tension.

12.4 Fluid Flow

We are now ready to consider *motion* of a fluid. Fluid flow can be extremely complex, as shown by the currents in river rapids or the swirling flames of a campfire. But we can represent some situations by relatively simple idealized models. An **ideal fluid** is a fluid that is *incompressible* (that is, its density cannot change) and has no internal friction (called **viscosity**). Liquids are approximately incompressible in most situations, and we may also treat a gas as incompressible if the pressure differences from one region to another are not too great. Internal friction in a fluid causes shear stresses when two adjacent layers of fluid move relative to each other, as when fluid flows inside a tube or around an obstacle. In some cases we can ignore these shear forces in comparison with forces arising from gravitation and pressure differences.

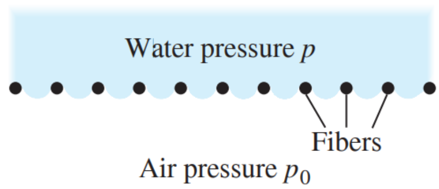


Figure 12.13 -Surface tension makes it difficult to force water through small crevices. The required water pressure p can be reduced by using hot, soapy water, which has less surface tension

The path of an individual particle in a moving fluid is called a **flow line**. In **steady flow**, the overall flow pattern does not change with time, so every element passing through a given point follows the same flow line. In this case the "map" of the fluid velocities at various points in space remains constant, although the velocity of a particular particle may change in both magnitude and direction during its motion. A **streamline** is a curve whose tangent at any point is in the direction of the fluid velocity at that point. When the flow pattern changes with time, thestreamlines do not coincide with the flow lines. We'll consider only steady-flow situations, for which flow lines and streamlines are identical. The flow lines passing through the edge of an imaginary element of area form a tube called a **flow tube**. From the definition of a flow line, in steady flow no fluid can cross the side walls of a given flow tube.

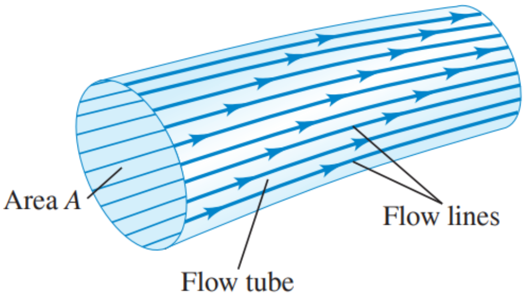


Figure 12.14 - A flow tube bounded by flow lines. In steady flow, fluid cannot cross the walls of a flow tube

Figure 12.14 shows the pattern of fluid flow from left to right around an obstacle. The photograph was made by injecting dye into water plates. This adjacent layers the flow is sufficiently high cause abrupt irregular and **12.15**). In pattern; the flow

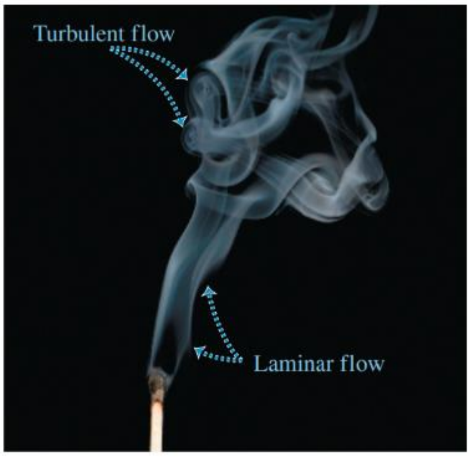


Figure 12.15 - The flow of smoke rising from this burnt match is laminar up to a certain point, and then becomes turbulent

flowing between two closely spaced glass pattern is typical of **laminar flow**, in which of fluid slide smoothly past each other and steady. (A *lamina* is a thin sheet). At flow rates, or when boundary surfaces changes in velocity, the flow can become chaotic. This is called **turbulent flow** (**Fig.** turbulent flow there is no steady-state pattern changes continuously.

The

The mass flows. This **continuity** between two

Continuity Equation

of a moving fluid doesn't change as it leads to an important relationship called the **equation**. Consider a portion of a flow tube stationary cross sections with areas A_1 and A_2 (**Fig. 12.17**). The fluid speeds at these sections are v_1 and v_2, respectively. As we mentioned above, no fluid flows in or out across the side walls of such a tube. During a small time interval dt, the fluid at A_1 moves a distance $ds_1 = v_1 dt$, so a cylinder of fluid with height $v1\ dt$ and volume $dV_1 = A_1 v_1$ dt flows into the tube across A_1. During this same interval, a cylinder of volume $dV_2 = A_2 v_2 dt$ flows out of the tube across A_2.

Let's first consider the case of an incompressible fluid so that the density ρ has the same value at all points. The mass dm_1 flowing into the tube across A_1 in time dt is $dm_1 = \rho A_1 \upsilon_1 dt$. Similarly, the mass dm_2 that flows out across A_2 in the same time is $dm_2 = \rho A_2 \upsilon_2 dt$. In steady flow the total mass in the tube is constant, so $dm_1 = dm_2$ and

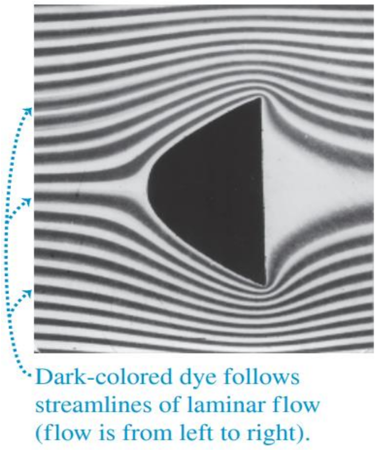


Figure 12.16 - Laminar flow around an obstacle **Де посилання на цей малюнок?**

$$\rho A_1 v_1 dt = \rho A_2 v_2 dt \quad \text{or}$$

Continuity equation for an **incompressible fluid:**

$$A_1 v_1 = A_2 v_2 \qquad (12.10)$$

Cross-sectional area of flow tube at two points (see Fig. 12.21)

Speed of flow at the two points

The product Av is the *volumeflow ratedV/dt*, the rate at which volume crosses a section of the tube:

Volume flow rate of a fluid

$$\frac{dV}{dt} = Av \qquad (12.11)$$

Cross-sectional area of flow tube

Speed of flow

The *mass flow rate* is the mass flow per unit time through a cross section. This is equal to the density r times the volume flow rate *dV*/*dt*.

Equation (12.10) shows that the volume flow rate has the same value at all points along any flow tube. When the cross section of a flow tube decreases, the speed increases, and vice versa. A broad, deep part of a river has a larger cross section and slower current than a narrow, shallow part, but the volume flow rates are the same in both. This is the essence of the familiar maxim, "Still waters run deep". If a water pipe with 2 cm diameter is connected to a pipe with 1 cm diameter, the flow speed is four times as great in the 1 cm part as in the 2 cm part.

We can generalize Eq. (12.10) for the case in which the fluid is *not* incompressible. If ρ_1 and ρ_2 are the densities at sections 1 and 2, then

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 \text{ (continuity equation, compressible fluid).} \quad (12.12)$$

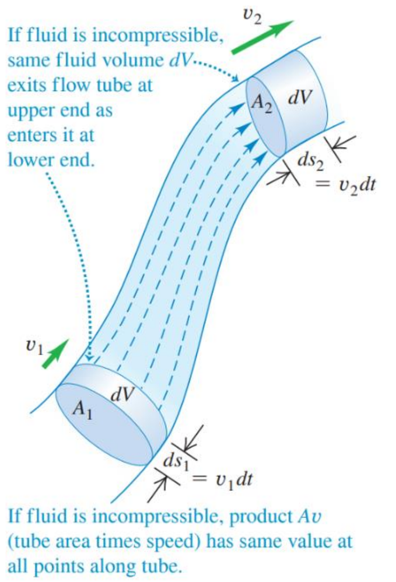


Figure 12.17 - A flow tube with changing cross-sectional area

If the fluid is denser at point 2 than at point 1 ($\rho_2 > \rho_1$), the volume flow rate at point 2 will be less than at point 1 ($A_2 \upsilon_2 < A_1 \upsilon_1$). We leave the details to you. If the fluid is incompressible so that ρ_1 and ρ_2 are always equal, Eq. (12.12) reduces to Eq. (12.10).

12.5 Bernoulli's Equation

According to the continuity equation, the speed of fluid flow can vary along the paths of the fluid. The pressure can also vary; it depends on height as in the static situation, and it also depends on the speed of flow. We can derive an important relationship called *Bernoulli's equation*, which relates the pressure, flow speed, and height for flow of an ideal, incompressible fluid. Bernoulli's equation is useful in analyzing many kinds of fluid flow.

The dependence of pressure on speed follows from the continuity equation, Eq. (12.10). When an incompressible fluid flows along a flow tube with varying cross section, its speed *must* change, and so an element of fluid must have an acceleration. If the tube is horizontal, the force that causes this acceleration has to be applied by the surrounding fluid. This means that the pressure *must* be different in regions of different cross section; if it were the same everywhere, the net force on every fluid element would be zero. When a horizontal flow tube narrows and a fluid element speeds up, it must be moving toward a region of lower pressure in order to have a net forward force to accelerate it. If the elevation also changes, this causes an additional pressure difference.

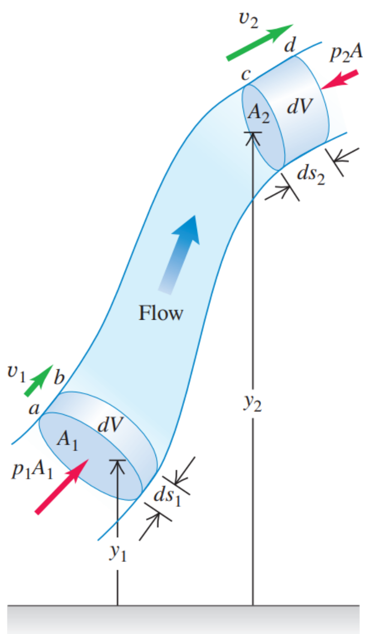


Figure 12.18 - Deriving Bernoulli's equation. The net work done on a fluid element by the pressure of the surrounding fluid equals the change in the kinetic energy plus the change in the gravitational potential energy

Deriving Bernoulli's Equation

To derive Bernoulli's equation, we apply the work–energy theorem to the fluid in a section of a flow tube. In **Fig. 12.18** we consider the element of fluid that at some initial time lies between the two cross sections *a* and *c*. The speeds at the lower and upper ends are υ_1 and υ_2. In a small time interval *dt*, the fluid that is initially at *a* moves to *b*, a distance $ds_1 = \upsilon_1 dt$, and the fluid that is initially at *c* moves to *d*, a distance $ds_2 = \upsilon_2 dt$. The cross-sectional areas at the two ends are A_1 and A_2, as shown. The fluid is incompressible; hence by the continuity equation, Eq. (12.10), the volume of fluid *dV* passing *any* cross section during time *dt* is the same. That is,

$$dV = A_1 ds_1 = A_2 ds_2 .$$

Let's compute the *work* done on this fluid element during *dt*. If there is negligible internal friction in the fluid (i.e., no viscosity), the only nongravitational forces that do work on the element are due to the pressure of the surrounding fluid. The pressures at the two ends are p_1 and p_2 the force on the cross section at *a* is $p_1 A_1$, and the force at *c* is $p_2 A_2$. The net work *dW* done on the element by the surrounding fluid during this displacement is therefore

$$dW = p_1 A_1 ds_1 - p_2 A_2 ds_2 = (p_1 - p_2) dV . \tag{12.13}$$

The term $p_2 A_2 ds_2$ has a negative sign because the force at *c* opposes the displacement of the fluid.

The work *dW* is due to forces other than the conservative force of gravity, so it equals the change in the total mechanical energy (kinetic energy plus gravitational potential energy) associated with the fluid element. The mechanical energy for the fluid between sections *b* and *c* does not change. At the beginning of *dt* the fluid between *a* and *b* has volume $A_1 ds_1$, mass $\rho A_1 ds_1$, and kinetic energy $\frac{1}{2}\rho (A_1 ds_1) \upsilon_1^2$. At the end of *dt* the fluid between *c* and *d* has kinetic energy $\frac{1}{2}\rho (A_2 ds_2) \upsilon_2^2$. The net change in kinetic energy *dK* during time *dt* is

$$dK = \tfrac{1}{2}\rho dV (\upsilon_2^2 - \upsilon_1^2) . \tag{12.14}$$

What about the change in gravitational potential energy? At the beginning of time interval *dt*, the potential energy for the mass between *a* and *b* is $dm\, gy_1 = \rho\, dV\, gy_1$. At the end of *dt*, the potential energy for the mass between *c* and *d* is $dm\, gy_2 = \rho\, dV\, gy_2$. The net change in potential energy *dU* during *dt* is

$$dU = \rho dV g (y_2 - y_1) . \tag{12.15}$$

Combining Eqs. (12.13), (12.14), and (12.15) in the energy equation *dW* = *dK* + *dU*, we obtain

$$\begin{gathered}(p_1 - p_2) dV = \tfrac{1}{2}\rho dV (\upsilon_2^2 - \upsilon_1^2) + \rho dV g (y_2 - y_1) \\ p_1 - p_2 = \tfrac{1}{2}\rho (\upsilon_2^2 - \upsilon_1^2) + \rho g (y_2 - y_1)\end{gathered} \tag{12.16}$$

This is **Bernoulli's equation**. It states that the work done on a unit volume of fluid by the surrounding fluid is equal to the sum of the changes in kinetic and potential energies per unit volume that occur during the flow. We may also interpret Eq. (12.16) in terms of pressures. The first term on the right is the pressure difference associated with the change of speed of the fluid. The second term on the right is the additional pressure difference caused by the weight of the fluid and the difference in elevation of the two ends.

We can also express Eq. (12.16) in a more convenient form as

$$p_1 + \rho g y_1 + \tfrac{1}{2}\rho \upsilon_1^2 = p_2 + \rho g y_2 + \tfrac{1}{2}\rho \upsilon_2^2 . \tag{12.17}$$

Subscripts 1 and 2 refer to *any* two points along the flow tube, so we can write

Bernoulli's equation for an ideal, incompressible fluid:

Pressure Fluid density Value is **same** at all points in flow tube.

$$p + \rho g y + \tfrac{1}{2}\rho v^2 = \text{constant} \tag{12.18}$$

Acceleration due to gravity Elevation Flow speed

Note that when the fluid is not moving (so $\upsilon_1 = \upsilon_2 = 0$), Eq. (12.17) reduces to the pressure relationship we derived for a fluid at rest, Eq. (12.5).

CAUTION! **Bernoulli's equation applies in certain situations onlyю** We stress again that Bernoulli's equation is valid for only incompressible, steady flow of a fluid with no internal friction (no viscosity). It's a simple equation, but don't be tempted to use it in situations in which it doesn't apply!

PROBLEM-SOLVING STRATEGY 12.1 Bernoulli's Equation

Bernoulli's equation is derived from the work–energy theorem, so much of Problem-Solving Strategy 7.1 (Section 7.1) applies here.

IDENTIFY *the relevant concepts:*

Bernoulli's equation is applicable to steady flow of an incompressible fluid that has no internal friction (see Section 12.6). It is generally applicable to flows through large pipes and to flows within bulk fluids (e.g., air flowing around an airplane or water flowing around a fish).

SET UP *the problem* :

- Identify the points 1 and 2 referred to in Bernoulli's equation, Eq. (12.17).
- Define your coordinate system, particularly the level at which $y = 0$. Take the positive y-direction to be upward.
- List the unknown and known quantities in Eq. (12.17). Decide which unknowns are the target variables.

EXECUTE *the solution*:

Write Bernoulli's equation and solve for the unknowns. You may need the continuity equation, Eq. (12.10), to relate the two speeds in terms of cross-sectional areas of pipes or containers. You may also need Eq. (12.11) to find the volume flow rate.

EVALUATE *your answer:*

- Verify that the results make physical sense. Check that you have used consistent units: In SI units, pressure is in pascals, density in kilograms per cubic meter, and speed in meters per second. The pressures must be either *all* absolute pressures or *all* gauge pressures.

12.6 Viscosity and Turbulence

In our discussion of fluid flow we assumed that the fluid had no internal friction and that the flow was laminar. While these assumptions are often quite valid, in many important physical situations the effects of viscosity (internal friction) and turbulence (nonlaminar flow) are extremely important. Let's take a brief look at some of these situations.

Viscosity is internal friction in a fluid. Viscous forces oppose the motion of one portion of a fluid relative to another. Viscosity is the reason it takes effort to paddle a canoe through calm water, but it is also the reason the paddle works. Viscous effects are important in the flow of fluids in pipes, the flow of blood, the lubrication of engine parts, and many other situations.

Fluids that flow readily, such as water or petrol have smaller viscosities than do "thick" liquids such as honey or motor oil. Viscosities of all fluids are strongly temperature dependent, increasing for gases and decreasing for liquids as the temperature increases. Oils for engine lubrication must flow equally well in cold and warm conditions, and so are designed to have as *little* temperature variation of viscosity as possible.

A viscous fluid always tends to cling to a solid surface in contact with it. There is always a thin *boundary layer* of fluid near the surface, in which the fluid is nearly at rest with respect to the surface. That's why dust particles can cling to a fan blade even when it is rotating rapidly, and why you can't get all the dirt off your car by just squirting a hose at it.

Viscosity has important effects on the flow of liquids through pipes, including the flow of blood in the circulatory system. First think about a fluid with zero viscosity so that we can apply Bernoulli's equation, Eq. (12.17). If the two ends of a long cylindrical pipe are at the same height ($y_1 = y_2$) and the flow speed is the same at both ends ($\upsilon_1 = \upsilon_2$), Bernoulli's equation tells us that the pressure is the same at both ends of the pipe. But this isn't true if we account for viscosity. To see why, consider **Fig. 12.19**, which shows the flow-speed profile for laminar flow of a viscous fluid in a long cylindrical pipe. Due to viscosity, the speed is *zero* at the pipe walls (to which the fluid clings) and is greatest at the center of the pipe. The motion is like a lot of concentric tubes sliding relative to one another, with the central tube moving fastest and the outermost tube at rest. Viscous forces between the tubes oppose this sliding, so to keep the flow going we must apply a greater pressure at the back of the flow than at the front. That's why you have to keep squeezing a tube of toothpaste or a packet of ketchup (both viscous fluids) to keep the fluid coming out of its container. Your fingers provide a pressure at the back of the flow that is far greater than the atmospheric pressure at the front of the flow.

The pressure difference required to sustain a given volume flow rate through a cylindrical pipe of length L and radius R turns out to be proportional to L/R^4 . If we decrease R by one-half, the required pressure increases by $2^4 = 16$; decreasing R by a factor of 0.90 (a 10% reduction) increases the required pressure difference by a factor of $(1/0.90)^4 = 1.52$ (a 52% increase). This simple relationship explains the connection between a highcholesterol diet (which tends to narrow the arteries) and high blood pressure. Due to the R^4 dependence, even a small narrowing of the arteries can result in substantially elevated blood pressure and added strain on the heart muscle.

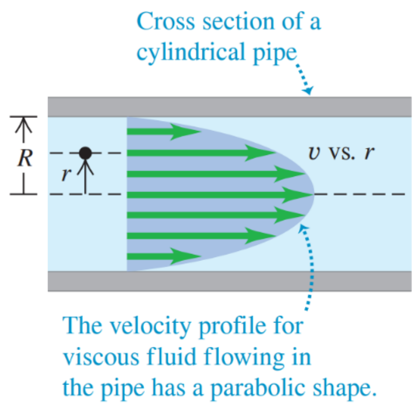


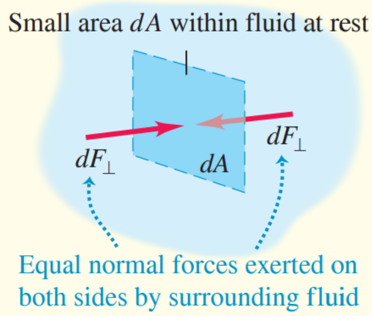
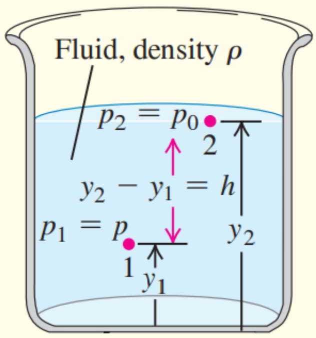
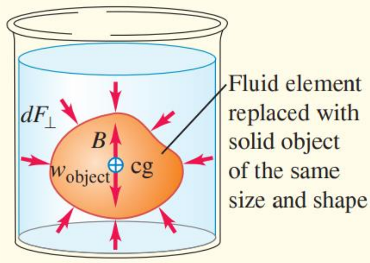
Figure 12.19 - Velocity profile for a viscous fluid in a cylindrical pipe

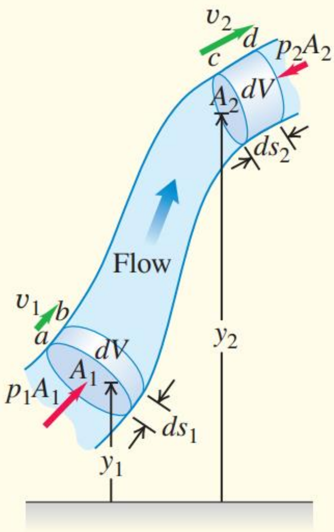
Turbulence

When the speed of a flowing fluid exceeds a certain critical value, the flow is no longer laminar. Instead, the flow pattern becomes extremely irregular and complex, and it changes continuously with time; there is no steady-state pattern. This irregular, chaotic flow is called **turbulence**. Figure 12.15 shows the contrast between laminar and turbulent flow for smoke rising in air. Bernoulli's equation is *not* applicable to regions where turbulence occurs because the flow is not steady.

Whether a flow is laminar or turbulent depends in part on the fluid's viscosity. The greater the viscosity, the greater the tendency for the fluid to flow in sheets (laminae) and the more likely the flow is to be laminar. (When we discussed Bernoulli's equation in Section 12.5, we assumed that the flow was laminar and that the fluid had zero viscosity. In fact, a *little* viscosity is needed to ensure that the flow is laminar).

For a fluid of a given viscosity, flow speed is a determining factor for the onset of turbulence. A flow pattern that is stable at low speeds suddenly becomes unstable when a critical speed is reached. Irregularities in the flow pattern can be caused by roughness in the pipe wall, variations in the density of the fluid, and many other factors. At low flow speeds, these disturbances damp out; the flow pattern is *stable* and tends to maintain its laminar nature. When the critical speed is reached, however, the flow pattern becomes unstable. The disturbances no longer damp out but grow until they destroy the entire laminar-flow pattern.

CHAPTER 12: SUMMARY		
Density and pressure: Density is mass per unit volume. If a mass *m* of homogeneous material has volume *V*, its density ρ is the ratio *m*/*V*. Specific gravity is the ratio of the density of a material to the density of water. Pressure is normal force per unit area. Pascal's law states that pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid. Absolute pressure is the total pressure in a fluid; gauge pressure is the difference between absolute pressure and atmospheric pressure. The SI unit of pressure is the pascal (Pa): 1 Pa = 1 N/m^2	$\rho = \frac{m}{V}$ $p = \frac{dF_\perp}{dA}$	
Pressures in a fluid at rest: The pressure difference between points 1 and 2 in a static fluid of uniform density ρ (an incompressible fluid) is proportional to the difference between the elevations y_1 and y_2. If the pressure at the surface of an incompressible liquid at rest is p_0, then the pressure at a depth *h* is greater by an amount ρgh	$p_2 - p_1 = -\rho g(y_2 - y_1)$ (pressure in a fluid of uniform density) $p = p_0 + \rho gh$ (pressure in a fluid of uniform density)	
Buoyancy: Archimedes's principle states that when an object is immersed in a fluid, the fluid exerts an upward buoyant force on the object equal to the weight of the fluid that the object displaces.		

Fluid flow: An ideal fluid is incompressible and has no viscosity (no internal friction). A flow line is the path of a fluid particle; a streamline is a curve tangent at each point to the velocity vector at that point. A flow tube is a tube bounded at its sides by flow lines. In laminar flow, layers of fluid slide smoothly past each other. In turbulent flow, there is great disorder and a constantly changing flow pattern. Conservation of mass in an incompressible fluid is expressed by the continuity equation, which relates the flow speeds υ_1 and υ_2 for two cross sections A_1 and A_2 in a flow tube. The product *Av* equals the volume flow rate, *dV*/*dt*, the rate at which volume crosses a section of the tube. Bernoulli's equation states that a quantity involving the pressure *p*, flow speed υ, and elevation *y* has the same value anywhere in a flow tube, assuming steady flow in an ideal fluid. This equation can be used to relate the properties of the flow at any two points.	$A_1\upsilon_1 = A_2\upsilon_2$ (continuity equation, incompressible fluid) $\frac{dV}{dt} = A\upsilon$ (volume flow rate) $p + \rho g y + \frac{1}{2}\rho\upsilon^2 = \text{constant}$ (Bernoulli's equation)	

13 GRAVITATION

Some of the earliest investigations in physical science started with questions that people asked about the night sky. Why doesn't the moon fall to earth? Why do the planets move across the sky? Why doesn't the earth fly off into space rather than remaining in orbit around the sun? The study of gravitation provides the answers to these and many related questions.

As we remarked in Chapter 5, gravitation is one of the four classes of interactions found in nature, and it was the earliest of the four to be studied extensively. Newton discovered in the 17th century that the same interaction that makes an apple fall out of a tree also keeps the planets in their orbits around the sun. This was the beginning of *celestial mechanics*, the study of the dynamics of objects in space. Today, our knowledge of celestial mechanics allows us to determine how to put a satellite into any desired orbit around the earth or to choose just the right trajectory to send a spacecraft to another planet.

In this chapter you'll learn the basic law that governs gravitational interactions. This law is *universal*: Gravity acts in the same fundamental way between the earth and your body, between the sun and a planet, and between a planet and one of its moons. We'll apply the law of gravitation to phenomena such as the variation of weight with altitude, the orbits of satellites around the earth, and the orbits of planets around the sun.

13.1 Newton's Law of Gravitation

The gravitational attraction that's most familiar to you is your *weight*, the force that attracts you toward the earth. By studying the motions of the moon and planets, Newton discovered a fundamental **law of gravitation** that describes the gravitational attraction between *any* two objects. Newton published this law in 1687 along with his three laws of motion. In modern language, it says

NEWTON'S LAW OF GRAVITATION. Every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them.

Figure 13.1 depicts this law, which we can express as an equation:

Newton's **law of gravitation:** Magnitude of attractive gravitational force between any two particles

Gravitational constant (same for any two particles)

Masses of particles

Distance between particles

$$F_g = \frac{Gm_1m_2}{r^2} \tag{13.1}$$

The **gravitational constant** G in Eq. (13.1) is a fundamental physical constant that has the same value for any two particles. We'll see shortly what the value of G is and how this value is measured.

Equation (13.1) tells us that the gravitational force between two particles decreases with increasing distance r: If the distance is doubled, the force is only one-fourth as great, and so on. Although many of the stars in the night sky are far more massive than the sun, they are so far away that their gravitational force on the earth is negligibly small.

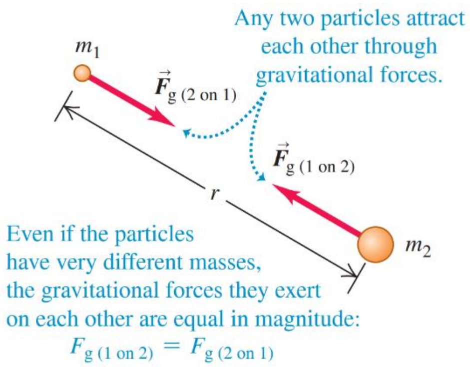


Figure 13.1 - The gravitational forces that two particles of masses m_1 and m_2 exert on each other

CAUTION! **Don't confuse *g* and *G*.** The symbols g and G are similar, but they represent two very different gravitational quantities. Lowercase g is the acceleration due to gravity, which relates the weight w of an object to its mass m: $w = mg$. The value of g is different at different locations on the earth's surface and on the surfaces of other planets. By contrast, capital G relates the gravitational force between any two objects to their masses and the distance between them. We call G a *universal* constant because it has the same value for any two objects, no matter where in space they are located. We'll soon see how the values of g and G are related.

Gravitational forces always act along the line joining the two particles and form an action–reaction pair. Even when the masses of the particles are different, the two interaction forces have equal magnitude (Fig. 13.1). The attractive force that your body exerts on the earth has the same magnitude as the force that the earth exerts on you. When you fall from a diving board into a swimming pool, the entire earth rises up to meet you! (You don't notice this because the earth's mass is greater than yours by a factor of about 10^{23}. Hence the earth's acceleration is only 10^{-23} as great as yours).

Gravitation and Spherically Symmetric Objects

We have stated the law of gravitation in terms of the interaction between two *particles*. It turns out that the gravitational interaction of any two objects that have *sphericallysymmetric* mass distributions (such as solid spheres or spherical shells) is the same as though we concentrated all the mass of each at its center, as in **Fig. 13.2**. Thus, if we model the earth as a spherically symmetric object with mass m_E , the force it exerts on a particle or on a spherically symmetric object with mass *m*, at a distance *r* between centers, is

$$F_g = \frac{Gm_E m}{r^2}, \tag{13.2}$$

provided that the object lies outside the earth. A force of the same magnitude is exerted *on* the earth by the object. (We'll prove these statements in Section 13.6).

At points *inside* the earth the situation is different. If we could drill a hole to the center of the earth and measure the gravitational force on an object at various depths, we would find that toward the center of the earth the force *decreases*, rather than increasing as $1/r^2$. As the object enters the interior of the earth (or other spherical object), some of the earth's mass is on the side of the object opposite from the center and pulls in the opposite direction. Exactly at the center, the earth's gravitational force on the object is zero.

Spherically symmetric objects are an important case because moons, planets, and stars all tend to be spherical. Since all particles in an object gravitationally attract each other, the particles tend to move to minimize the distance between them. As a result, the object naturally tends to assume a spherical shape, just as a lump of clay forms into a sphere if you squeeze it with equal forces on all sides. This effect is greatly reduced in celestial objects of low mass, since the gravitational attraction is less, and these objects tend *not* to be spherical.

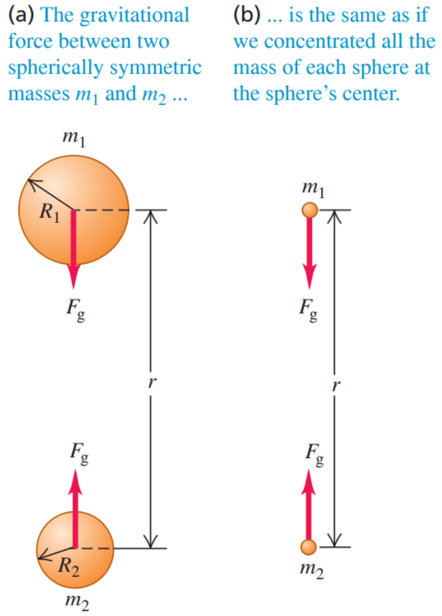


Figure 13.2 - The gravitational effect outside any spherically symmetric mass distribution is the same as though all of the mass were concentrated at its center

Determining the Value of *G*

To determine the value of the gravitational constant *G*, we have to *measure* the gravitational force between two objects of known masses m_1 and m_2 at a known distance *r*. The force is extremely small for objects that are small enough to be brought into the laboratory, but it can be measured with an instrument called a *torsion balance*, which Sir Henry Cavendish used in 1798 to determine *G*.

Figure 13.3 shows a modern version of the Cavendish torsion balance. A light, rigid rod shaped like an inverted T is supported by a very thin, vertical quartz fiber. Two small spheres, each of mass m_1, are mounted at the ends of the horizontal arms of the T. When we bring two large spheres, each of mass

m_2, to the positions shown, the attractive gravitational forces twist the T through a small angle. To measure this angle, we shine a beam of light on a mirror fastened to the T. The reflected beam strikes a scale, and as the T twists, the reflected beam moves along the scale.

After calibrating the Cavendish balance, we can measure gravitational forces and thus determine *G*. The accepted value as of this writing (2018) is

$$G = 6.67408(31)\times10^{-11}\,\mathrm{N{\bullet}m^2/kg^2}.$$

To three significant figures, $G = 6.67\times10^{-11}\,\mathrm{N{\bullet}m^2/kg^2}$. Because 1 N = 1 kg • m/s^2 , the units of *G* can also be expressed as m^3/(kg • s^2).

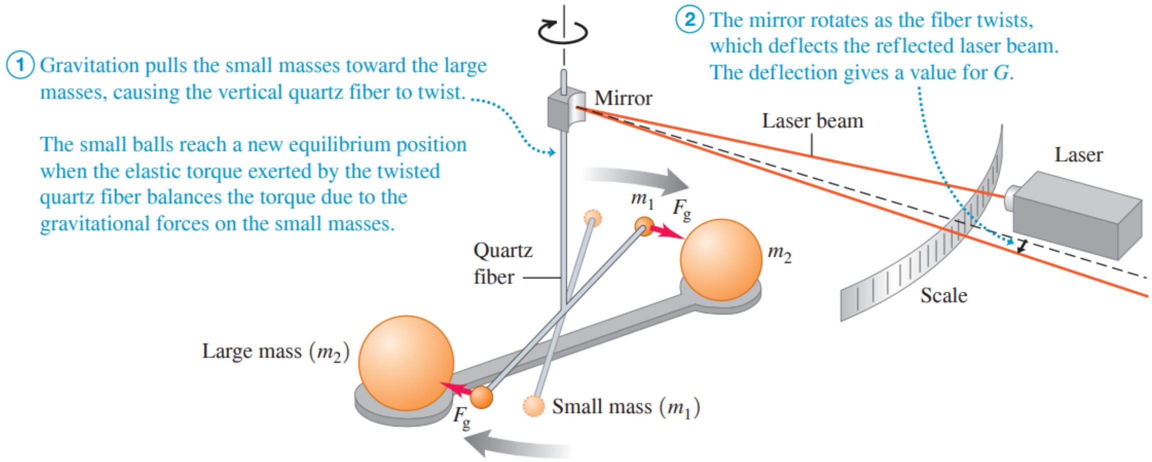


Figure 13.3 - The principle of the Cavendish balance, used for determining the value of *G*. The angle of deflection has been exaggerated here for clarity

Gravitational forces combine vectorially. If each of two masses exerts a force on a third, the *total* force on the third mass is the vector sum of the individual forces of the first two. Example 13.3 makes use of this property, which is often called *superposition of forces* (see Section 4.1).

EXAMPLE 13.1 Calculating gravitational force

The mass m_1 of one of the small spheres of a Cavendish balance is 0.0100 kg, the mass m_2 of the nearest large sphere is 0.500 kg, and the center-to-center distance between them is 0.0500 m. Find the gravitational force F_g on each sphere due to the other.

IDENTIFY, SET UP and EXECUTE

Because the spheres are spherically symmetric, we can calculate F_g by treating them as *particles* separated by 0.0500 m, as in Fig. 13.2. Each sphere experiences the same magnitude of force from the other sphere. We use Newton's law of gravitation, Eq. (13.1), to determine F_g:

$$F_g = \frac{(6.67\times10^{-11}\,\mathrm{N{\bullet}m^2/kg^2})(0.0100\mathrm{kg})(0.500\mathrm{kg})}{(0.500\mathrm{m})^2} = 1.33\times10^{-10}\,\mathrm{N}.$$

EVALUATE It's remarkable that such a small force could be measured— or even detected— more than 200 years ago. Only a very massive object such as the earth exerts a gravitational force we can feel.

CAUTION! **Newton's third law applies to gravitational forces, too.** Even though the large sphere in this example has 50 times the mass of the small sphere, each sphere feels the *same* magnitude of force from the other. But because their masses are different, the accelerations of the two objects in response to that magnitude of force are different (see the following example).

KEYCONCEPT

Any two objects exert attractive gravitational forces on each other that are proportional to the product of the masses of the two objects and inversely proportional to the square of the distance between their centers (Newton's law of gravitation).

Why Gravitational Forces Are Important

Comparing Examples 13.1 and 13.3 shows that gravitational forces are negligible between ordinary household-sized objects but very substantial between objects that are the size of stars. Indeed, gravitation is *the* most important force on the scale of planets, stars, and galaxies. It is responsible for holding our earth together and for keeping the planets in orbit about the sun. The mutual gravitational attraction between different parts of the sun compresses material at the sun's core to very high densities and temperatures, making it possible for nuclear reactions to take place there. These reactions generate the sun's energy output, which makes it possible for life to exist on earth and for you to read these words.

The gravitational force is so important on the cosmic scale because it acts *at a distance*, without any direct contact between objects. Electric and magnetic forces have this same remarkable property, but they are less important on astronomical scales because large accumulations of matter are electrically neutral; that is, they contain equal amounts of positive and negative charge. As a result, the electric and magnetic forces between stars or planets are very small or zero. The strong and weak interactions that we discussed in Section 5.5 also act at a distance, but their influence is negligible at distances much greater than the diameter of an atomic nucleus (about 10^{-14} m).

A useful way to describe forces that act at a distance is in terms of a *field*. One object sets up a disturbance or field at all points in space, and the force that acts on a second object at a particular point is its response to the first object's field at that point. There is a field associated with each force that acts at a distance, and so we refer to gravitational fields, electric fields, magnetic fields, and so on. We won't need the field concept for our study of gravitation in this chapter, so we won't discuss it further here. But in later chapters we'll find that the field concept is an extraordinarily powerful tool for describing electric and magnetic interactions.

13.2Weight

We defined the *weight* of an object in Section 4.4 as the attractive gravitational force exerted on it by the earth. We can now broaden our definition and say that *the weight of an object is the total gravitational force exerted on the object by all other objects in the universe*. When the object is near the surface of the earth, we can ignore all other gravitational forces and consider the weight as just the earth's gravitational attraction. At the surface of the *moon* we consider an object's weight to be the gravitational attraction of the moon, and so on.

If we again model the earth as a spherically symmetric object with radius R_E , the weight of a small object at the earth's surface (a distance R_E from its center) is

Gravitational constant

Mass of the earth

Weight of an object at the earth's surface ...

$$w = F_g = \frac{Gm_E m}{R_E^2} \tag{13.3}$$

Mass of object

... equals gravitational force the earth exerts on object.

Radius of the earth

But we also know from Section 4.4 that the weight w of an object is the force that causes the acceleration g of free fall, so by Newton's second law, $w = mg$. Equating this with Eq. (13.3) and dividing by m, we find

Gravitational constant — Mass of the earth

Acceleration due to gravity at the earth's surface

$$g = \frac{Gm_E}{R_E^2} \tag{13.4}$$

Radius of the earth

The acceleration due to gravity g is independent of the mass m of the object because m doesn't appear in this equation. We already knew that, but we can now see how it follows from the law of gravitation.

We can *measure* all the quantities in Eq. (13.4) except for m_E , so this relationship allows us to compute the mass of the earth. Solving Eq. (13.4) for m_E and using R_E = 6370 km = 6.37×10^6 m and $g = 9.80\ \text{m/s}^2$, we find

$$m_E = \frac{gR_E^2}{G} = 5.96 \times 10^{24}\,\text{kg}\,.$$

This is very close to the currently accepted value of 5.972×10^{24} kg. Once Cavendish had measured G, he computed the mass of the earth in just this way.

At a point above the earth's surface a distance r from the center of the earth (a distance r - R_E above the surface), the weight of an object is given by Eq. (13.3) with R_E replaced by r:

$$w = F_g = \frac{Gm_E m}{r^2}\,. \tag{13.5}$$

The weight of an object decreases inversely with the square of its distance from the earth's center.

The *apparent* weight of an object on earth differs slightly from the earth's gravitational force because the earth rotates and is therefore not precisely an inertial frame of reference. We've ignored this relatively small effect in our discussion but will consider it carefully in Section 13.7.

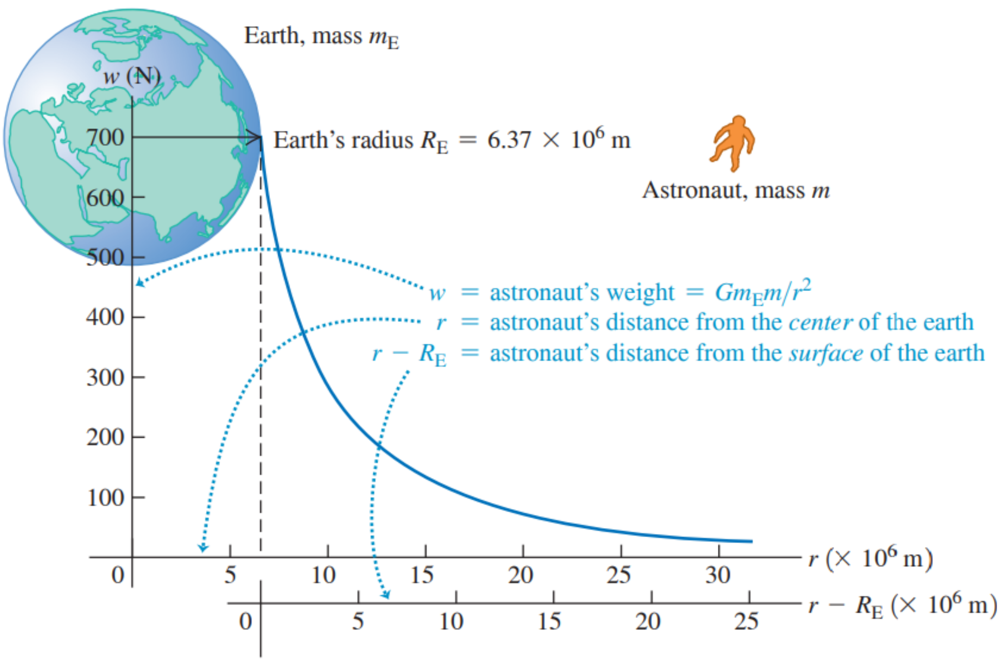


Figure 13.4 - An astronaut who weighs 700 N at the earth's surface experiences less gravitational attraction when above the surface. The relevant distance r is from the astronaut to the *center* of the earth (*not* from the astronaut to the earth's surface)

While the earth is an approximately spherically symmetric distribution of mass, it is *not* uniform throughout its volume. To demonstrate this, let's first calculate the average *density*, or mass per unit volume, of the earth. If we assume a spherical earth, the volume is

$$V_E = \tfrac{4}{3}\pi R_E^3 = \tfrac{4}{3}\pi(6.63\times10^6\,\text{m})^3 = 1.08\times10^{21}\,\text{m}^3 .$$

The average density ρ (the Greek letter rho) of the earth is the total mass divided by the total volume:

$$\rho = \frac{m_E}{V_E} = \frac{5.97\times10^{24}\,\text{kg}}{1.08\times10^{21}\,\text{m}^3} = 5500\,\text{kg/m}^3 = 5.5\,\text{g/cm}^3 .$$

(Compare to the density of water, 1000 kg/m^3 = 1.00 g/cm^3). If the earth were uniform, rocks near the earth's surface would have this same density. In fact, the density of surface rocks is substantially lower, ranging from about 2000 kg/m^3 for sedimentary rocks to about 3300 kg/m^3 for basalt. So the earth *cannot* be uniform, and its interior must be much more dense than its surface in order that the *average* density be 5500 kg/m^3. According to geophysical models of the earth's interior, the maximum density at the center is about 13,000 kg/m^3. **Figure 13.5** is a graph of density as a function of distance from the center.

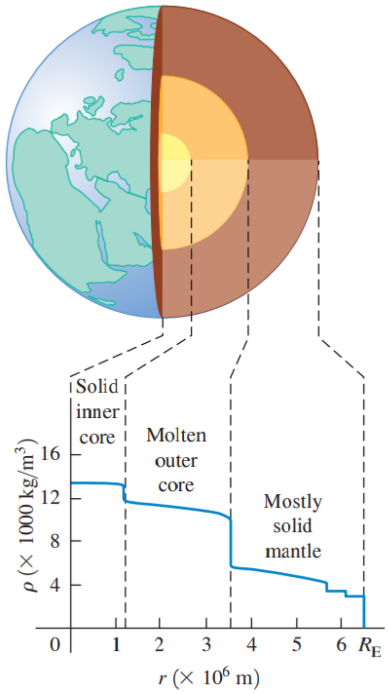


Figure 13.5 - The density ρ of the earth decreases with increasing distance r from its center

13.3Gravitational Potential Energy

When we first introduced gravitational potential energy in Section 7.1, we assumed that the earth's gravitational force on an object of mass m doesn't depend on the object's height. This led to the expression $U = mgy$. But Eq. (13.2), $F_g = Gm_E m / r^2$, shows that the gravitational force exerted by the earth (mass m_E) *does* in general depend on the distance r from the object to the earth's center. For problems in which an object can be far from the earth's surface, we need a more general expression for gravitational potential energy. To find this expression, we follow the same steps as in Section 7.1. We consider an object of mass m outside the earth, and first compute the work W_{grav} done by the gravitational force when the object moves directly away from or toward the center of the earth from $r = r_1$ to $r = r_2$, as in **Fig. 13.6**. This work is given by

$$W_{grav} = \int_{r_1}^{r_2} F_r dr , \qquad (13.6)$$

where F_r is the radial component of the gravitational force $\vec{F}$—that is, the component in the direction *outward* from the center of the earth. Because $\vec{F}$ points directly *inward* toward the center of the earth, F_r is negative. It differs from Eq. (13.2), the magnitude of the gravitational force, by a minus sign:

$$F_r = \frac{Gm_E m}{r^2} \text{ (13.7).}$$

Substituting Eq. (13.7) into Eq. (13.6), we see that W_{grav} is given by

$$W_{grav} = -G m_E m \int_{r_1}^{r_2} \frac{dr}{r^2} = \frac{G m_E m}{r_2} - \frac{G m_E m}{r_1}. \qquad (13.8)$$

The path doesn't have to be a straight line; it could also be a curve like the one in **Fig. 13.6.** By an argument similar to that in Section 7.1, this work depends on only the initial and final values of r, not on the path taken. This also proves that the gravitational force is always *conservative*.

We now define the corresponding potential energy U so that $W_{grav} = U_1 - U_2$, as in Eq. (7.3). Comparing this with Eq. (13.8),
we see that the appropriate definition for **gravitational potential energy** is

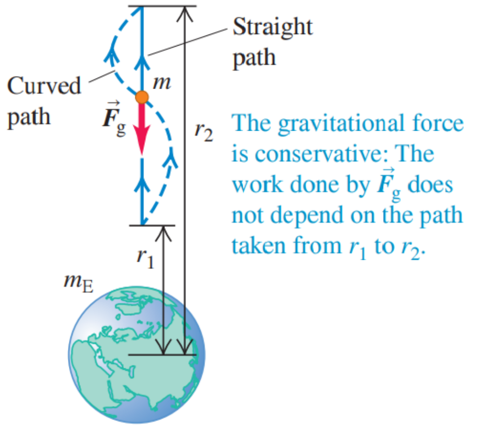


Figure 13.6 - Calculating the work done on an object by the gravitational force as the object moves from radial coordinate r_1 to r_2

Gravitational constant; Mass of the earth; Mass of object; Distance of object from the earth's center

Gravitational potential energy (general expression)

$$U = -\frac{G m_E m}{r}. \qquad (13.9)$$

CAUTION! **Gravitational force vs. gravitational potential energy.** Don't confuse the expressions for gravitational force, Eq. (13.7), and gravitational potential energy, Eq. (13.9). The force F_r is proportional to $1/r^2$, while potential energy U is proportional to $1/r$.

Figure 13.7 shows how the gravitational potential energy depends on the distance r between the object of mass m and the center of the earth. When the object moves away from the earth, r increases, the gravitational force does negative work, and U increases (i.e., becomes less negative). When the object "falls" toward earth, r decreases, the gravitational work is positive, and the potential energy decreases (i.e., becomes more negative).

CAUTION! **Don't worry about gravitational potential energy being negative.** You may be troubled by Eq. (13.9) because it states that gravitational potential energy is always negative. But in fact you've seen negative values of U before. In using the formula $U = mgy$ in Section 7.1, we found that U was negative whenever the object of mass m was at a value of y below the arbitrary height we chose to be $y = 0$—that is, whenever the object and the earth were closer together than some arbitrary distance. (See, for instance, Example 7.2 in Section 7.1). In defining U by Eq. (13.9), we have chosen U to be zero when the object of mass m is infinitely far from the earth ($r = \infty$). As the object moves toward the earth, gravitational potential energy decreases and so becomes negative.

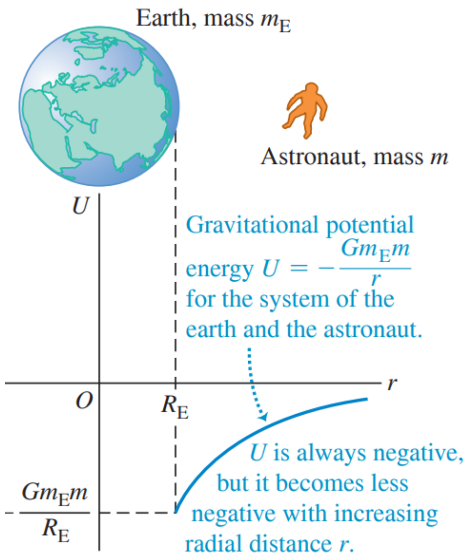


Figure 13.7 - A graph of the gravitational potential energy U for the system of the earth (mass m_E) and an astronaut (mass m) versus the astronaut's distance r from the center of the earth

If we wanted, we could make $U = 0$ at the earth's surface, where $r = R_E$, by adding the quantity $G m_E m / R_E$ to Eq. (13.9). This would make U positive when $r > R_E$. We won't do this for two reasons: One, it would complicate the expression for U; two, the added term would not affect the *difference* in potential energy between any two points, which is the only physically significant quantity.

If the earth's gravitational force on an object is the only force that does work, then the total mechanical energy of the system of the earth and object is constant, or *conserved*. In the following example we'll use this principle to calculate **escape speed**, the speed required for an object to escape completely from a planet.

More on Gravitational Potential Energy

As a final note, let's show that when we are close to the earth's surface, Eq. (13.9) reduces to the familiar $U = mgy$ from Chapter 7. We first rewrite Eq. (13.8) as

$$W_{\text{grav}} = Gm_{\text{E}}m\frac{r_1 - r_2}{r_1 r_2} .$$

If the object stays close to the earth, then in the denominator we may replace r_1 and r_2 by R_{E}, the earth's radius, so

$$W_{\text{grav}} = Gm_{\text{E}}m\frac{r_1 - r_2}{R_{\text{E}}^2} .$$

According to Eq. (13.4), $g = Gm_{\text{E}} / R_{\text{E}}^2$, so

$$W_{\text{grav}} = mg(r_1 - r_2) .$$

If we replace the *r*'*s* by *y*'*s*, this is just Eq. (7.1) for the work done by a constant gravitational force. In Section 7.1 we used this equation to derive Eq. (7.2), *U* = *mgy*, so we may consider Eq. (7.2) for gravitational potential energy to be a special case of the more general Eq. (13.9).

13.4 The Motion of Satellites

Artificial satellites orbiting the earth are a familiar part of technology. But how do they stay in orbit, and what determines the properties of their orbits? We can use Newton's laws and the law of gravitation to provide the answers. In the next section we'll analyze the motion of planets in the same way.

To begin, think back to the discussion of projectile motion in Section 3.3. In Example 3.6 a motorcycle rider rides horizontally off the edge of a cliff, launching himself into a parabolic path that ends on the flat ground at the base of the cliff. If he survives and repeats the experiment with increased launch speed, he will land farther from the starting point. We can imagine him launching himself with great enough speed that the earth's curvature becomes significant. As he falls, the earth curves away beneath him. If he is going fast enough, and if his launch point is high enough that he clears the mountaintops, he may be able to go right on around the earth without ever landing.

Figure 13.8 shows a variation on this theme. We launch a projectile from point *A* in the direction *AB*, tangent to the earth's surface. Trajectories 1 through 7 show the effect of increasing the initial speed. In trajectories 3 through 5 the projectile misses the earth and becomes a satellite. If there is no retarding force such as air resistance, the projectile's speed when it returns to point *A* is the same as its initial speed and it repeats its motion indefinitely.

Trajectories 1 through 5 close on themselves and are called **closed orbits**. All closed orbits are ellipses or segments of ellipses; trajectory 4 is a circle, a special case of an ellipse. (We'll discuss the properties of an ellipse in Section 13.5). Trajectories 6 and 7 are **open orbits**. For these paths the projectile never returns to its starting point but travels ever farther away from the earth.

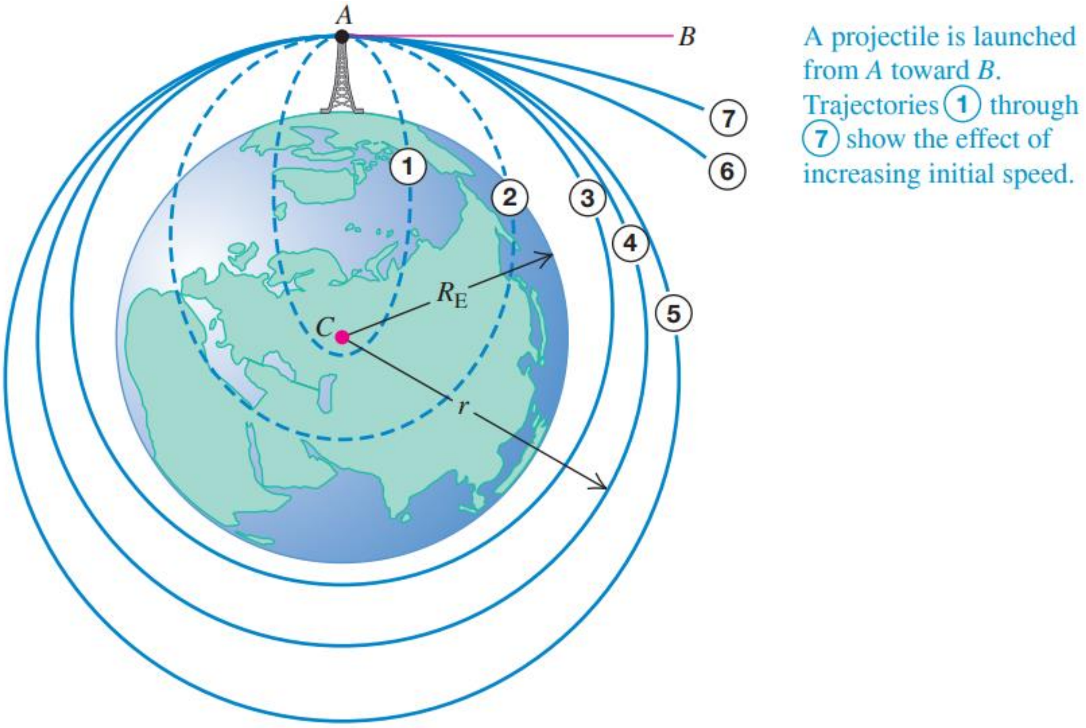


Figure 13.8 - Trajectories of a projectile launched from a great height (ignoring air resistance). Orbits 1 and 2 would be completed as shown if the earth were a point mass at *C*. (This illustration is based on one in Isaac Newton's *Principia*)

Satellites: Circular Orbits

A *circular* orbit, like trajectory 4 in Fig. 13.8, is the simplest case. It is also an important case, since many artificial satellites have nearly circular orbits and the orbits of the planets around the sun are also fairly circular. The only force acting on a satellite in circular orbit around the earth is the earth's gravitational attraction, which is directed toward the center of the earth and hence toward the center of the orbit (**Fig. 13.9**). As we discussed in Section 5.4, this means that the satellite is in *uniform* circular motion and its speed is constant. The satellite isn't falling *toward* the earth; rather, it's constantly falling *around* the earth. In a circular orbit the speed is just right to keep the distance from the satellite to the center of the earth constant.

Let's see how to find the constant speed υ of a satellite in a circular orbit. The radius of the orbit is *r*, measured from the *center* of the earth; the acceleration of the satellite has magnitude $a_{\mathrm{rad}} = \upsilon^2 / r$ and is always directed toward the center of the circle. By the law of gravitation, the net force (gravitational force) on the satellite of mass *m* has magnitude $F_{\mathrm{g}} = Gm_{\mathrm{E}} m / r^2$ and is in the same direction as the acceleration. Newton's second law $(\sum \vec{F} = m\vec{a})$ then tells us that

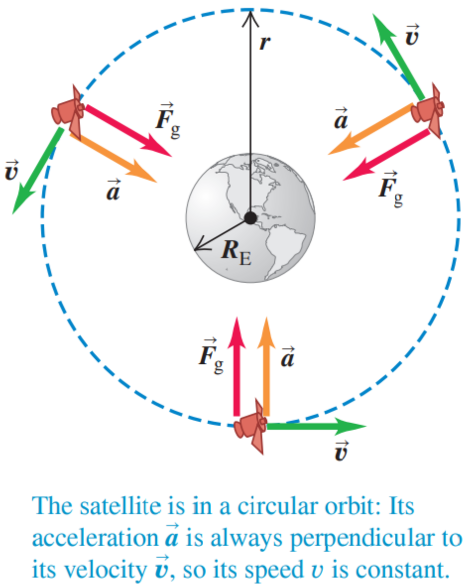


Figure 13.9 - The force $\vec{F}_{\mathrm{g}}$ due to the earth's gravitational attraction provides the centripetal acceleration that keeps a satellite in orbit. Compare to Fig. 5.12

$$\frac{Gm_{\mathrm{E}} m}{r^2} = \frac{m\upsilon^2}{r}.$$

Solving this for υ, we find

Speed of satellite in a circular orbit around the earth

Gravitational constant

Mass of the earth

Radius of orbit

$$\upsilon = \sqrt{\frac{Gm_{\mathrm{E}}}{r}} \tag{13.10}$$

This relationship shows that we can't choose the orbit radius r and the speed υ independently; for a given radius r, the speed υ for a circular orbit is determined.

The satellite's mass m doesn't appear in Eq. (13.10), which shows that the motion of a satellite does not depend on its mass. An astronaut on board an orbiting space station is herself a satellite of the earth, held by the earth's gravity in the same orbit as the station. The astronaut has the same velocity and acceleration as the station, so nothing is pushing her against the station's floor or walls. She is in a state of *apparent weightlessness*, as in a freely falling lift; see the discussion following Example 5.9 in Section 5.2. (*True* weightlessness would occur only if the astronaut were infinitely far from any other masses, so that the gravitational force on her would be zero). Indeed, every part of her body is apparently weightless; she feels nothing pushing her stomach against her intestines or her head against her shoulders.

Apparent weightlessness is not just a feature of circular orbits; it occurs whenever gravity is the only force acting on a spacecraft. Hence it occurs for orbits of any shape, including open orbits such as trajectories 6 and 7 in Fig. 13.8.

We can derive a relationship between the radius r of a circular orbit and the period T, the time for one revolution. The speed v is the distance $2\pi r$ traveled in one revolution, divided by the period:

$$\upsilon = \frac{2\pi r}{T}. \tag{13.11}$$

We solve Eq. (13.11) for T and substitute υ from Eq. (13.10):

Period of a circular orbit around the earth

Radius of orbit

$$T = \frac{2\pi r}{\upsilon} = 2\pi r\sqrt{\frac{r}{Gm_{\mathrm{E}}}} = \frac{2\pi r^{3/2}}{\sqrt{Gm_{\mathrm{E}}}} \tag{13.12}$$

Orbital speed

Gravitational constant

Mass of the earth

Equations (13.10) and (13.12) show that larger orbits correspond to slower speeds and longer periods. As an example, the International Space Station orbits 6800 km from the center of the earth (400 km above the earth's surface) with an orbital speed of 7.7 km/s and an orbital period of 93 min. The moon orbits the earth in a much larger orbit of radius 384,000 km, and so has a much slower orbital speed (1.0 km/s) and a much longer orbital period (27.3 days).

It's interesting to compare Eq. (13.10) to the calculation of escape speed in Example 13.5. We see that the escape speed from a spherical object with radius R is $\sqrt{2}$ times greater than the speed of a satellite in a circular orbit at that radius. If our spacecraft is in circular orbit around *any* planet, we have to multiply our speed by a factor of $\sqrt{2}$ to escape to infinity, regardless of the planet's mass.

Since the speed υ in a circular orbit is determined by Eq. (13.10) for a given orbit radius r, the total mechanical energy $E = K + U$ is determined as well. Using Eqs. (13.9) and (13.10), we have

$$\begin{aligned} E &= K + U = \tfrac{1}{2}m\upsilon^2 + \left(-\frac{Gm_{\mathrm{E}}m}{r}\right) \\ &= \tfrac{1}{2}m\left(\frac{Gm_{\mathrm{E}}}{r}\right) - \frac{Gm_{\mathrm{E}}m}{r} = -\frac{Gm_{\mathrm{E}}m}{2r} \end{aligned} \quad \text{(circular orbit).} \tag{13.13}$$

The total mechanical energy in a circular orbit is negative and equal to one-half the potential energy. Increasing the orbit radius r means increasing the total mechanical energy (that is, making E less negative). If the satellite is in a relatively low orbit that encounters the outer fringes of earth's atmosphere, the total mechanical energy decreases due to negative work done by the force of air resistance; as a result, the orbit radius decreases until the satellite hits the ground or burns up in the atmosphere. We have talked mostly about earth satellites, but we can apply the same analysis to the circular motion of *any* object under its gravitational attraction to a stationary object.

13.5 Kepler's Laws and the Motion of Planets

The name *planet* comes from a Greek word meaning "wanderer", and indeed the planets continuously change their positions in the sky relative to the background of stars. One of the great intellectual accomplishments of the 16th and 17th centuries was the threefold realization that the earth is also a planet, that all planets orbit the sun, and that the apparent motions of the planets as seen from the earth can be used to determine their orbits precisely.

The first and second of these ideas were published by Nicolaus Copernicus in Poland in 1543. The nature of planetary orbits was deduced between 1601 and 1619 by the German astronomer and mathematician Johannes Kepler, using precise data on apparent planetary motions compiled by his mentor, the Danish astronomer Tycho Brahe. By trial and error, Kepler discovered three empirical laws that accurately described the motions of the planets:

KEPLER'S LAWS

1. Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.

2. A line from the sun to a given planet sweeps out equal areas in equal times.

3. The periods of the planets are proportional to the $\frac{3}{2}$ powers of the major axis lengths of their orbits.

Kepler did not know *why* the planets moved in this way. Three generations later, when Newton turned his attention to the motion of the planets, he discovered that each of Kepler's laws can be *derived*; they are consequences of Newton's laws of motion and the law of gravitation. Let's see how each of Kepler's laws arises.

Kepler's First Law

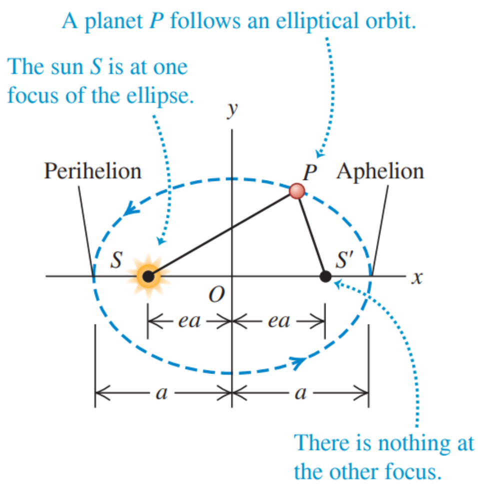


Figure 13.10 - Geometry of an ellipse. The sum of the distances *SP* and *S'P* is the same for every point on the curve. The sizes of the sun (*S*) and planet (*P*) are exaggerated for clarity

First consider the elliptical orbits described in Kepler's first law. **Figure 13.10** shows the geometry of an ellipse. The longest dimension is the *major axis*, with half-length a; this half-length is called the **semi-major axis**. The sum of the distances from S to P and from S' to P is the same for all points on the curve. S and S' are the *foci* (plural of *focus*). The sun is at S (*not* at the center of the ellipse) and the planet is at P; we think of both as points because the size of each is very small in comparison to the distance between them. There is nothing at the other focus, S'.

The distance of each focus from the center of the ellipse is ea, where e is a dimensionless number between 0 and 1 called the **eccentricity**. If $e = 0$, the two foci coincide and the ellipse is a circle. The actual orbits of the planets are fairly circular; their eccentricities range from 0.007 for Venus to 0.206 for Mercury. (The earth's orbit has $e = 0.017$). The point in the planet's orbit closest to the sun is the *perihelion*, and the point most distant is the *aphelion*.

Newton showed that for an object acted on by an attractive force proportional to $1/r^2$, the only possible closed orbits are a circle or an ellipse; he also showed that open orbits (trajectories 6 and 7 in Fig. 13.8) must be parabolas or hyperbolas. These results can be derived from Newton's laws and the law of gravitation, together with a lot more differential equations than we're ready for.

Kepler's Second Law

Figure 13.11 shows Kepler's second law. In a small time interval *dt*, the line from the sun *S* to the planet *P* turns through an angle $d\theta$. The area swept out is the colored triangle with height *r*, base length $r\,d\theta$, and area $dA = \frac{1}{2}r^2 d\theta$ in Fig. 13.19b. The rate at which area is swept out, *dA*/*dt*, is called the *sector velocity*:

$$\frac{dA}{dt} = \tfrac{1}{2} r^2 \frac{d\theta}{dt}. \tag{13.14}$$

The essence of Kepler's second law is that the sector velocity has the same value at all points in the orbit. When the planet is close to the sun, *r* is small and $d\theta / dt$ is large; when the planet is far from the sun, r is large and du > dt is small.

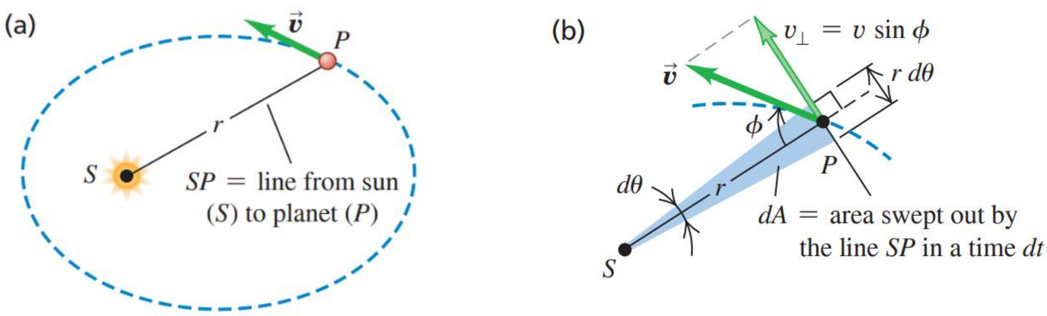
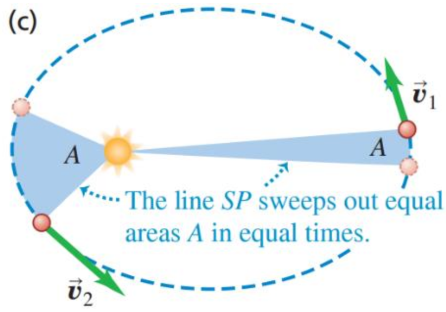


Figure 13.11 - (a) The planet (*P*) moves about the sun (*S*) in an elliptical orbit.
(b) In a time *dt* the line *SP* sweeps out an area $dA = \frac{1}{2}(r d\theta) r = \frac{1}{2} r^2 d\theta$.
(c) The planet's speed varies so that the line *SP* sweeps out the same area *A* in a given time *t* regardless of the planet's position in its orbit



To see how Kepler's second law follows from Newton's laws, we express *dA*/*dt* in terms of the velocity vector v S of the planet *P*. The component of $\vec{v}$ perpendicular to the radial line is $v_\perp = v \sin\phi$. From Fig. 13.11b the displacement along the direction of $v_\perp$ during time *dt* is $r\,d\theta$, so we also have $v_\perp = r\,d\theta / dt$. Using this relationship in Eq. (13.14), we find

$$\frac{dA}{dt} = \tfrac{1}{2} r v \sin\phi \ \text{(sector velocity)}. \tag{13.15}$$

Now $r v \sin\phi$ is the magnitude of the vector product $\vec{r} \times \vec{v}$, which in turn is 1/*m* times the angular momentum $\vec{L} = \vec{r} \times m\vec{v}$ of the planet with respect to the sun. So we have

$$\frac{dA}{dt} = \frac{1}{2m}\left|\vec{r} \times m\vec{v}\right| = \frac{L}{2m}. \tag{13.16}$$

Thus Kepler's second law—that sector velocity is constant—means that angular momentum is constant!

It is easy to see why the angular momentum of the planet *must* be constant.

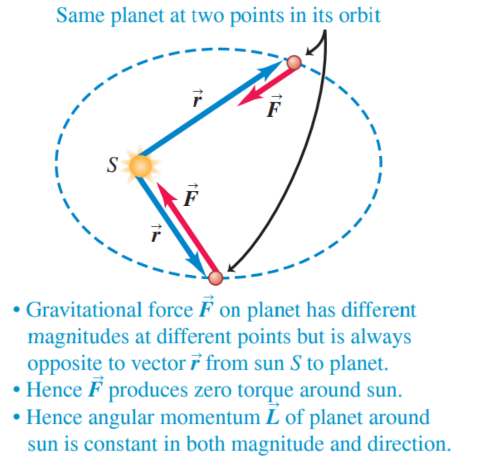


Figure 13.12 - Because the gravitational force that the sun exerts on a planet produces zero torque around the sun, the planet's angular momentum around the sun remains constant

According to Eq. (10.26), the rate of change of $\vec{L}$ equals the torque of the gravitational force $\vec{F}$ acting on the planet:

$$\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{r} \times \vec{F}.$$

In our situation, $\vec{r}$ is the vector from the sun to the planet, and the force $\vec{F}$ is directed from the planet to the sun (**Fig. 13.12**). So these vectors always lie along the same line, and their vector product $\vec{r} \times \vec{F}$ is zero. Hence $d\vec{L}/dt = 0$. This conclusion does not depend on the $1/r^2$ behavior of the force; angular momentum is conserved for *any* force that acts always along the line joining the particle to a fixed point. Such a force is called a *central force*. (Kepler's first and third laws are valid for a $1/r^2$ force *only*).

Conservation of angular momentum also explains why the orbit lies in a plane. The vector $\vec{L} = \vec{r} \times m\vec{\upsilon}$ is always perpendicular to the plane of the vectors $\vec{r}$ and $\vec{\upsilon}$; since $\vec{L}$ is constant in magnitude *and* direction, $\vec{r}$ and $\vec{\upsilon}$ always lie in the same plane, which is just the plane of the planet's orbit.

Kepler's Third Law

We have already derived Kepler's third law for the particular case of circular orbits. Equation (13.12) shows that the period of a satellite or planet in a circular orbit is proportional to the $\frac{3}{2}$ power of the orbit radius. Newton was able to show that this same relationship holds for an *elliptical* orbit, with the orbit radius *r* replaced by the semi-major axis *a*:

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_S}} \text{ (elliptical orbit around the sun).} \tag{13.17}$$

Since the planet orbits the sun, not the earth, we have replaced the earth's mass m_E in Eq. (13.12) with the sun's mass m_S. Note that the period does not depend on the eccentricity *e*. An asteroid in an elongated elliptical orbit with semi-major axis *a* will have the same orbital period as a planet in a circular orbit of radius *a*. The key difference is that the asteroid moves at different speeds at different points in its elliptical orbit (Fig. 13.11c), while the planet's speed is constant around its circular orbit.

EXAMPLE 13.2 Orbital speeds

At what point in an elliptical orbit (see Fig. 13.11) does a planet move the fastest? The slowest?

SOLUTION

Total mechanical energy is conserved as a planet moves in its orbit. The planet's kinetic energy $K = \frac{1}{2}m\upsilon^2$ is maximum when the potential energy $U = -Gm_S m/r$ >r is minimum (that is, most negative; see Fig. 13.7), which occurs when the sun–planet distance *r* is a minimum. Hence the speed υ is greatest at perihelion. Similarly, *K* is minimum when *r* is maximum, so the speed is slowest at aphelion.

Your intuition about falling objects is helpful here. As the planet falls inward toward the sun, it picks up speed, and its speed is maximum when closest to the sun. The planet slows down as it moves away from the sun, and its speed is minimum at aphelion.

KEYCONCEPT

Total mechanical energy remains constant for a planet in an elliptical orbit. The planet's kinetic energy (and speed) therefore increases as it moves closer to the sun and the gravitational potential energy decreases (becomes more negative).

Planetary Motions and the Center of Mass

We have assumed that as a planet or comet orbits the sun, the sun remains absolutely stationary. This can't be correct; because the sun exerts a gravitational force on the planet, the planet exerts a gravitational force on the sun of the same magnitude but opposite direction. In fact, *both* the sun and the planet orbit around their common center of mass (**Fig. 13.13**). We've made only a small error by ignoring this effect, however; the sun's mass is about 750 times the total mass of all the planets combined, so the center of mass of the solar system is not far from the center of the sun. Remarkably, astronomers have used this effect to detect the presence of planets orbiting other stars. Sensitive telescopes are able to detect the apparent "wobble" of a star as it orbits the common center of mass of the star and an unseen companion planet. (The planets are too faint to observe directly). By analyzing these "wobbles," astronomers have discovered planets in orbit around hundreds of other stars.

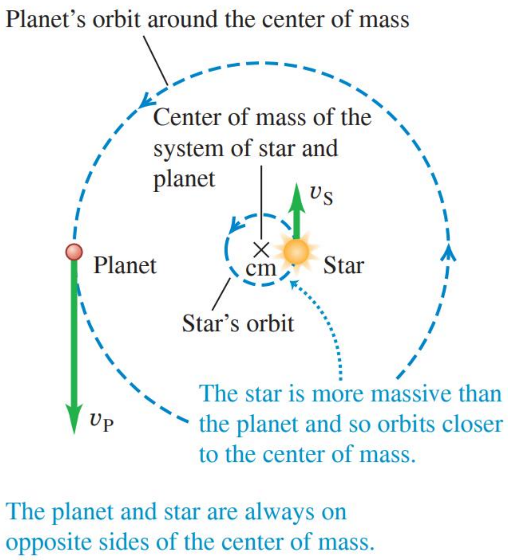


Figure 13.13 - Both a star and its planet orbit about their common center of mass

The most remarkable result of Newton's analysis of planetary motion is that objects in the heavens obey the *same* laws of motion as do objects on the earth. This *Newtonian synthesis*, as it has come to be called, is one of the great unifying principles of science. It has had profound effects on the way that humanity looks at the universe - not as a realm of impenetrable mystery, but as a direct extension of our everyday world, subject to scientific study and calculation.

13.6 Spherical Mass Distributions

We have stated without proof that the gravitational interaction between two spherically symmetric mass distributions is the same as though all the mass of each were concentrated at its center. Now we're ready to prove this statement. Newton searched for a proof for several years, and he delayed publication of the law of gravitation until he found one.

Rather than starting with two spherically symmetric masses, we'll tackle the simpler problem of a point mass *m* interacting with a thin spherical shell with total mass *M*. We'll show that when *m* is outside the sphere, the *potential energy* associated with this gravitational interaction is the same as though *M* were concentrated in a point at the center of the sphere. We learned in Section 7.4 that the force is the negative derivative of the potential energy, so the *force* on *m* is also the same as for a point mass *M*. Our result will also hold for *any* spherically symmetric mass distribution, which we can think of as being made of many concentric spherical shells.

A Point Mass Outside a Spherical Shell

We start by considering a ring on the surface of a shell (Fig. 13.14a, next page), centered on the line from the center of the shell to *m*. We do this because all of the particles that make up the ring are the same distance *s* from the point mass *m*. From Eq. (13.9) the potential energy of interaction between the earth (mass m_E) and a point mass *m*, separated by a distance *r*, is $U = -GmEm > r$. From this expression, we see that the potential energy of interaction between the point mass *m* and a particle of mass m_i within the ring is

$$U_i = -\frac{Gmm_i}{s}.$$

To find the potential energy *dU* of interaction between *m* and the entire ring of mass $dU = \sum_i m_i$ we sum this expression for U_i over all particles in the ring:

$$dU = \sum_i U_i = \sum_i \left(-\frac{Gmm_i}{s} \right) = -\frac{Gm}{s} \sum_i m_i = -\frac{GmdM}{s}. \tag{13.18}$$

To proceed, we need to know the mass *dM* of the ring. We can find this with the aid of a little geometry. The radius of the shell is *R*, so in terms of the angle ϕ f shown in the figure, the radius of the ring is $R\sin\phi$, and its circumference is $2\pi R\sin\phi$. The width of the ring is $Rd\phi$, and its area *dA* is approximately equal to its width times its circumference:

$$dA = 2\pi R^2 \sin\phi \, d\phi.$$

The ratio of the ring mass *dM* to the total mass *M* of the shell is equal to the ratio of the area *dA* of the ring to the total area $A = 2\pi R^2$ of the shell:

$$\frac{dM}{M} = \frac{2\pi R^2 \sin\phi \, d\phi}{4\pi R^2} = \tfrac{1}{2}\sin\phi \, d\phi. \tag{13.19}$$

Now we solve Eq. (13.19) for *dM* and substitute the result into Eq. (13.18) to find the potential energy of interaction between point mass m and the ring:

$$dU = -\frac{GMm\sin\phi \, d\phi}{2s}. \tag{13.20}$$

The total potential energy of interaction between the point mass and the *shell* is the integral of Eq. (13.20) over the whole sphere as ϕ varies from 0 to π (*not* 2π!) and *s* varies from *r* – *R*to*r* + *R*. To carry out the integration, we have to express the integrand in terms of a single variable; we choose *s*. To express ϕ and $d\phi$ in terms of *s*, we have to do a little more geometry. Figure 13.14b shows that *s* is the hypotenuse of a right triangle with sides ($r - R\cos\phi$) and $R\sin\phi$, so the Pythagorean theorem gives

$$s^2 = (r - R\cos\phi)^2 + (R\sin\phi)^2 = r^2 - 2rR\cos\phi + R^2. \tag{13.21}$$

We take differentials of both sides:

$$2s \; ds = 2rR\cos\phi \; d\phi.$$

Next we divide this by 2*rR* and substitute the result into Eq. (13.20):

$$dU = -\frac{GMm}{2s}\frac{s \; ds}{rR} = -\frac{GMm}{2rR}ds. \tag{13.22}$$

We can now integrate Eq. (13.22), recalling that *s* varies from *r* - *R* to *r* + *R*:

$$U = -\frac{GMm}{2rR}\int_{r-R}^{r+R} ds = -\frac{GMm}{2rR}\left[(r+R)-(r-R)\right]. \tag{13.23}$$

Finally, we have

$$U = -\frac{GMm}{r} \quad \text{(point mass } m \text{ outside spherical shell } M\text{).} \tag{13.24}$$

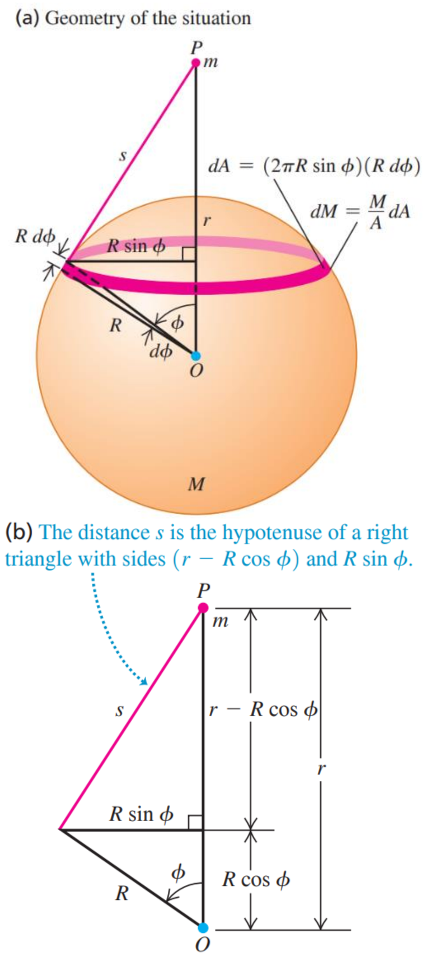


Figure 13.14 Calculating the gravitational potential energy of interaction between a point mass m outside a spherical shell and a ring on the surface of the shell of mass M

This is equal to the potential energy of two point masses m and M at a distance r. So we have proved that the gravitational potential energy of spherical shell M and point mass m at any distance r is the same as though they were point masses. Because the force is given by $F_r = -dU/dr$, the force is also the same.

The Gravitational Force Between Spherical Mass Distrutions

Any spherically symmetric mass distribution can be thought of as a combination of concentric spherical shells. Because of the principle of superposition of forces, what is true of one shell is also true of the combination. So we have proved half of what we set out to prove: that the gravitational interaction between any spherically symmetric mass distribution and a point mass is the same as though all the mass of the spherically symmetric distribution were concentrated at its center.

The other half is to prove that *two* spherically symmetric mass distributions interact as though both were points. That's easier. In Fig. 13.14a the forces the two objects exert on each other are an action–reaction pair, and they obey Newton's third law. So we have also proved that the force that m exerts on sphere M is the same as though M were a point. But now if we replace m with a spherically symmetric mass distribution centered at m's location, the resulting gravitational force on any part of M is the same as before, and so is the total force. This completes our proof.

A Point Mass Inside a Spherical Shell

We assumed at the beginning that the point mass m was outside the spherical shell, so our proof is valid only when m is outside a spherically symmetric mass distribution. When m is *inside* a spherical shell, the geometry is as shown in **Fig. 13.15**. The entire analysis goes just as before; Eqs. (13.18) through (13.22) are still valid. But when we get to Eq. (13.23), the limits of integration have to be changed to R - r and R + r. We then have

$$U = -\frac{GMm}{2rR}\int_{R-r}^{R+r} ds = -\frac{GMm}{2rR}\left[(R+r)-(R-r)\right]. \tag{13.25}$$

and the final result is

$$U = -\frac{GMm}{r} \text{ (point mass } m \text{ outside spherical shell } M). \tag{13.26}$$

Compare this result to Eq. (13.24): Instead of having *r*, the distance between *m* and the center of *M*, in the denominator, we have *R*, the radius of the shell. This means that *U* in Eq. (13.26) doesn't depend on *r* and thus has the same value everywhere inside the shell. When *m* moves around inside the shell, no work is done on it, so the force on *m* at any point inside the shell must be zero.

More generally, at any point in the interior of any spherically symmetric mass distribution (not necessarily a shell), at a distance *r* from its center, the gravitational force on a point mass *m* is
the same as though we removed all the mass at points farther than *r* from the center and concentrated all the remaining mass at the center.

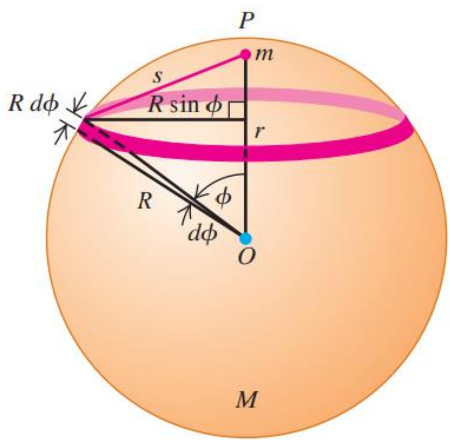


Figure 13.15 - When a point mass *m* is *inside* a uniform spherical shell of mass *M*, the potential energy is the same no matter where inside the shell the point mass is located. The force from the masses' mutual gravitational interaction is zero

13.7Apparent weight and the Earth's Rotation

Because the earth rotates on its axis, it is not precisely an inertial frame of reference. For this reason the apparent weight of an object on earth is not precisely equal to the earth's gravitational attraction, which we'll call the **true weight** $\vec{w}_0$ of the object. **Figure 13.16** is a cutaway view of the earth, showing three observers. Each one holds a spring scale with an object of mass *m* hanging from it. Each scale applies a tension force $\vec{F}$ to the object hanging from it, and the reading on each scale is the magnitude *F* of this force. If the observers are unaware of the earth's rotation, each one *thinks* that the scale reading equals the weight of the object because he thinks the object on his spring scale is in equilibrium. So each observer thinks that the tension $\vec{F}$ must be opposed by an equal and opposite force $\vec{w}$, which we call the **apparent weight**. But if the objects are rotating with the earth, they are *not* precisely in equilibrium. Our problem is to find the relationship between the apparent weight $\vec{w}$ and the true weight $\vec{w}_0$.

If we assume that the earth is spherically symmetric, then the true weight $\vec{w}_0$ has magnitude $Gm_E m / R_E^2$, where m_E and R_E are the mass and radius of the earth. This value is the same for all points on the earth's surface. If the center of the earth can be taken as the origin of an inertial coordinate system, then the object at the north pole really is in equilibrium in an inertial system, and the reading on that observer's spring scale is equal to w_0. But the object at the equator is moving in a circle of radius R_E with speed υ, and there must be a net inward force equal to the mass times the centripetal acceleration:

$$w_0 - F = \frac{m\upsilon^2}{R_E}.$$

So the magnitude of the apparent weight (equal to the magnitude of *F*) is

$$w = w_0 - \frac{m\upsilon^2}{R_E} \text{ (at the equator).} \tag{13.27}$$

If the earth were not rotating, the object when released would have a free-fall acceleration $g_0 = w_0 / m$. Since the earth *is* rotating, the falling object's actual acceleration relative to the observer at the equator is $g = w / m$. Dividing Eq. (13.27) by *m* and using these relationships, we find

$$g = g_0 - \frac{\upsilon^2}{R_E} \text{ (at the equator).}$$

To evaluate υ^2 / R_E , we note that in 86,164 s a point on the equator moves a distance equal to the earth's circumference, $2\pi R_E = 2\pi(6.37\times10^6\,\mathrm{m})$. (The solar day, 86,400 s, is $\frac{1}{365}$ longer than this because in one day the earth also completes $\frac{1}{365}$ of its orbit around the sun). Thus we find

$$\upsilon = \frac{2\pi(6.37\times10^6)}{86{,}164\ \mathrm{s}} = 465\ \mathrm{m/s},$$

$$\frac{\upsilon^2}{R_E} = \frac{(465\ \mathrm{m/s})^2}{6.37\times10^6\,\mathrm{m}} = 0.0339\ \mathrm{m/s^2}.$$

So for a spherically symmetric earth the acceleration due to gravity should be about $0.03\ \mathrm{m/s^2}$ less at the equator than at the poles.

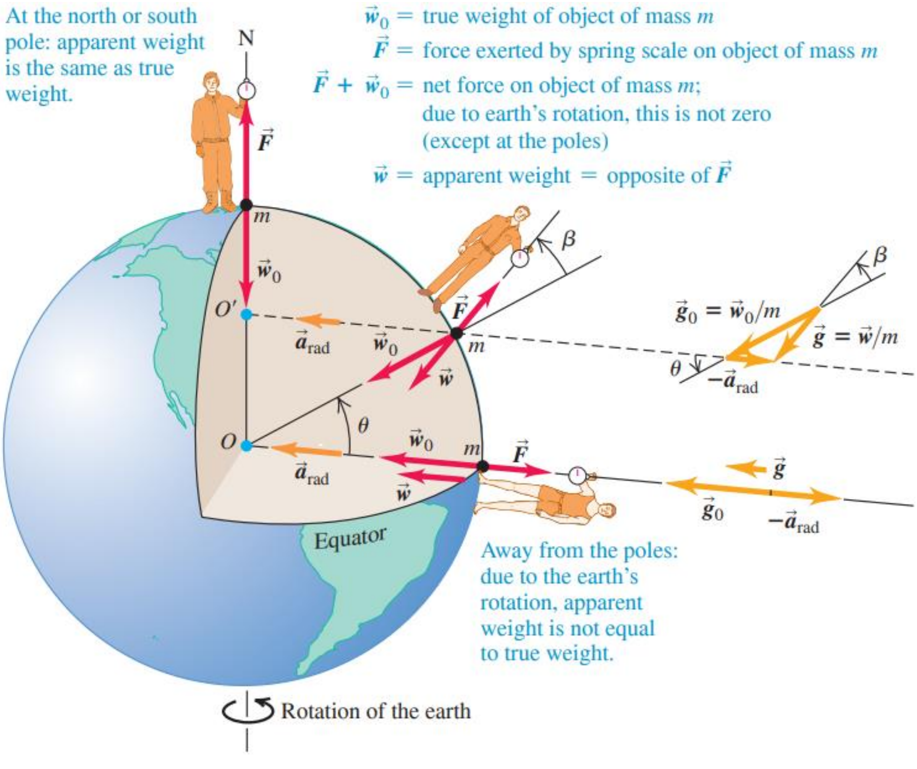


Figure 13.16 - Except at the poles, the reading for an object being weighed on a scale (the *apparent weight*) is less than the gravitational force of attraction on the object (the *true weight*). The reason is that a net force is needed to provide a centripetal acceleration as the object rotates with the earth. For clarity, the illustration greatly exaggerates the angle β between the true and apparent weight vectors

At locations intermediate between the equator and the poles, the true weight $\vec{w}_0$ and the centripetal acceleration are not along the same line, and we need to write a vector equation corresponding to Eq. (13.27). From Fig. 13.16 we see that the appropriate equation is

$$\vec{w}_0 = \vec{w}_0 - m\vec{a}_{\mathrm{rad}} = m\vec{g}_0 - m\vec{a}_{\mathrm{rad}}. \tag{13.28}$$

The difference in the magnitudes of g and g_0 lies between zero and 0.0339 m/s^2. As Fig. 13.16 shows, the *direction* of the apparent weight differs from the direction toward the center of the earth by a small angle β, which is 0.1° or less.

Table 13.1 gives the values of g at several locations. In addition to moderate variations with latitude, there are small variations due to elevation, differences in local density, and the earth's deviation from perfect spherical symmetry.

Table13.1 - Variations of g with Latitude and Elevation

Station	**North Latitude**	**Elevation (m)**	**g(m/s^2)**
Canal Zone	09°	0	9.78243
Jamaica	18°	0	9.78591
Bermuda	32°	0	9.79806
Denver, CO	40°	1638	9.79609
Pittsburgh, PA	40.5°	235	9.80118
Cambridge, MA	42°	0	9.80398
Greenland	70°	0	9.82534

13.8 Black Holes

In 1916 Albert Einstein presented his general theory of relativity, which included a new concept of the nature of gravitation. In his theory, a massive object actually changes the geometry of the space around it. Other objects sense this altered geometry and respond by being attracted to the first object. The general theory of relativity is beyond our scope in this chapter, but we can look at one of its most startling predictions: the existence of **black holes**, objects whose gravitational influence is so great that nothing - not even light - can escape them. We can understand the basic idea of a black hole by using Newtonian principles.

The Escape Speed from a Star

Think first about the properties of our own sun. Its mass M = 1.99 × 10^{30} kg and radius R = 6.96 × 10^8 m are much larger than those of any planet, but compared to other stars, our sun is not exceptionally massive. You can find the sun's average density ρ in the same way we found the average density of the earth in Section 13.2:

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{1.99\times10^{30}\,\text{kg}}{\frac{4}{3}\pi(6.96\times10^{8}\,\text{m})^3} = 1410\,\text{kg/m}^3 .$$

The sun's temperatures range from 5800 K (about 5500°C or 10,000°F) at the surface up to 1.5 × 10^7 K (about 2.7 × 10^7 °F) in the interior, so it surely contains no solids or liquids. Yet gravitational attraction pulls the sun's gas atoms together until the sun is, on average, 41% denser than water and about 1200 times as dense as the air we breathe.

Now think about the escape speed for an object at the surface of the sun. In Example 13.5 (Section 13.3) we found that the escape speed from the surface of a spherical mass M with radius R is $\upsilon = \sqrt{2GM/R}$. Substituting $M = \rho V = \rho(\frac{4}{3}\pi R^3)$ into the expression for escape speed gives

$$\upsilon = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{8\pi G\rho}{3}}R . \tag{13.29}$$

Using either form of this equation, you can show that the escape speed for an object at the surface of our sun is $\upsilon = 6.18\times10^5\,\text{m/s}$ (about 2.2 million km/h). This value, roughly $\frac{1}{500}$ the speed of light in vacuum, is independent of the mass of the escaping object; it depends on only the mass and radius (or average density and radius) of the sun.

Now consider various stars with the same average density ρ and different radii *R*. Equation (13.29) shows that for a given value of density ρ, the escape speed υ is directly proportional to *R*. In 1783 the Rev. John Michell noted that if an object with the same average density as the sun had about 500 times the radius of the sun, its escape speed would be greater than the speed of light in vacuum, *c*. With his statement that “all light emitted from such a body would be made to return towards it,” Michell became the first person to suggest the existence of what we now call a black hole.

Black Holes, the Schwarzschild Radius, and the Event Horizon

The first expression for escape speed in Eq. (13.29) suggests that an object of mass *M* will act as a black hole if its radius *R* is less than or equal to a certain critical radius. How can we determine this critical radius? You might think that you can find the answer by simply setting υ = *c* in Eq. (13.29). As a matter of fact, this does give the correct result, but only because of two compensating errors. The kinetic energy of light is *not*mc^2/2, and the gravitational potential energy near a black hole is *not* given by Eq. (13.9). In 1916, Karl Schwarzschild used Einstein’s general theory of relativity to derive an expression for the critical radius R_S, now called the **Schwarzschild radius**. The result turns out to be the same as though we had set υ = *c* in Eq. (13.29), so

$$c = \sqrt{\frac{2GM}{R_S}}\,.$$

Solving for the Schwarzschild radius R_S, we find

Gravitational constant

Schwarzschild radius of a black hole

Mass of black hole

Speed of light in vacuum

$$R_S = \frac{2GM}{c^2}\,. \tag{13.30}$$

If a spherical, nonrotating object with mass *M* has a radius less than R_S, then *nothing* (not even light) can escape from the surface of the object, and the object is a black hole (**Fig. 13.17**). In this case, any other object within a distance R_S of the center of the black hole is trapped by the gravitational attraction of the black hole and cannot escape from it.

The surface of the sphere with radius R_S surrounding a black hole is called the **event horizon**: Since light can’t escape from within that sphere, we can’t see events occurring inside. All that an observer outside the event horizon can know about a black hole is its mass (from its gravitational effects on other objects), its electric charge (from the electric forces it exerts on other charged objects), and its angular momentum (because a rotating black hole tends to drag space—and everything in that space—around with it). All other information about the object is irretrievably lost when it collapses inside its event horizon.

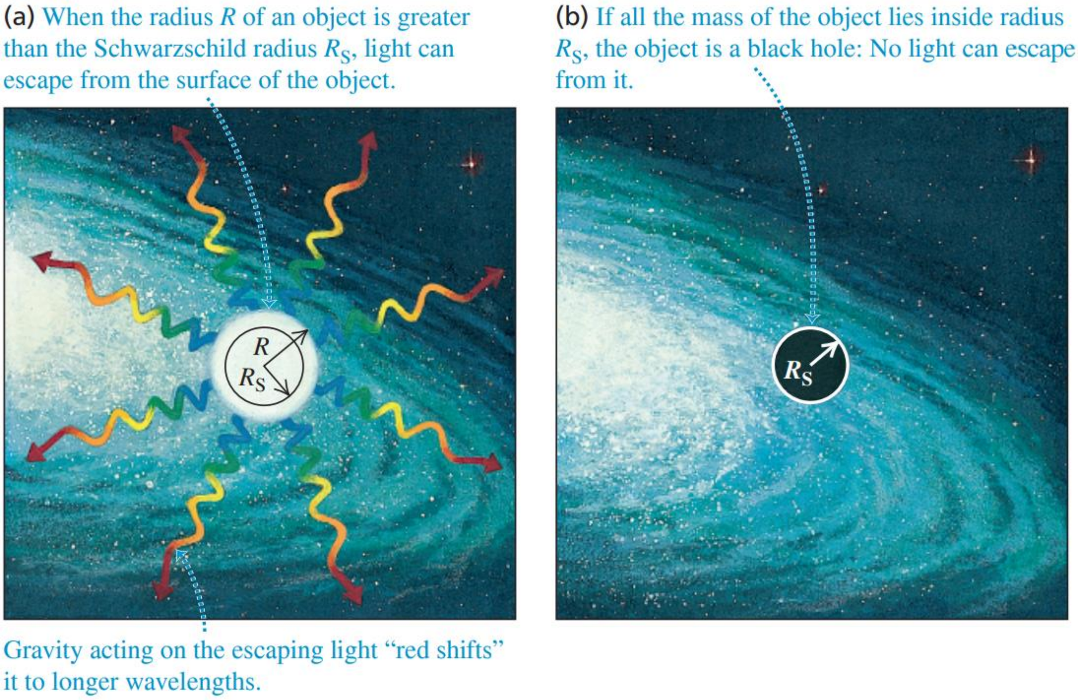


Figure 13.17 - (a) An object with a radius R greater than the Schwarzschild radius R_S. (b) If the object collapses to a radius smaller than R_S, it is a black hole with an escape speed greater than the speed of light. The surface of the sphere of radius R_S is called the event horizon of the black hole

A Visit to a Black Hole

At points far from a black hole, its gravitational effects are the same as those of any normal object with the same mass. If the sun collapsed to form a black hole, the orbits of the planets would be unaffected. But things get dramatically different close to the black hole. If you decided to become a martyr for science and jump into a black hole, the friends you left behind would notice several odd effects as you moved toward the event horizon, most of them associated with effects of general relativity.

If you carried a radio transmitter to send back your comments on what was happening, your friends would have to retune their receiver continuously to lower and lower frequencies, an effect called the *gravitational red shift*. Consistent with this shift, they would observe that your clocks (electronic or biological) would appear to run more and more slowly, an effect called *time dilation*. In fact, during their lifetimes they would never see you make it to the event horizon.

In your frame of reference, you would make it to the event horizon in a rather short time but in a rather disquieting way. As you fell feet first into the black hole, the gravitational pull on your feet would be greater than that on your head, which would be slightly farther away from the black hole. The *differences* in gravitational force on different parts of your body would be great enough to stretch you along the direction toward the black hole and compress you perpendicular to it. These effects (called *tidal forces*) would rip you to atoms, and then rip your atoms apart, before you reached the event horizon.

Detecting Black Holes

If light cannot escape from a black hole and if black holes are as small as Example 13.11 suggests, how can we know that such things exist? The answer is that any gas or dust near the black hole tends to be pulled into an *accretion disk* that swirls around and into the black hole, rather like a whirlpoo. Friction within the accretion disk's gas causes it to lose mechanical energy and spiral into the black hole; as it moves inward, it is compressed together. This causes heating of the gas, just as air compressed in a bicycle pump gets hotter. Temperatures in excess of 106 K can occur in the accretion disk, so hot that the disk emits not just visible light (as do objects that are "red-hot" or "white-hot") but x rays. Astronomers look for these x rays (emitted by the gas material *before* it crosses the event horizon) to signal the presence of a black hole. Several promising candidates have been found, and astronomers now express considerable confidence in the existence of black holes.

Black holes in binary star systems like the one depicted in Fig. 13.18 have masses a few times greater than the sun's mass. There is also mounting evidence for the existence of much larger *supermassive black holes*. One example lies at the center of our Milky Way galaxy, some 26,000 light-years from earth in the direction of the constellation Sagittarius. High-resolution images of the galactic center reveal stars moving at speeds greater than 1500 km/s about an unseen object that lies at the position of a source of radio waves called Sgr A*. By analyzing these motions, astronomers can infer the period T and semi-major axis a of each star's orbit. The mass m_X of the unseen object can be calculated from Kepler's third law in the form given in Eq. (13.17), with the mass of the sun m_S replaced by m_X :

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_X}} \qquad \text{so} \qquad m_X = \frac{4\pi^2 a^3}{GT^2}.$$

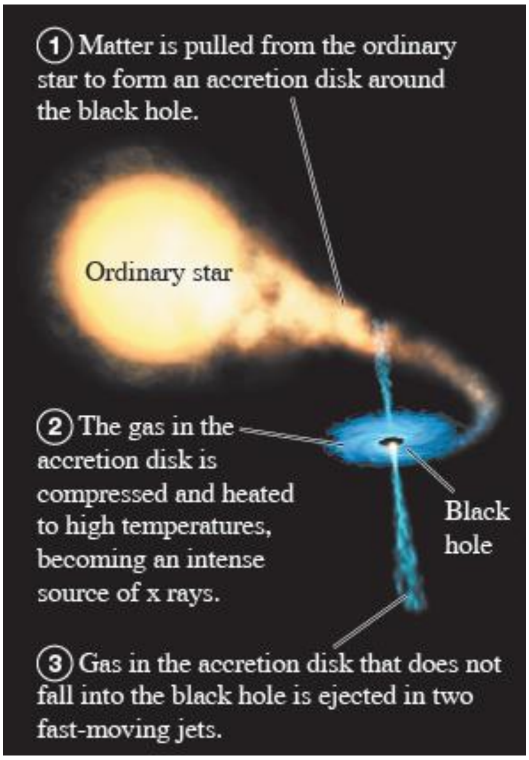


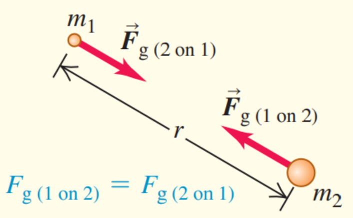
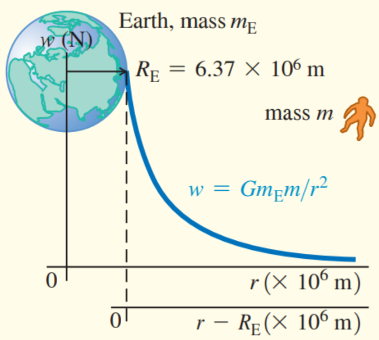
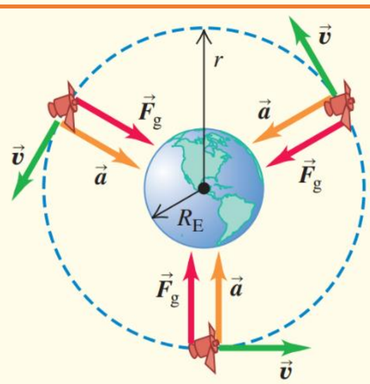
Figure 13.18 - In a *binary* star system, two stars orbit each other; in the special case shown here, one of the stars is a black hole. The black hole itself cannot be seen, but the x rays from its accretion disk can be detected

The conclusion is that the mysterious dark object at the galactic center has a mass of $8.2\times10^{36}\,\text{kg}$, or 4.1 *million* times the mass of the sun. Yet observations with radio telescopes show that it has a radius no more than 4.4×10^{10} m, about one-third of the distance from the earth to the sun. These observations suggest that this massive, compact object is a black hole with a Schwarzschild radius of 1.1×10^{10} m. Astronomers hope to improve the resolution of their observations so that they can actually see the event horizon of this black hole.

Additional evidence for the existence of black holes has come from observations of *gravitational radiation*. Einstein's general theory of relativity, which we'll discuss in Chapter 37, predicts that space itself is curved by the presence of massive objects like a planet, star, or black hole. If a massive object accelerates in a certain manner, it produces ripples in the curvature of space that radiate outward from the object. The disturbances caused by such gravitational radiation are incredibly feeble, but can be measured with sensitive detectors if the objects that produce them are very massive and have tremendous accelerations. This can happen when two massive black holes are in close orbits around each other. Each black hole has an acceleration as it moves around its curved orbit, so it emits gravitational radiation that carries away energy. This makes the orbits of the black holes smaller, so they move faster and emit gravitational radiation at an ever faster rate. The orbits keep shrinking until the two black holes finally merge. Since 2015 scientists have detected the gravitational radiation from several such black hole mergers. From the data, they conclude that the merging black holes have masses from 7 to 36 times the mass of the sun. (Rainer Weiss, Kip Thorne, and Barry Barish were awarded the 2017 Nobel Prize in Physics for their contributions to these discoveries).

Other lines of research suggest that even larger black holes, in excess of 10^9 times the mass of the sun, lie at the centers of other galaxies. Observational and theoretical studies of black holes of all sizes continue to be an exciting area of research in both physics and astronomy.

CHAPTER 13: SUMMARY

Newton's law of gravitation: *Any* two particles with masses m_1 and m_2, a distance r apart, attract each other with forces inversely proportional to r^2. These forces form an action– reaction pair and obey Newton's third law. When two or more objects exert gravitational forces on a particular object, the total gravitational force on that individual object is the vector sum of the forces exerted by the other objects. The gravitational interaction between spherical mass distributions, such as planets or stars, is the same as if all the mass of each distribution were concentrated at the center.	$F_g = \dfrac{Gm_1m_2}{r^2}$	
Gravitational force, weight, and gravitational potential energy: The weight w of an object is the total gravitational force exerted on it by all other objects in the universe. Near the surface of the earth (mass m_E and radius R_E), the weight is essentially equal to the gravitational force of the earth alone. The gravitational potential energy U of two masses m and m_E separated by a distance r is inversely proportional to r. The potential energy is never positive; it is zero only when the two objects are infinitely far apart.	$w = F_g = \dfrac{Gm_Em}{R_E^2}$ (weight at earth's surface) $g = \dfrac{Gm_E}{R_E^2}$ (acceleration due to gravity at earth's surface) $U = -\dfrac{Gm_Em}{r}$	
Orbits: When a satellite moves in a circular orbit, the centripetal acceleration is provided by the gravitational attraction of the earth. Kepler's three laws describe the more general case: an elliptical orbit of a planet around the sun or a satellite around a planet.	$\upsilon = \sqrt{\dfrac{Gm_E}{r}}$ (speed in circular orbit) $T = \dfrac{2\pi r}{\upsilon} = 2\pi r\sqrt{\dfrac{r}{Gm_E}} = \dfrac{2\pi r^{3/2}}{\sqrt{Gm_E}}$ (period in circular orbit)	

Black holes: If a nonrotating spherical mass distribution with total mass M has a radius less than its Schwarzschild radius R_S, it is called a black hole. The gravitational interaction prevents anything, including light, from escaping from within a sphere with radius R_S.	$R_S = \dfrac{2GM}{c^2}$ (Schwarzschild radius)	 If all of the object is inside its Schwarzschild radius $R_S = 2GM/c^2$, the object is a black hole.

14 PERIODIC MOTION

Many kinds of motion repeat themselves over and over: the vibration of a quartz crystal in a watch, the swinging pendulum of a grandfather clock, the sound vibrations produced by a clarinet or an organ pipe, and the back-and-forth motion of the pistons in a car engine. This kind of motion, called **periodic motion** or **oscillation**, is the subject of this chapter. Understanding periodic motion will be essential for our later study of waves, sound, alternating electric currents, and light.

An object that undergoes periodic motion always has a stable equilibrium position. When it is moved away from this position and released, a force or torque comes into play to pull it back toward equilibrium. But by the time it gets there, it has picked up some kinetic energy, so it overshoots, stopping somewhere on the other side, and is again pulled back toward equilibrium. Picture a ball rolling back and forth in a round bowl or a pendulum that swings back and forth past its straight-down position.

In this chapter we'll concentrate on two simple examples of systems that can undergo periodic motions: spring-mass systems and pendulums. We'll also study why oscillations often tend to die out with time and why some oscillations can build up to greater and greater displacements from equilibrium when periodically varying forces act.

14.1 Describing Oscillation

Figure 14.1 shows one of the simplest systems that can have periodic motion. An object with mass *m* rests on a frictionless horizontal guide system, such as a linear air track, so it can move along the *x*-axis only. The object is attached to a spring of negligible mass that can be either stretched or compressed. The left end of the spring is held fixed, and the right end is attached to the object. The spring force is the only horizontal force acting on the object; the vertical normal and gravitational forces always add to zero.

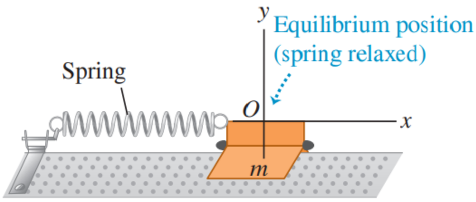


Figure 14.1 - A system that can have periodic motion

It's simplest to define our coordinate system so that the origin *O* is at the equilibrium position, where the spring is neither stretched nor compressed. Then *x* is the *x*-component of the displacement of the object from equilibrium and is also the change in the length of the spring. The spring exerts a force on the object with *x*-component F_x, and the *x*-component of acceleration is $a_x = F_x / m$.

Figure 14.2 shows the object for three different displacements of the spring. Whenever the object is displaced from its equilibrium position, the spring force tends to restore it to the equilibrium position. We call a force with this character a **restoring force**. Oscillation can occur only when there is a restoring force tending to return the system to equilibrium.

Let's analyze how oscillation occurs in this system. If we displace the object to the right to $x = A$ and then let go, the net force and the acceleration are to the left (Fig. 14.2a). The speed increases as the object approaches the equilibrium position *O*. When the object is at *O*, the net force acting on it is zero (Fig. 14.2b), but because of its motion it *overshoots* the equilibrium position. On the other side of the equilibrium position the object is still moving to the left, but the net force and the acceleration are to the right (Fig. 14.2c); hence the speed decreases until the object comes to a stop. We'll show later that with an ideal spring, the stopping point is at *x* = -*A*. The object then accelerates to the right, overshoots equilibrium again, and stops at the starting point *x* = *A*, ready to repeat the whole process. The object is oscillating! If there is no friction or other force to remove mechanical energy from the system, this motion repeats forever; the restoring force perpetually draws the object back toward the equilibrium position, only to have the object overshoot time after time.

In different situations the force may depend on the displacement *x* from equilibrium in different *ways*. But oscillation always occurs if the force is a *restoring* force that tends to return the system to equilibrium.

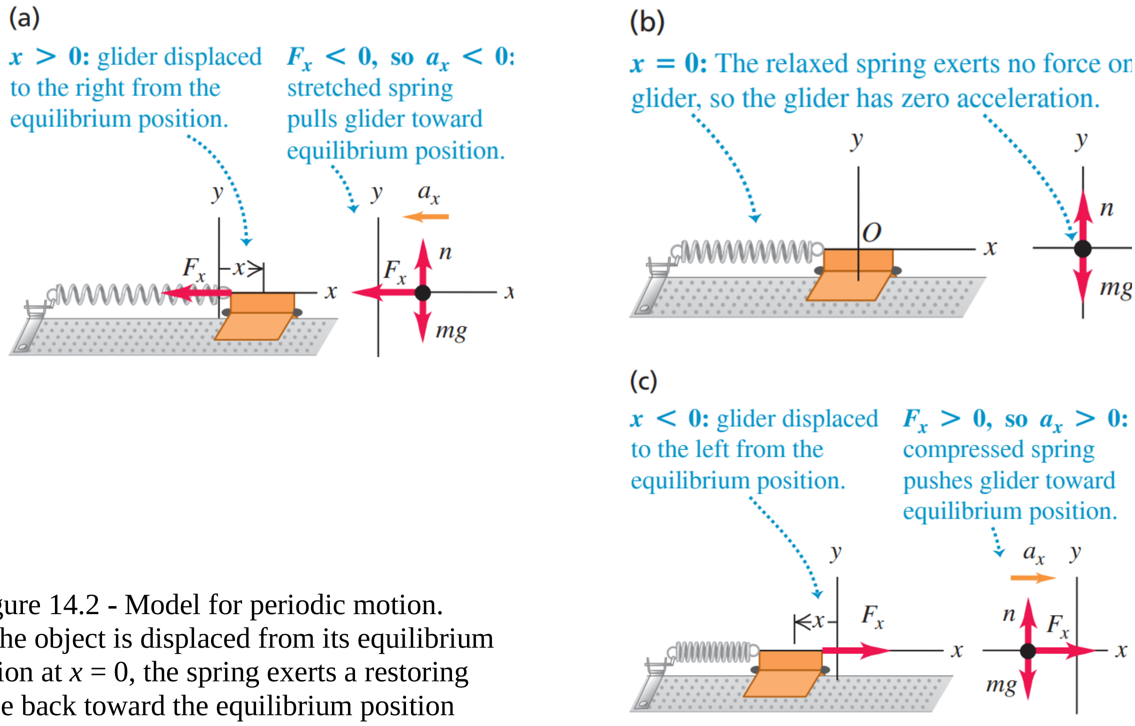


Figure 14.2 - Model for periodic motion. When the object is displaced from its equilibrium position at $x = 0$, the spring exerts a restoring force back toward the equilibrium position

Amplitude, Period, Frequency, and Angular Frequency

Here are some terms that we'll use in discussing periodic motions of all kinds:

The **amplitude** of the motion, denoted by *A*, is the maximum magnitude of displacement from equilibrium - that is, the maximum value of $|x|$. It is always positive. If the spring in Fig. 14.2 is an ideal one, the total overall range of the motion is 2*A*. The SI unit of *A* is the meter. A complete vibration, or **cycle**, is one complete round trip—say, from *A* to -*A* and back to *A*, or from *O* to *A*, back through *O* to -*A*, and back to *O*. Note that motion from one side to the other (say, -*A* to *A*) is a half-cycle, not a whole cycle.

The **period**, *T*, is the time to complete one cycle. It is always positive. The SI unit is the second, but it is sometimes expressed as "seconds per cycle."

The **frequency**, *f*, is the number of cycles in a unit of time. It is always positive. The SI unit of frequency is the *hertz*, named for the 19th-century German physicist Heinrich Hertz:

$$1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ cycle/s} = 1 \text{ s}^{-1}.$$

The **angular frequency**, ω, is 2π times the frequency:

$$\omega = 2\pi f\ .$$

We'll learn shortly why ω is a useful quantity. It represents the rate of change of an angular quantity (not necessarily related to a rotational motion) that is always measured in radians, so its units are rad/s. Since *f* is in cycle/s, we may regard the number 2π as having units rad/cycle.

By definition, period and frequency are reciprocals of each other:

In periodic motion **frequency and period** are reciprocals of each other.

$$\underbrace{f}_{\text{Frequency}} = \frac{1}{\underbrace{T}_{\text{Period}}} \qquad T = \frac{1}{f} \tag{14.1}$$

Also, from the definition of ω

$$\underbrace{\omega}_{\text{Angular frequency related to frequency and period}} = 2\pi \overbrace{f}^{\text{Frequency}} = \frac{2\pi}{\underbrace{T}_{\text{Period}}} \qquad (14.2)$$

CAUTION! **One period spans a complete cycle.** Keep in mind that the period of an oscillation is the time for a complete cycle—for example, the time to travel from $x = -A$ to $x = +A$ and back again to $x = -A$.

EXAMPLE 14.1 Period, frequency, and angular freguency

An ultrasonic transducer used for medical diagnosis oscillates at 6.7 MHz = 6.7×10^6 Hz. How long does each oscillation take, and what is the angular frequency?

IDENTIFY and SET UP

The target variables are the period T and the angular frequency ω. We can find these from the given frequency f in Eqs. (14.1) and (14.2).

EXECUTE

From Eqs. (14.1) and (14.2),

$$T = \frac{1}{f} = \frac{1}{6.7\times10^{6}\,\text{Hz}} = 1.5\times10^{-7}\,\text{s} = 0.15\ \mu\text{s},$$

$$\omega = 2\pi f = 2\pi(6.7\times10^{6}\,\text{Hz}) = (2\pi\ \text{rad/cycle})(6.7\times10^{6}\,\text{cycle/s}) = 4.2\times10^{7}\,\text{rad/s}.$$

EVALUATE

This is a very rapid vibration, with large f and ω and small T. A slow vibration has small f and ω and large T.

KEYCONCEPT

The period of an oscillation is the reciprocal of the oscillation frequency. The angular frequency equals the frequency multiplied by 2π.

14.2 Simple Harmonic Motion

The simplest kind of oscillation occurs when the restoring force F_x is *directly proportional* to the displacement from equilibrium x. This happens if the spring in Figs. 14.1 and 14.2 is an ideal one that obeys *Hooke's law* (see Section 6.3). The constant of proportionality between F_x and x is the force constant k. On either side of the equilibrium position, F_x and x always have opposite signs. In Section 6.3 we represented the force acting *on* a stretched ideal spring as $F_x = kx$. The x-component of force the spring exerts *on the object* is the negative of this, so

$$\underbrace{\overbrace{F_x}^{x\text{-component of force}}}_{\text{Restoring force exerted by an ideal spring}} = -\underbrace{k}_{\text{Force constant of spring}}\overbrace{x}^{\text{Displacement}}. \qquad (14.3)$$

This equation gives the correct magnitude and sign of the force, whether x is positive, negative, or zero (**Fig. 14.3**). The force constant k is always positive and has units of N/m (a useful alternative set of units is kg/s^2). We are assuming that there is no friction, so Eq. (14.3) gives the *net* force on the object.

When the restoring force is directly proportional to the displacement from equilibrium, as given by Eq. (14.3), the oscillation is called **simple harmonic motion (SHM)**. The acceleration $a_x = d^2x/dt = F_x/m$ of an object in SHM is

Equation for simple harmonic motion

$$a_x = \frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad (14.4)$$

x-component of acceleration; Second derivative of displacement; Force constant of restoring force; Displacement; Mass of object

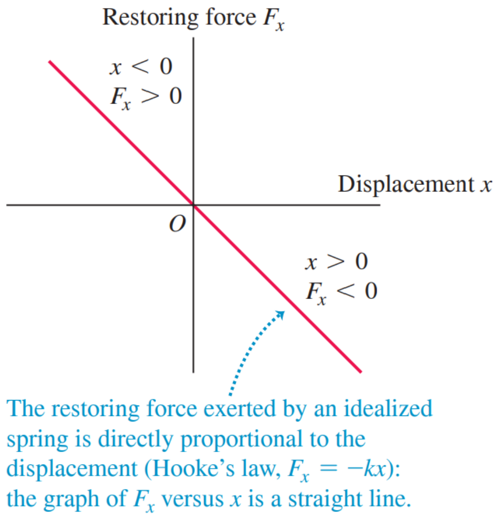


Figure 14.3 - An idealized spring exerts a restoring force that obeys Hooke's law, $F_x = -kx$. Oscillation with such a restoring force is called simple harmonic motion

The minus sign means that, in SHM, the acceleration and displacement always have opposite signs. This acceleration is *not* constant, so don't even think of using the constantacceleration equations from Chapter 2. We'll see shortly how to solve this equation to find the displacement x as a function of time. An object that undergoes simple harmonic motion is called a **harmonic oscillator**.

Why is simple harmonic motion important? Not all periodic motions are simple harmonic; in periodic motion in general, the restoring force depends on displacement in a more complicated way than in Eq. (14.3). But in many systems the restoring force is *approximately* proportional to displacement if the displacement is sufficiently small (**Fig. 14.4**). That is, if the amplitude is small enough, the oscillations of such systems are approximately simple harmonic and therefore approximately described by Eq. (14.4). Thus we can use SHM as an approximate model for many different periodic motions, such as the vibration of a tuning fork, the electric current in an alternating-current circuit, and the oscillations of atoms in molecules and solids.

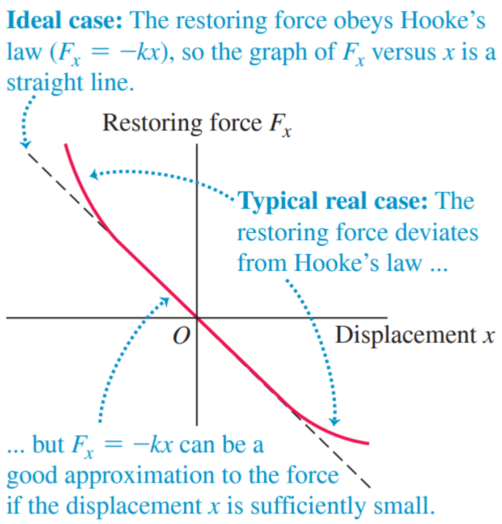


Figure 14.4 - In most real oscillations Hooke's law applies provided the object doesn't move too far from equilibrium. In such a case small-amplitude oscillations are approximately simple harmonic

Circular Motion and the Equations of SHM

To explore the properties of simple harmonic motion, we must express the displacement x of the oscillating object as a function of time, $x(t)$. The second derivative of this function, d^2x/dt^2, must be equal to $(-k/m)$ times the function itself, as required by Eq. (14.4). As we mentioned, the formulas for constant acceleration from Section 2.4 are no help because the acceleration changes constantly as the displacement x changes. Instead, we'll find $x(t)$ by noting that SHM is related to *uniform circular motion*, which we studied in Section 3.4.

Figure 14.5a shows a top view of a horizontal disk of radius A with a ball attached to its rim at point Q. The disk rotates with constant angular speed ω (measured in rad/s), so the ball moves in uniform circular motion. A horizontal light beam casts a shadow of the ball on a screen. The shadow at point P oscillates back and forth as the ball moves in a circle. We then arrange an object attached to an ideal spring, like the combination shown in Figs. 14.1 and 14.2, so that the object oscillates parallel to the shadow. We'll prove that the motions of the object and of the ball's shadow are *identical* if the amplitude of the object's oscillation is equal to the disk radius *A*, and if the angular frequency $2\pi f$ of the oscillating object is equal to the angular speed ω of the rotating disk. That is, *simple harmonic motion is the projection of uniform circular motion onto a diameter*.

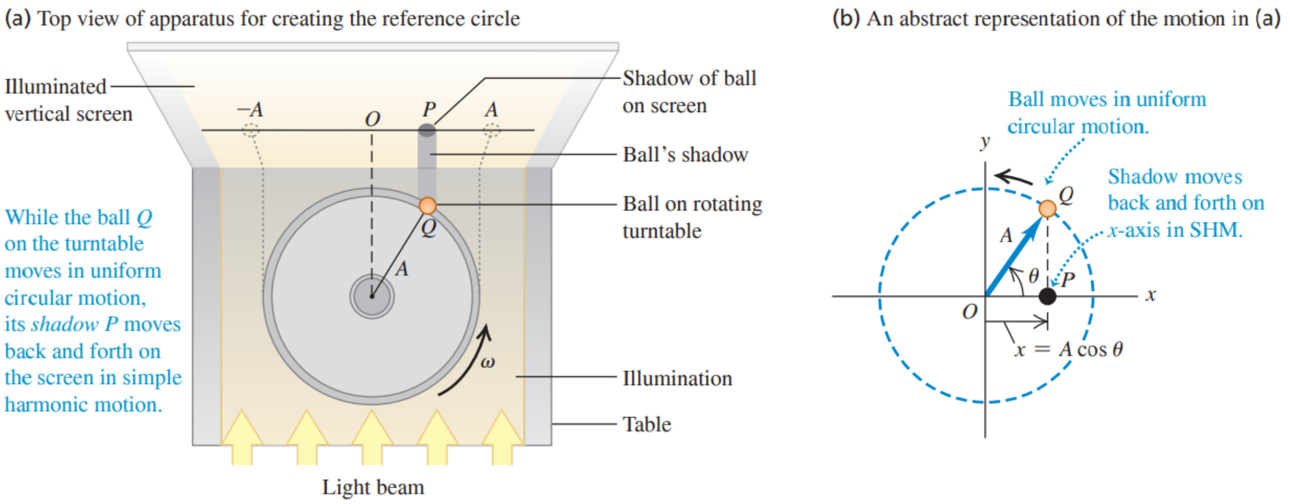


Figure 14.5 - (a) Relating uniform circular motion and simple harmonic motion. (b) The ball's shadow moves exactly like an object oscillating on an ideal spring

We can verify this remarkable statement by finding the acceleration of the shadow at *P* and comparing it to the acceleration of an object undergoing SHM, given by Eq. (14.4). The circle in which the ball moves so that its projection matches the motion of the oscillating object is called the **reference circle**; we'll call the point *Q* the *reference point*. We take the reference circle to lie in the *xy*-plane, with the origin *O* at the center of the circle (Fig.14.5b). At time t the vector *OQ* from the origin to reference point *Q* makes an angle θ with the positive *x*-axis. As point *Q* moves around the reference circle with constant angular speed ω, vector *OQ* rotates with the same angular speed. Such a rotating vector is called a **phasor**. (This term was in use long before the invention of the *Star Trek* stun gun with a similar name).

The *x*-component of the phasor at time *t* is just the *x*-coordinate of the point *Q*:

$$x = A\cos\theta\,. \tag{14.5}$$

This is also the *x*-coordinate of the shadow *P*, which is the *projection* of *Q* onto the *x*-axis. Hence the *x*-velocity of the shadow *P* along the *x*-axis is equal to the *x*-component of the velocity vector of point *Q* (Fig. 14.6a), and the *x*-acceleration of *P* is equal to the *x*-component of the acceleration vector of *Q* (Fig. 14.6b). Since point *Q* is in uniform circular motion, its acceleration vector $\vec{a}_Q$ is always directed toward *O*. Furthermore, the magnitude of $\vec{a}_Q$ is constant and given by the angular speed squared times the radius of the circle (see Section 9.3):

$$a_Q = \omega^2 A\,. \tag{14.6}$$

Figure 14.6b shows that the *x*-component of $\vec{a}_Q$ is $a_x = -a_Q\cos\theta$. Combining this with Eqs. (14.5) and (14.6), we get that the acceleration of point *P* is

$$a_x = -a_Q\cos\theta = -\omega^2 A\cos\theta \quad \text{or} \tag{14.7}$$

$$a_x = -\omega^2 x\,. \tag{14.8}$$

The acceleration of point *P* is directly proportional to the displacement *x* and always has the opposite sign. These are precisely the hallmarks of simple harmonic motion.

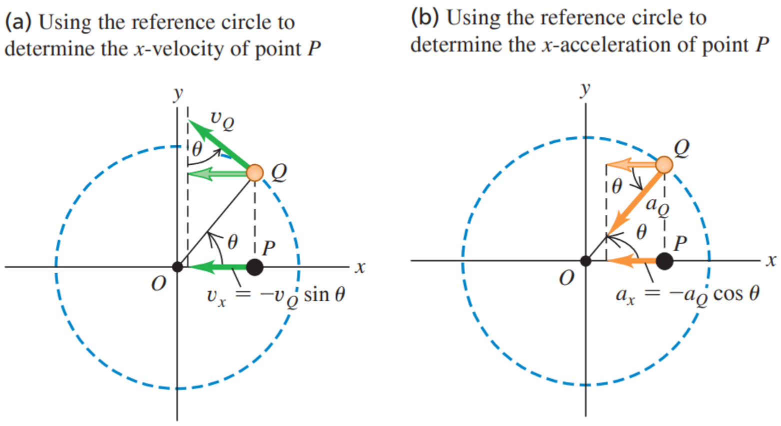


Figure 14.6 - The (a) x-velocity and (b) x-acceleration of the ball's shadow P (see Fig. 14.5) are the x-components of the velocity and acceleration vectors, respectively, of the ball Q

Equation (14.8) is *exactly* the same as Eq. (14.4) for the acceleration of a harmonic oscillator, provided that the angular speed ω of the reference point Q is related to the force constant k and mass m of the oscillating object by

$$\omega^2 = \frac{k}{m} \quad \text{or} \quad \omega = \sqrt{\frac{k}{m}} \qquad (14.9)$$

We have been using the same symbol ω for the angular *speed* of the reference point Q and the angular *frequency* of the oscillating point P. The reason is that these quantities are equal! If point Q makes one complete revolution in time T, then point P goes through one complete cycle of oscillation in the same time; hence T is the period of the oscillation. During time T the point Q moves through 2π radians, so its angular speed is $\omega = 2\pi / T$. But this is the same as Eq. (14.2) for the angular frequency of the point P, which verifies our statement about the two interpretations of ω. This is why we introduced angular frequency in Section 14.1; this quantity makes the connection between oscillation and circular motion. So we reinterpret Eq. (14.9) as an expression for the angular frequency of simple harmonic motion:

Angular frequency for simple harmonic motion

$$\omega = \sqrt{\frac{k}{m}} \qquad (14.10)$$

k: Force constant of restoring force
m: Mass of object

When you start an object oscillating in SHM, the value of ω is not yours to choose; it is predetermined by the values of k and m. The units of k are N/m or kg/s^2 , so k/m is in (kg/s^2)/kg = s^{-2} . When we take the square root in Eq. (14.10), we get s^{-1}, or more properly rad/s because this is an *angular* frequency (recall that a radian is not a true unit). According to Eqs. (14.1) and (14.2), the frequency f and period T are

Frequency for simple harmonic motion

$$f = \underbrace{\frac{\omega}{2\pi}}_{\omega:\ \text{Angular frequency}} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad (14.11)$$

k: Force constant of restoring force
m: Mass of object

Period for simple harmonic motion

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad (14.12)$$

f: Frequency
ω: Angular frequency
m: Mass of object
k: Force constant of restoring force

CAUTION! **Don't confuse frequency and angular frequency.** You can run into trouble if you don't make the distinction between frequency f and angular frequency $\omega = 2\pi f$. Frequency tells you how many cycles of oscillation occur per second, while angular frequency tells you how many radians per second this corresponds to on the reference circle. In solving problems, pay careful attention to whether the goal is to find f or ω .

We see from Eq. (14.12) that a larger mass m will have less acceleration and take a longer time for a complete cycle (Fig. 14.7). A stiffer spring (one with a larger force constant k) exerts a greater force at a given deformation x, causing greater acceleration and a shorter time T per cycle.

Period and Amplitude in SHM

Equations (14.11) and (14.12) show that the period and frequency of simple harmonic motion are completely determined by the mass m and the force constant k. *In simple harmonic motion the period and frequency do not depend on the amplitude A*. For given values of m and k, the time of one complete oscillation is the same whether the amplitude is large or small. Equation (14.3) shows why we should expect this. Larger A means that the object reaches larger values of 0 x 0 and is subjected to larger restoring forces. This increases the average speed of the object over a complete cycle; this exactly compensates for having to travel a larger distance, so the same total time is involved.

The oscillations of a tuning fork are essentially simple harmonic motion, so it always vibrates with the same frequency, independent of amplitude. This is why a tuning fork can be used as a standard for musical pitch. If it were not for this characteristic of simple harmonic motion, it would be impossible to play most musical instruments in tune. If you encounter an oscillating object with a period that *does* depend on the amplitude, the oscillation is *not* simple harmonic motion.

Displacement, Velocity, and Acceleration in SHM

We still need to find the displacement x as a function of time for a harmonic oscillator. Equation (14.4) for an object in SHM along the x-axis is identical to Eq. (14.7) for the x-coordinate of the reference point in uniform circular motion with constant angular speed $\omega = \sqrt{k/m}$. Hence Eq. (14.5), $x = A\cos\theta$, describes the x-coordinate for both situations. If at t = 0 the phasor OQ makes an angle ϕ (the Greek letter phi) with the positive x-axis, then at any later time t this angle is $\theta = \omega t + \phi$. We substitute this into Eq. (14.5) to obtain

Displacement in simple harmonic motion as a function of time

$$x = \overset{\text{Amplitude}}{A} \cos(\underset{\text{Angular frequency} = \sqrt{k/m}}{\omega}\overset{\text{Time}}{t} + \overset{\text{Phase angle}}{\phi}) \qquad (14.13)$$

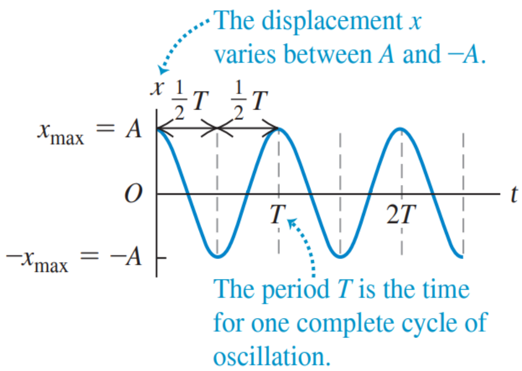


Figure 14.7 Graph of x versus t [see Eq. (14.13)] for simple harmonic motion. The case shown has ϕ = 0

Figure 14.7 shows a graph of Eq. (14.13) for the particular case ϕ = 0. We could also have written Eq. (14.13) in terms of a sine function rather than a cosine by using the identity $\cos\alpha = \sin(\alpha + \pi/2)$. *In simple harmonic motion the displacement is a periodic, sinusoidal function of time*. There are many other periodic functions, but none so simple as a sine or cosine function.

The value of the cosine function is always between -1 and 1, so in Eq. (14.13), x is always between -A and A. This confirms that A is the amplitude of the motion.

The cosine function in Eq. (14.13) repeats itself whenever time t increases by one period T, or when $\omega t + \phi$ increases by 2π radians. Thus, if we start at time t = 0, the time T to complete one cycle is

$$\omega T = \sqrt{\frac{k}{m}}T = 2\pi \quad \text{or} \quad T = 2\pi\sqrt{\frac{m}{k}},$$

which is just Eq. (14.12). Changing either m or k changes the period T (Figs. 14.8a and 14.8b), but T does not depend on the amplitude A (Fig. 14.8c).

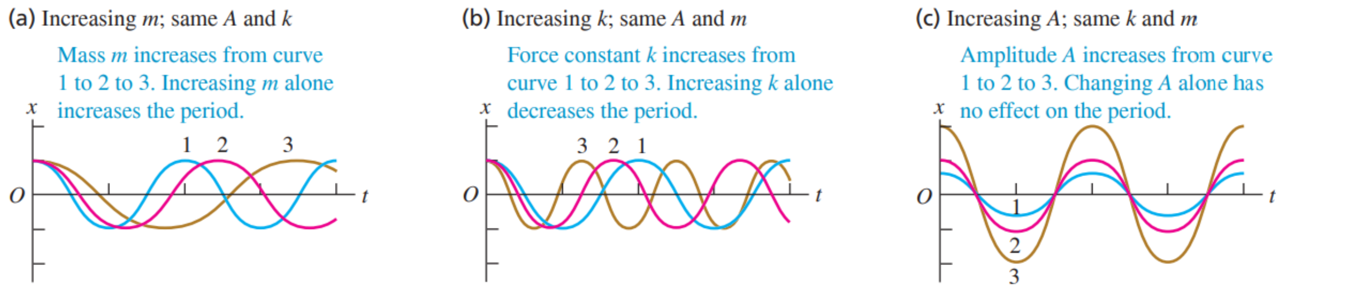


Figure 14.8 - Variations of simple harmonic motion. All cases shown have ϕ = 0 [see Eq. (14.13)]

The constant ϕ in Eq. (14.13) is called the **phase angle**. It tells us at what point in the cycle the motion was at t = 0 (equivalent to where around the circle the point Q was at t = 0). We denote the displacement at t = 0 by x_0. Putting t = 0 and x = x_0 in Eq. (14.13), we get

$$x_0 = A\cos\phi . \qquad (14.14)$$

If ϕ = 0, then $x_0 = A\cos 0 = A$, and the object starts at its maximum positive displacement. If $\phi = \pi$, then $x_0 = A\cos\pi = -A$, and the particle starts at its maximum *negative* displacement. If f$\phi = \pi/2$, then $x_0 = A\cos(\pi/2) = 0$, and the particle is initially at the origin. **Figure 14.9** shows the displacement x versus time for three different phase angles.

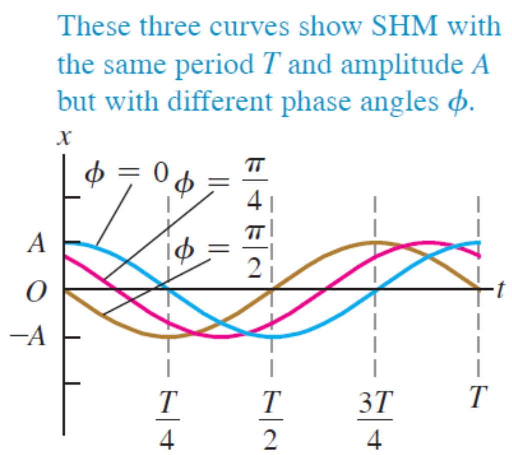


Figure 14.9 - Variations of simple harmonic motion: same m, k, and A but different phase angles ϕ

We find the velocity υ_x and acceleration a_x as functions of time for a harmonic oscillator by taking derivatives of Eq. (14.13) with respect to time:

$$\upsilon_x = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi) \quad \text{(velocity in SHM).} \qquad (14.15)$$

$$a_x = \frac{d\upsilon_x}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi) \quad \text{(acceleration in SHM).} \qquad (14.16)$$

The velocity υ_x oscillates between

$$\upsilon_{\max} = +\omega A \text{ and } -\upsilon_{\max} = -\omega A,$$

and the acceleration a_x oscillates between

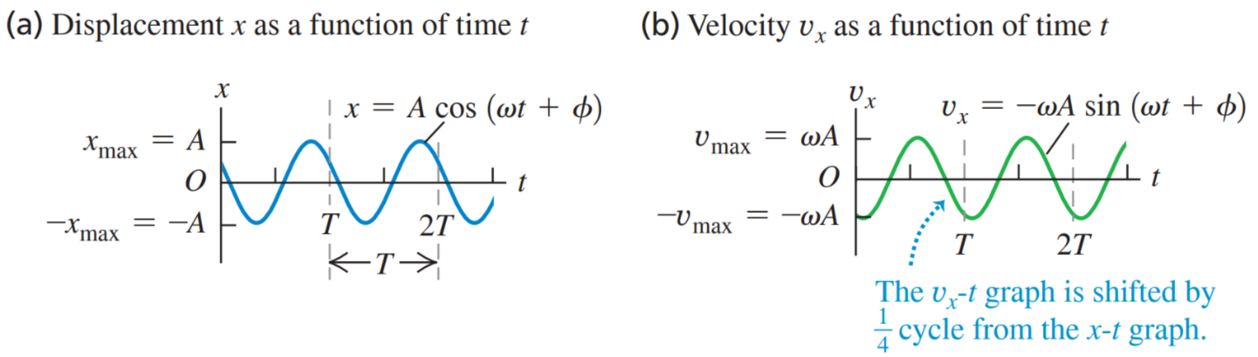
$$a_{\max} = +\omega^2 A \text{ and } -a_{\max} = -\omega^2 A$$

(Fig. 14.10). Comparing Eq. (14.16) with Eq. (14.13) and recalling that $\omega^2 = k/m$ from Eq. (14.9), we see that

$$a_{\max} = -\omega^2 x = -\frac{k}{m}x,$$

which is just Eq. (14.4) for simple harmonic motion. This confirms that Eq. (14.13) for x as a function of time is correct.

We actually derived Eq. (14.16) earlier in a geometrical way by taking the *x*- component of the acceleration vector of the reference point *Q*. This was done in Fig. 14.6b and Eq. (14.7) (recall that $\theta = \omega t + \phi$). In the same way, we could have derived Eq. (14.15) by taking the *x*-component of the velocity vector of *Q*, as shown in Fig. 14.6b. We'll leave the details for you to work out.



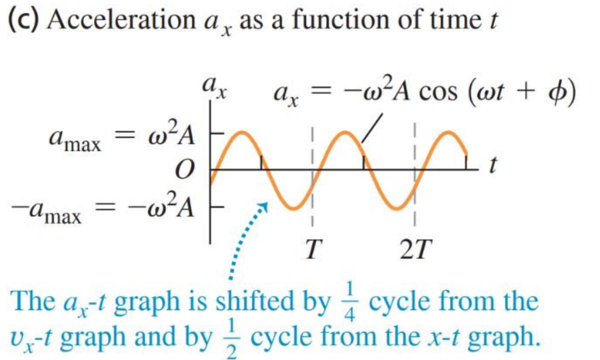


Figure 14.10 - Graphs of (a) *x* versus *t*, (b) v_x versus *t*, and (c) a_x versus *t* for an object in SHM. For the motion depicted in these graphs, $\phi = \pi / 3$

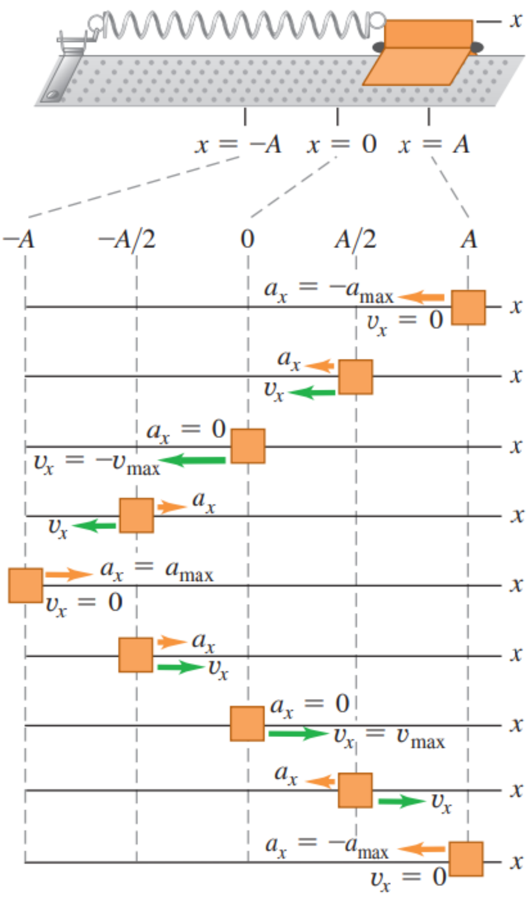


Figure 14.11 - How *x*-velocity v_x and *x*-acceleration a_x vary during one cycle of SHM

Note that the sinusoidal graph of displacement versus time (Fig. 14.10a) is shifted by one-quarter period from the graph of velocity versus time (Fig. 14.10b) and by one-half period from the graph of acceleration versus time (Fig. 14.10 c). Figure 14.11 shows why this is so. When the object is passing through the equilibrium position so that *x* = 0, the velocity equals either v_{max} or $-v_{max}$ (depending on which way the object is moving) and the acceleration is zero. When the object is at either its most positive displacement, *x* = +*A*, or its most negative displacement, $x = -A$, the velocity is zero and the object is instantaneously at rest. At these points, the restoring force $F_x = -kx$ and the acceleration of the object have their maximum magnitudes. At *x* = +*A* the acceleration is negative and equal to $-a_{max}$. At *x* = –*A* the acceleration is positive: $a_x = +a_{max}$.

Here's how we can determine the amplitude *A* and phase angle ϕ for an oscillating object if we are given its initial displacement x_0 and initial velocity v_{0x} . The initial velocity v_{0x} is the velocity at time *t* = 0; putting $v_x = v_{0x}$ and *t* = 0 in Eq. (14.15), we find

$$v_{0x} = -\omega A \sin\phi . \tag{14.17}$$

To find ϕ, we divide Eq. (14.17) by Eq. (14.14). This eliminates *A* and gives an equation that we can solve for ϕ:

$$\frac{\upsilon_{0x}}{x_0} = \frac{-\omega A \sin\phi}{A\cos\phi} = -\omega \tan\phi,$$

$$\phi = \arctan\left(-\frac{\upsilon_{0x}}{\omega x_0}\right) \qquad \text{(phase angle in SHM).} \qquad (14.18)$$

It is also easy to find the amplitude A if we are given x_0 and υ_{0x}. We'll sketch the derivation, and you can fill in the details. Square Eq. (14.14); then divide Eq. (14.17) by ω, square it, and add to the square of Eq. (14.14). The right side will be $A^2(\sin^2\phi + \cos^2\phi)$, which is equal to A^2. The final result is

$$A = \sqrt{x_0^2 + \frac{\upsilon_{0x}{}^2}{\omega^2}} \qquad \text{(amplitude in SHM).} \qquad (14.19)$$

Note that when the object has both an initial displacement x_0 and a nonzero initial velocity υ_{0x}, the amplitude A is *not* equal to the initial displacement. That's reasonable; if you start the object at a positive x_0 but give it a positive velocity υ_{0x}, it will go *farther* than x_0 before it turns and comes back, and so $A > x_0$.

PROBLEM-SOLVING STRATEGY

14.1 Simple Harmonic Motion I: Describing Motion

IDENTIFY *the relevant concepts:*

An oscillating system undergoes simple harmonic motion (SHM) *only* if the restoring force is directly proportional to the displacement.

SET UP *the problem*:

- Identify the known and unknown quantities, and determine which are the target variables.
- Distinguish between two kinds of quantities. *Properties of the system* include the mass *m*, the force constant *k*, and quantities derived from *m* and *k*, such as the period *T*, frequency *f*, and angular frequency ω. These are independent of *properties of the motion*, which describe how the system behaves when it is set into motion in a particular way; they include the amplitude *A*, maximum velocity $\upsilon_{\max}$, and phase angle ϕ f, and values of *x*, υ_x, and a_x at particular times.
- If necessary, define an *x*-axis as in Fig. 14.14, with the equilibrium position at *x* = 0.

EXECUTE *the solution*:

1. Use the equations given in Sections 14.1 and 14.2 to solve for the target variables.
2. To find the values of *x*, υ_x, and a_x at particular times, use Eqs. (14.13), (14.15), and (14.16), respectively. If both the initial displacement x_0 and initial velocity υ_{0x} are given, determine ϕ and *A* from Eqs. (14.18) and (14.19). If the object has an initial positive displacement x_0 but zero initial velocity (x_0 = 0), then the amplitude is *A* = x_0 and the phase angle is ϕ = 0. If it has an initial positive velocity υ_{0x} but no initial displacement (x_0 = 0), the amplitude is $A = \upsilon_{0x}/\omega$ and the phase angle is $\phi = -\pi/2$. Express all phase angles in *radians*.

EVALUATE *your answer:*

Make sure that your results are consistent. For example, suppose you used x_0 and υ_{0x} to find general expressions for x and υ_x at time t. If you substitute $t = 0$ into these expressions, you should get back the given values of x_0 and υ_{0x}.

14.3 Energy in Simple Harmonic Motion

We can learn even more about simple harmonic motion by using energy considerations. The only horizontal force on the object in SHM in Figs. 14.2 and 14.14 is the conservative force exerted by an ideal spring. The vertical forces do no work, so the total mechanical energy of the system is *conserved*. We also assume that the mass of the spring itself is negligible.

The kinetic energy of the object is $K = \frac{1}{2}m\upsilon^2$ and the potential energy of the spring is $K = \frac{1}{2}kx^2$, just as in Section 7.2. There are no nonconservative forces that do work, so the total mechanical energy $E = K + U$ is conserved:

$$E = \tfrac{1}{2}m\upsilon_x^2 + \tfrac{1}{2}kx^2 = \text{constant}\,. \qquad (14.20)$$

(Since the motion is one-dimensional, $\upsilon^2 = \upsilon_x^2$).

The total mechanical energy E is also directly related to the amplitude A of the motion. When the object reaches the point $x = A$, its maximum displacement from equilibrium, it momentarily stops as it turns back toward the equilibrium position. That is, when $x = A$ (or $-A$), $\upsilon_x = 0$. At this point the energy is entirely potential, and $E = \frac{1}{2}kA^2$. Because E is constant, it is equal to $\frac{1}{2}kA^2$ at any other point. Combining this expression with Eq. (14.20), we get

$$\text{Total mechanical energy in simple harmonic motion} \rightarrow E = \tfrac{1}{2}\overset{\text{Mass}}{m}\underset{\text{Velocity}}{\upsilon_x^2} + \tfrac{1}{2}\overset{\text{Force constant of restoring force}}{k}\underset{\text{Displacement}}{x^2} = \tfrac{1}{2}k\underset{\text{Amplitude}}{A^2} = \text{constant} \qquad (14.21)$$

We can verify this equation by substituting x and υ_x from Eqs. (14.13) and (14.15) and using $\omega^2 = k/m$ from Eq. (14.9):

$$\begin{aligned} E &= \tfrac{1}{2}m\upsilon_x^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}m\left[-\omega A\sin(\omega t+\phi)\right]^2 + \tfrac{1}{2}k\left[S\cos(\omega t+\phi)\right]^2 \\ &= \tfrac{1}{2}kA^2\sin^2(\omega t+\phi) + \tfrac{1}{2}kA^2\cos^2(\omega t+\phi) = \tfrac{1}{2}kA^2 \end{aligned}$$

(Recall that $\sin^2\alpha + \cos^2\alpha = 1$).

Hence our expressions for displacement and velocity in SHM are consistent with energy conservation, as they must be.

We can use Eq. (14.21) to solve for the velocity υ_x of the object at a given displacement x:

$$\upsilon_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2}\,. \qquad (14.22)$$

The $\pm$ sign means that at a given value of x the object can be moving in either direction. For example, when $x = \pm A/2$,

$$\upsilon_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - \left(\pm\frac{A}{2}\right)^2} = \pm\sqrt{\frac{3}{4}}\sqrt{\frac{k}{m}}A\,.$$

Equation (14.22) also shows that the *maximum* speed υ_{max} occurs at $x = 0$. Using Eq. (14.10), $\omega = \sqrt{k/m}$, we find that

$$v_x = \sqrt{\frac{k}{m}}A = \omega A. \tag{14.23}$$

This agrees with Eq. (14.15): v_x oscillates between $-\omega A$ and $+\omega A$.

Interpreting *E*, *K*, and *U* in SHM

Figure 14.12 shows the energy quantities E, K, and U at x = 0, $x = \pm A/2$, and $x = \pm A$. **Figure 14.13** is a graphical display of Eq. (14.21); energy (kinetic, potential, and total) is plotted vertically and the coordinate x is plotted horizontally. The parabolic curve in Fig. 14.13a represents the potential energy $U = \frac{1}{2}kx^2$. The horizontal line represents the total mechanical energy E, which is constant and does not vary with x. At any value of x between $-A$ and A, the vertical distance from the x-axis to the parabola is U; since $E = K + U$, the remaining vertical distance up to the horizontal line is K. Figure 14.13 shows both K and U as functions of x. The horizontal line for E intersects the potential-energy curve at $x = -A$ and $x = A$, so at these points the energy is entirely potential, the kinetic energy is zero, and the object comes momentarily to rest before reversing direction. As the object oscillates between $-A$ and A, the energy is continuously transformed from potential to kinetic and back again.

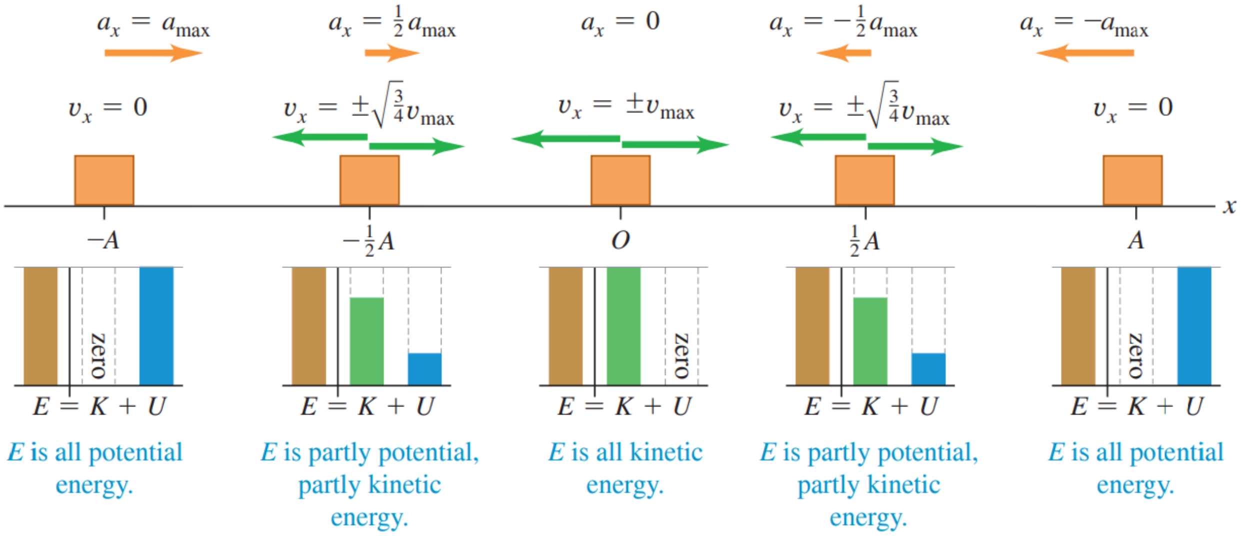


Figure 14.12 - Graphs of E, K, and U versus displacement in SHM. The velocity of the object is *not* constant, so these images of the object at equally spaced positions are *not* equally spaced in time

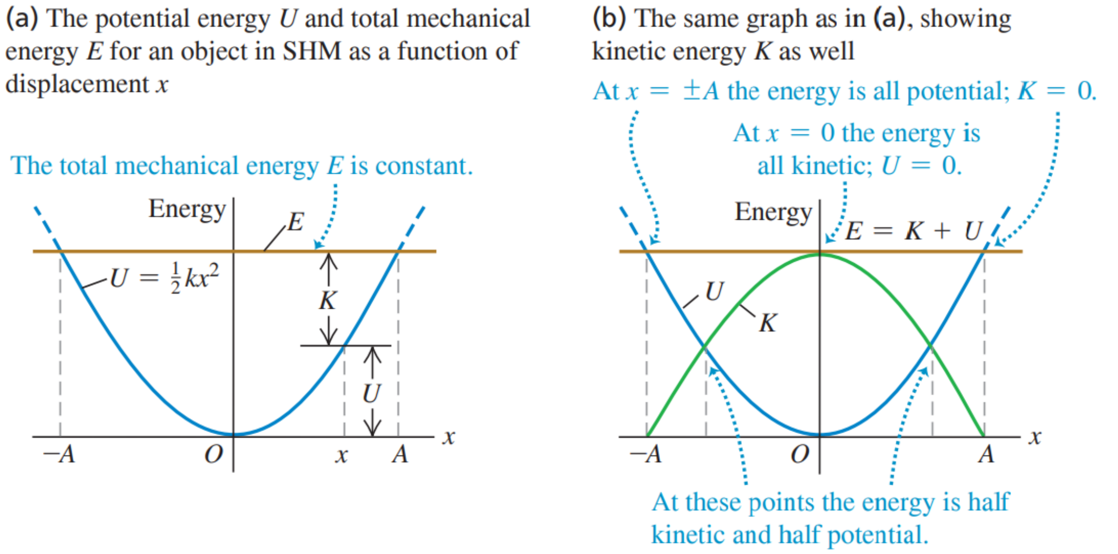


Figure 14.13 - Kinetic energy K, potential energy U, and total mechanical energy E as functions of displacement for SHM. At each value of x the sum of the values of K and U equals the constant value of E. Can you show that the energy is half kinetic and half potential at $x = \pm\sqrt{\frac{1}{2}}A$?

Figure 14.13a shows the connection between the amplitude A and the corresponding total mechanical energy $E = \frac{1}{2}kA^2$. If we tried to make x greater than A (or less than $-A$), U would be greater than E, and K would have to be negative. But K can never be negative, so x can't be greater than A or less than $-A$.

PROBLEM-SOLVING STRATEGY
14.1Simple Harmonic Motion I: Describing Motion

The SHM energy equation, Eq. (14.21), is a useful relationship among velocity, displacement, and total mechanical energy. If a problem requires you to relate displacement, velocity, and acceleration without reference to time, consider using Eq. (14.4) (from Newton's second law) or Eq. (14.21) (from energy conservation).

Because Eq. (14.21) involves x^2 and υ_x^2, you must infer the signs of x and υ_x from the situation. For instance, if the object is moving from the equilibrium position toward the point of greatest positive displacement, then x is positive and υ_x is positive.

14.4 Applications of Simple Harmonic Motion

So far, we've looked at a grand total of *one* situation in which simple harmonic motion (SHM) occurs: an object attached to an ideal horizontal spring. But SHM can occur in any system in which there is a restoring force that is directly proportional to the displacement from equilibrium, as given by Eq. (14.3), $F_x = -kx$. The restoring force originates in different ways in different situations, so we must find the force constant k for each case by examining the net force on the system. Once this is done, it's straightforward to find the angular frequency ω, frequency f, and period T; we just substitute the value of k into Eqs. (14.10), (14.11), and (14.12), respectively. Let's use these ideas to examine several examples of simple harmonic motion.

Vertical SHM

Suppose we hang a spring with force constant k (Fig. 14.14a) and suspend from it an object with mass m. Oscillations will now be vertical; will they still be SHM? In Fig. 14.14b the object hangs at rest, in equilibrium. In this position the spring is stretched an amount Δl just great enough that the spring's upward vertical force k Δl on the object balances its weight mg:

$$k\Delta l = mg\ .$$

Take $x = 0$ to be this equilibrium position and take the positive x-direction to be upward. When the object is a distance *xabove* its equilibrium position (Fig. 14.14c), the extension of the spring is Δl - x. The upward force it exerts on the object is then $k(\Delta l - x)$, and the net x-component of force on the object is

$$F_{\text{net}} = k(\Delta l - x) + (-mg) = -kx\ .$$

that is, a net downward force of magnitude kx. Similarly, when the object is *below* the equilibrium position, there is a net upward force with magnitude kx. In either case there is a restoring force with magnitude kx. If the object is set in vertical motion, it oscillates in SHM with the same angular frequency as though it were horizontal, $\omega = \sqrt{k/m}$. So vertical SHM doesn't differ in any essential way from horizontal SHM. The only real change is that the equilibrium position $x = 0$ no longer corresponds to the point at which the spring is unstretched. The same ideas hold if an object with weight mg is placed atop a compressible spring (Fig. 14.15) and compresses it a distance Δl.

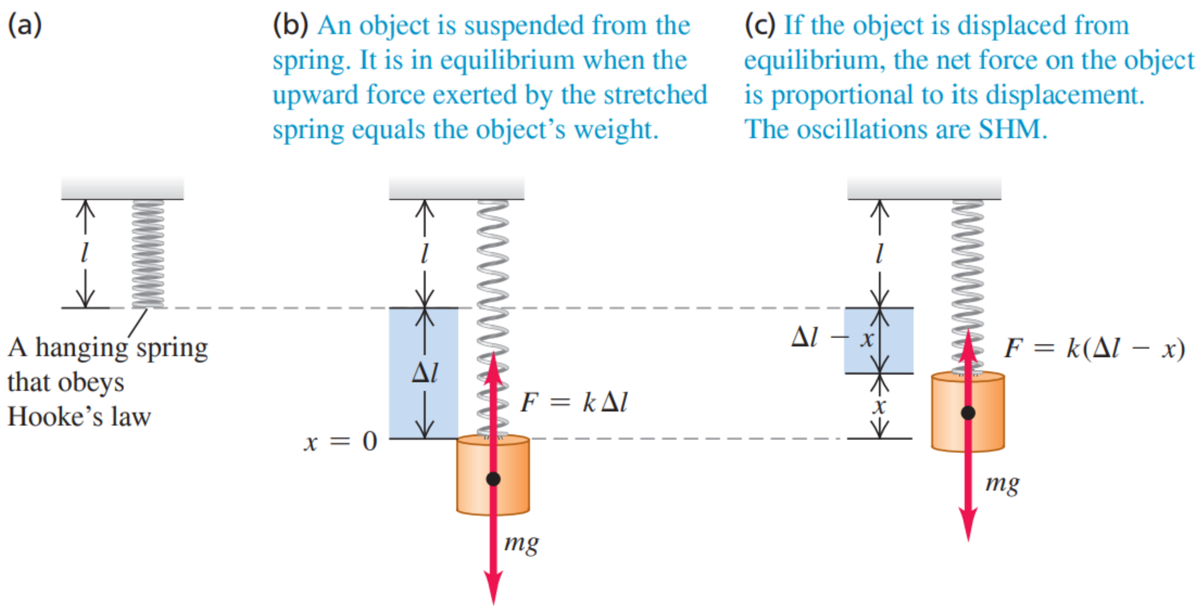


Figure 14.14 - An object attached to a hanging spring

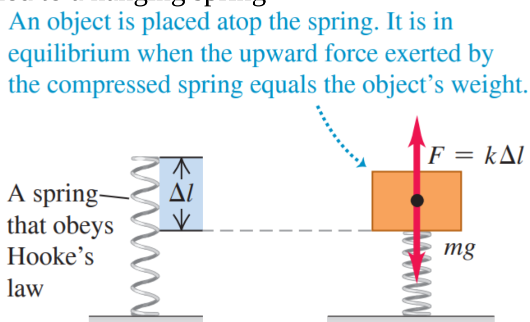


Figure 14.15 - If the weight mg compresses the spring by a distance Δl, the force constant is $k = mg/\Delta l$ and the angular frequency for vertical SHM is $\omega = \sqrt{k/m}$ - the same as if the object were suspended from the spring (see Fig. 14.14)

Angular SHM

A mechanical watch keeps time based on the oscillations of a balance wheel (Fig. 14.16). The wheel has a moment of inertia I about its axis. A coil spring exerts a restoring torque τ_z that is proportional to the angular displacement θ from the equilibrium position. We write $\tau_z = \kappa\theta$, where κ (the Greek letter kappa) is a constant called the *torsion constant*. Using the rotational analog of Newton's second law for a rigid body $\sum \tau_z = I\alpha_z = Id^2\theta/dt^2$, Eq. (10.7), we find

$$-\kappa\theta = I\alpha \quad \text{or} \quad \frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta .$$

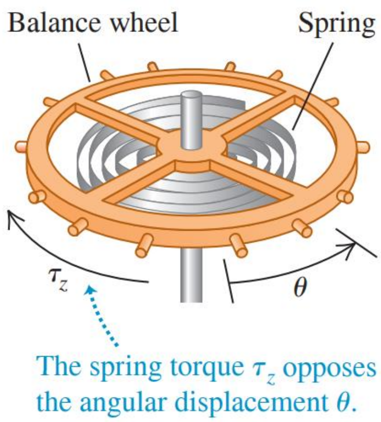


Figure 14.16 - The balance wheel of a mechanical watch. The spring exerts a restoring torque that is proportional to the angular displacement θ, so the motion is angular SHM

This equation is exactly the same as Eq. (14.4) for simple harmonic motion, with x replaced by θ and k/m replaced by κ/I. So we are dealing with a form of *angular* simple harmonic motion. The angular frequency ω and frequency f are given by Eqs. (14.10) and (14.11), respectively, with the same replacement:

Angular simple harmonic motion

$$\underbrace{\omega}_{\text{Angular frequency}} = \sqrt{\frac{\kappa}{I}} \quad \text{and} \quad \underbrace{f}_{\text{Frequency}} = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}} \qquad (14.24)$$

Torsion constant divided by moment of inertia

The angular displacement θ as a function of time is given by

$$\theta = \Theta \cos(\omega t + \phi),$$

where Θ (the capital Greek letter theta) plays the role of an angular amplitude.

It's a good thing that the motion of a balance wheel *is* simple harmonic. If it weren't, the frequency might depend on the amplitude, and the watch would run too fast or too slow as the spring ran down.

Vibrations of Molecules

The following discussion of the vibrations of molecules uses the binomial theorem. If you aren't familiar with this theorem, you should read about it in the appropriate section of a math textbook.

When two atoms are separated by a few atomic diameters, they can exert attractive forces on each other. But if the atoms are so close that their electron shells overlap, the atoms repel each other. Between these limits, there can be an equilibrium separation distance at which two atoms form a *molecule*. If these atoms are displaced slightly from equilibrium, they will oscillate.

Let's consider one type of interaction between atoms called the *van der Waals interaction*. Our immediate task here is to study oscillations, so we won't go into the details of how this interaction arises. Let the center of one atom be at the origin and let the center of the other atom be a distance r away (Fig. 14.17a); the equilibrium distance between centers is $r = R_0$. Experiment shows that the van der Waals interaction can be described by the potential-energy function

$$U = U_0\left[\left(\frac{R_0}{r}\right)^{12} - 2\left(\frac{R_0}{r}\right)^{6}\right], \qquad (14.25)$$

where U_0 is a positive constant with units of joules. When the two atoms are very far apart, $U = 0$; when they are separated by the equilibrium distance $r = R_0$, $U = -U_0$. From Section 7.4, the force on the second atom is the negative derivative of Eq. (14.25):

$$F_r = -\frac{dU}{dr} = U_0\left[\frac{12R_0^{12}}{r^{13}} - 2\frac{6R_0^{6}}{r^{7}}\right] = 12\frac{U_0}{R_0}\left[\left(\frac{R_0}{r}\right)^{13} - \left(\frac{R_0}{r}\right)^{7}\right]. \qquad (14.26)$$

Figures 14.17b and 14.17c plot the potential energy and force, respectively. The force is positive for $r < R_0$ and negative for $r > R_0$, so it is a *restoring* force.

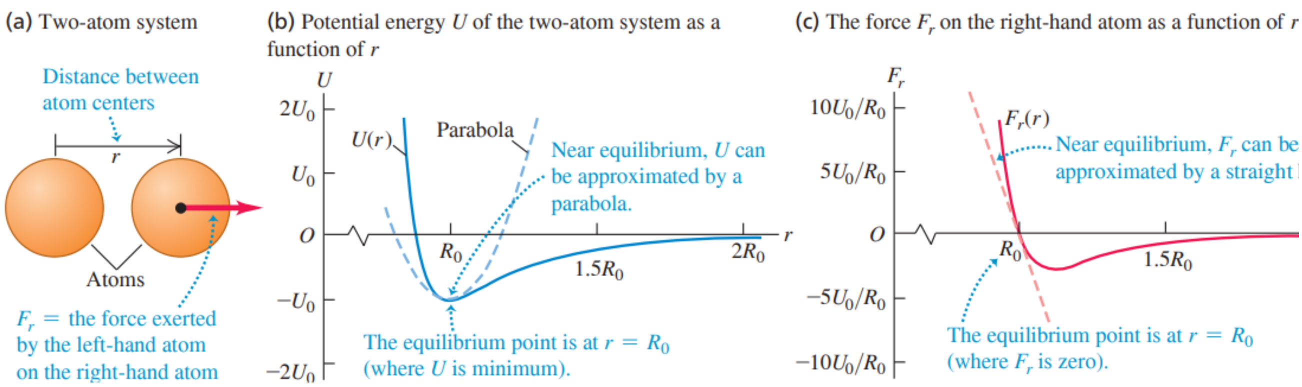


Figure 14.17 - (a) Two atoms with centers separated by r. (b) Potential energy U, and (c) Force F_r in the van der Waals interaction

Let's examine the restoring force F_r in Eq. (14.26). We let x represent the displacement from equilibrium:

$$x = r - R_0 \text{, so } r = R_0 + x .$$

In terms of x, the force F_r in Eq. (14.26) becomes

$$F_r = 12\frac{U_0}{R_0}\left[\left(\frac{R_0}{R_0+x}\right)^{13} - \left(\frac{R_0}{R_0+x}\right)^{7}\right] = 12\frac{U_0}{R_0}\left[\frac{1}{\left(1+x/R_0\right)^{13}} - \frac{1}{\left(1+x/R_0\right)^{7}}\right]. \qquad (14.27)$$

This looks nothing like Hooke's law, $F_x = -kx$, so we might be tempted to conclude that molecular oscillations cannot be SHM. But let us restrict ourselves to *small-amplitude* oscillations so that the absolute value of the displacement x is small in comparison to R_0 and the absolute value of the ratio x/R_0 is much less than 1. We can then simplify Eq. (14.27) by using the *binomial theorem*:

$$(1+u)^n = 1 + nu + \frac{n(n-1)}{2!}u^2 + \frac{n(n-1)(n-2)}{3!}u^3 + ... \qquad (14.28)$$

If $|u|$ is much less than 1, each successive term in Eq. (14.28) is much smaller than the one it follows, and we can safely approximate $\left(1+u\right)^n$ by just the first two terms. In Eq. (14.27), u is replaced by x/R_0 and n equals -13 or -7, so

$$\frac{1}{(1+x/R_0)^{13}} = (1+x/R_0)^{-13} \approx 1 + (-13)\frac{x}{R_0},$$

$$\frac{1}{(1+x/R_0)^{7}} = (1+x/R_0)^{-7} \approx 1 + (-7)\frac{x}{R_0},$$

$$F_r \approx 12\frac{U_0}{R_0}\left[\left(1+(-13)\frac{x}{R_0}\right) - \left(1+(-7)\frac{x}{R_0}\right)\right] = -\left(\frac{72U_0}{R_0^2}\right)x. \qquad (14.29)$$

This is just Hooke's law, with force constant $k = 72U_0/R_0^2$. (Note that k has the correct units, J/m^2 or N/m). So oscillations of molecules bound by the van der Waals interaction can be simple harmonic motion, provided that the amplitude is small in comparison to R_0 so that the approximation $|x/R_0| \ll 1$ used in the derivation of Eq. (14.29) is valid.

You can also use the binomial theorem to show that the potential energy U in Eq. (14.25) can be written as $U \approx \frac{1}{2}kx^2 + C$, where $C = -U_0$ and k is again equal to $72U_0/R_0^2$. Adding a constant to the potential-energy function has no effect on the physics, so the system of two atoms is fundamentally no different from a mass attached to a horizontal spring for which $U \approx \frac{1}{2}kx^2$.

14.5 The Simple Pendulum

A **simple pendulum** is an idealized model consisting of a point mass suspended by a massless, unstretchable string. When the point mass is pulled to one side of its straightdown equilibrium position and released, it oscillates about the equilibrium position. Familiar situations such as a wrecking ball on a crane's cable or a person on a swing can be modeled as simple pendulums.

The path of the point mass (sometimes called a *pendulum bob*) is not a straight line but the arc of a circle with radius L equal to the length of the string (**Fig. 14.18**). We use as our coordinate the distance x measured along the arc. If the motion is simple harmonic, the restoring force must be directly proportional to x or (because $x = L\theta$) to θ . Is it?

Figure 14.18 shows the radial and tangential components of the forces on the mass. The restoring force F_θ is the tangential component of the net force:

$$F_\theta = -mg\sin\theta . \qquad (14.30)$$

Gravity provides the restoring force F_θ ; the tension T merely acts to make the point mass move in an arc. Since F_θ is proportional to $\sin\theta$, not to θ, the motion is *not* simple harmonic. However, if angle θ is *small*, sin u is very nearly equal to θ in radians (**Fig.14.19**). (When $\theta = 0.1$ rad, about 6°, $\sin\theta = 0.0998$. That's only 0.2% different). With this approximation, Eq. (14.30) becomes

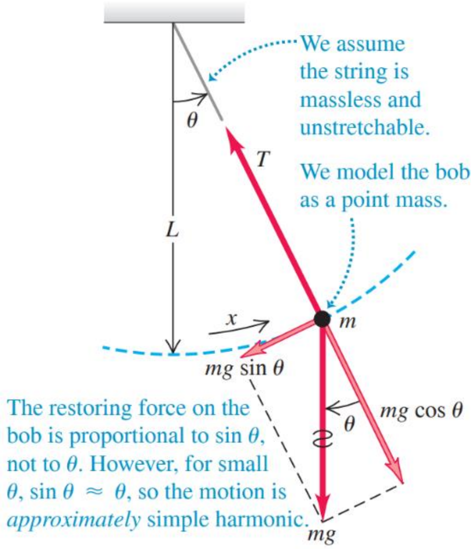


Figure 14.18 - The dynamics of a simple pendulum. An idealized simple pendulum

$F_\theta = -n$ 14.31)

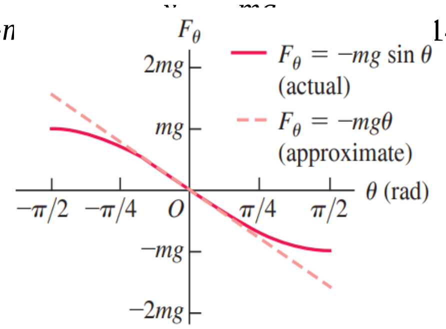


Figure 14.19 - For small angular displacements θ, the restoring force $F_\theta = -mg\sin\theta$ on a simple pendulum is approximately equal to $-mg\theta$; that is, it is approximately proportional to the displacement θ. Hence for small angles the oscillations are simple harmonic

The restoring force is then proportional to the coordinate for small displacements, and the force constant is $k = mg/L$. From Eq. (14.10) the angular frequency ω of a simple pendulum with small amplitude is

Angular frequency of simple pendulum, small amplitude

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg/L}{m}} = \sqrt{\frac{g}{L}} \qquad (14.32)$$

Acceleration due to gravity; Pendulum length; Pendulum mass (cancels)

The corresponding frequency and period relationships are

Frequency of simple pendulum, small amplitude

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \qquad (14.33)$$

Angular frequency; Acceleration due to gravity; Pendulum length

Period of simple pendulum, small amplitude

$$T = \frac{2\pi}{\omega} = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}} \qquad (14.34)$$

Pendulum length; Acceleration due to gravity; Angular frequency; Frequency

These expressions don't involve the *mass* of the particle. That's because the gravitational restoring force is proportional to *m*, so the mass appears on *both* sides of $\sum \vec{F} = m\vec{a}$ and cancels out. (The same physics explains why objects of different masses fall with the same acceleration in a vacuum). For small oscillations, the period of a pendulum for a given value of g is determined entirely by its length.

Equations (14.32) through (14.34) tell us that a long pendulum (large *L*) has a longer period than a shorter one. Increasing *g* increases the restoring force, causing the frequency to increase and the period to decrease.

The motion of a pendulum is only *approximately* simple harmonic. When the maximum angular displacement Θ (amplitude) is not small, the departures from simple harmonic motion can be substantial. In general, the period *T* is given by

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1^2}{2^2}\sin^2\frac{\Theta}{2} + \frac{1^2 \bullet 3^2}{2^2 \bullet 4^2}\sin^4\frac{\Theta}{2} + \ldots\right). \tag{14.35}$$

We can compute *T* to any desired degree of precision by taking enough terms in the series. You can confirm that when Θ = 15°, the true period is longer than that given by the approximate Eq. (14.34) by less than 0.5 %.

A pendulum is a useful timekeeper because the period is *very nearly* independent of amplitude, provided that the amplitude is small. Thus, as a pendulum clock runs down and the amplitude of the swings decreases a little, the clock still keeps very nearly correct time.

14.6 The Physical Pendulum

A **physical pendulum** is any *real* pendulum that uses an extended object, in contrast to the idealized *simple* pendulum with all of its mass concentrated at a point. **Figure 14.20** shows an object of irregular shape pivoted so that it can turn without friction about an axis through point *O*. In equilibrium the center of gravity (cg) is directly below the pivot; in the position shown, the object is displaced from equilibrium by an angle u, which we use as a coordinate for the system. The distance from *O* to the center of gravity is *d*, the moment of inertia of the object about the axis of rotation through *O* is *I*, and the total mass is *m*. When the object is displaced as shown, the weight *mg* causes a restoring torque.

$$\tau_z = -(mg)(d\sin\theta)\,. \tag{14.36}$$

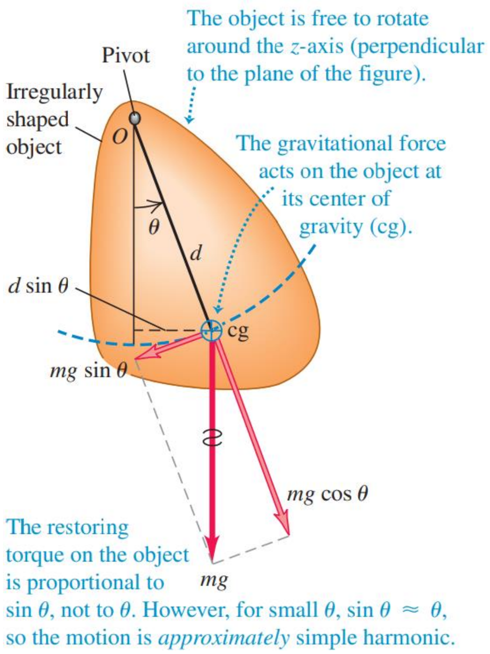


Figure 14.20 - Dynamics of a physical pendulum

The negative sign shows that the restoring torque is clockwise when the displacement is counterclockwise, and vice versa.

When the object is released, it oscillates about its equilibrium position. The motion is not simple harmonic because the torque τ_z is proportional to $\sin\theta$ rather than to u itself. However, if θ is small, we can approximate $\sin\theta$ by θ in radians, just as we did in analyzing the simple pendulum. Then the motion is *approximately* simple harmonic. With this approximation,

$$\tau_z = -(mgd)\theta\,.$$

From Section 10.2, the equation of motion is $\sum \tau_z = I\alpha_z$ so

$$-(mgd)\theta = I\alpha_z = I\frac{d^2\theta}{dt^2},$$

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\theta . \qquad (14.37)$$

Comparing this with Eq. (14.4), we see that the role of (k/m) for the spring-mass system is played here by the quantity (mgd/I). Thus the angular frequency is

Angular frequency of physical pendulum, small amplitude

$$\omega = \sqrt{\frac{mgd}{I}} \qquad (14.38)$$

Mass; Acceleration due to gravity; Distance from rotation axis to center of gravity; Moment of inertia

The frequency f is $1/2\pi$ times this, and the period T is $1>f$:

Period of physical pendulum, small amplitude

$$T = 2\pi\sqrt{\frac{I}{mgd}} \qquad (14.39)$$

Moment of inertia; Distance from rotation axis to center of gravity; Mass; Acceleration due to gravity

Equation (14.39) is the basis of a common method for experimentally determining the moment of inertia of an object with a complicated shape. First locate the center of gravity by balancing the object. Then suspend the object so that it is free to oscillate about an axis, and measure the period T of small-amplitude oscillations. Finally, use Eq. (14.39) to calculate the moment of inertia I of the object about this axis from T, the object's mass m, and the distance d from the axis to the center of gravity. Biomechanics researchers use this method to find the moments of inertia of an animal's limbs. This information is important for analyzing how an animal walks, as we'll see in the second of the two following examples.

14.7 Damped Oscillations

The idealized oscillating systems we have discussed so far are frictionless. There are no nonconservative forces, the total mechanical energy is constant, and a system set into motion continues oscillating forever with no decrease in amplitude.

Real-world systems always have some dissipative forces, however, and oscillations die out with time unless we replace the dissipated mechanical energy. A mechanical pendulum clock continues to run because potential energy stored in the spring or a hanging weight system replaces the mechanical energy lost due to friction in the pivot and the gears. But eventually the spring runs down or the weights reach the bottom of their travel. Then no more energy is available, and the pendulum swings decrease in amplitude and stop.

The decrease in amplitude caused by dissipative forces is called **damping** (*not* "dampening"), and the corresponding motion is called **damped oscillation**. The simplest case is a simple harmonic oscillator with a frictional damping force that is directly proportional to the *velocity* of the oscillating object. This behavior occurs in friction involving viscous fluid flow, such as in shock absorbers or sliding between oil-lubricated surfaces. We then have an additional force on the object due to friction, $F_x = -b\upsilon_x$, where $\upsilon_x = dx/dt$ is the velocity and b is a constant that describes the strength of the damping force. The negative sign shows that the force is always opposite in direction to the velocity. The *net* force on the object is then

$$\sum F_x = -kx - b\upsilon_x , \qquad (14.40)$$

and Newton's second law for the system is

$$-kx - bv_x = ma_x \quad \text{or} \quad -kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}\,. \qquad (14.41)$$

Equation (14.41) is a differential equation for x; it's the same as Eq. (14.4), the equation for the acceleration in SHM, but with the added term $-bdx/dt$. We won't go into how to solve this equation; we'll just present the solution. If the damping force is relatively small, the motion is described by

Initial amplitude · Damping constant · Mass · Time

Displacement of oscillator, little damping

$$x = Ae^{-(b/2m)t}\cos(\omega' t + \phi) \qquad (14.42)$$

Angular frequency of damped oscillations · Phase angle

The angular frequency of these damped oscillations is given by

Force constant of restoring force

Angular frequency of oscillator, little damping

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \qquad (14.43)$$

Damping constant · Mass

You can verify that Eq. (14.42) is a solution of Eq. (14.41) by calculating the first and second derivatives of x, substituting them into Eq. (14.41), and checking whether the left and right sides are equal.

The motion described by Eq. (14.42) differs from the undamped case in two ways. First, the amplitude $Ae^{-(b/2m)t}$ is not constant but decreases with time because of the exponential factor $e^{-(b/2m)t}$. **Figure 14.21** is a graph of Eq. (14.42) for $\phi = 0$; the larger the value of b, the more quickly the amplitude decreases.

Second, the angular frequency ω', given by Eq. (14.43), is no longer equal to $\omega = \sqrt{k/m}$ but is somewhat smaller. It becomes zero when b becomes so large that

$$\frac{k}{m} - \frac{b^2}{4m^2} = 0 \quad \text{or} \quad b = \sqrt{km}\,. \qquad (14.44)$$

When Eq. (14.44) is satisfied, the condition is called **critical damping**. The system no longer oscillates but returns to its equilibrium position without oscillation when it is displaced and released.

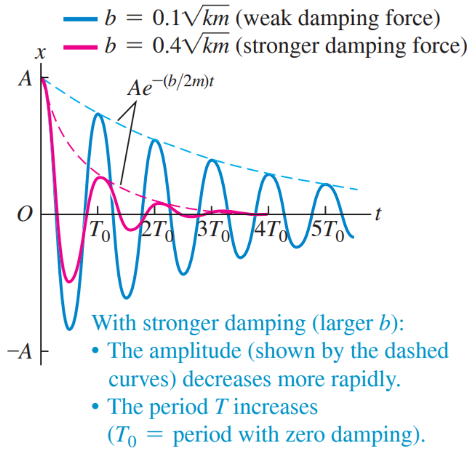


Figure 14.21 - Graph of displacement versus time for an oscillator with little damping [see Eq. (14.42)] and with phase angle $\phi = 0$. The curves are for two values of the damping constant b

CAUTION! **When frequencies are imaginary.** Note that when there is overdamping and b is greater than $2\sqrt{km}$, the argument of the square root in Eq. (14.43) is negative and the angular frequency of oscillation ω is an imaginary number. This is a mathematical clue that there is *no* oscillation in this case.

If b is greater than $2\sqrt{km}$, the condition is called **overdamping**. Again there is no oscillation, but the system returns to equilibrium more slowly than with critical damping. For the overdamped case the solutions of Eq. (14.41) have the form

$$x = C_1e^{-a_1t} + C_2e^{-a_2t},$$

where C_2 and C_2 are constants that depend on the initial conditions and a_1 and a_2 are constants determined by m, k, and b.

When b is less than the critical value, as in Eq. (14.42), the condition is called **underdamping**. The system oscillates with steadily decreasing amplitude.

In a vibrating tuning fork or guitar string, it is usually desirable to have as little damping as possible. By contrast, damping plays a beneficial role in the oscillations of an automobile's suspension system. The shock absorbers provide a velocity-dependent damping force so that when the car goes over a bump, it doesn't continue bouncing forever (**Fig.14.22**). For optimal passenger comfort, the system should be critically damped or slightly underdamped. Too much damping would be counterproductive; if the suspension is overdamped and the car hits a second bump just after the first one, the springs in the suspension will still be compressed somewhat from the first bump and will not be able to fully absorb the impact.

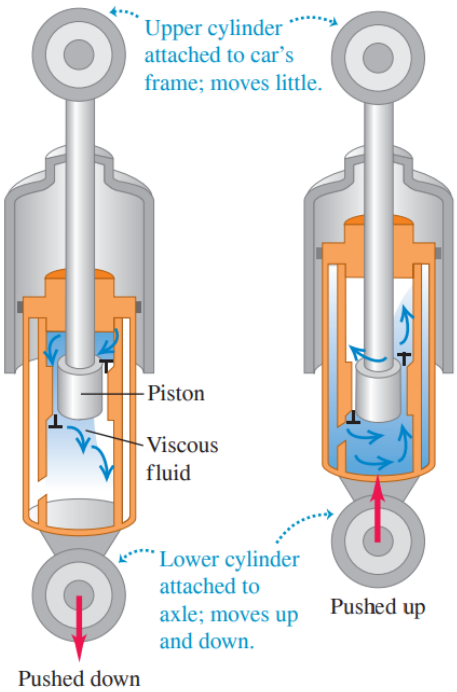


Figure 14.22 - An automobile shock absorber. The viscous fluid causes a damping force that depends on the relative velocity of the two ends of the unit

Vibrations of Molecules

In damped oscillations the damping force is nonconservative; the total mechanical energy of the system is not constant but decreases continuously, approaching zero after a long time. To derive an expression for the rate of change of energy, we first write an expression for the total mechanical energy E at any instant:

$$E=\tfrac{1}{2}m\upsilon_x^2+\tfrac{1}{2}kx^2 .$$

To find the rate of change of this quantity, we take its time derivative:

$$\frac{dE}{dt}=m\upsilon_x\frac{d\upsilon_x}{dt}+kx\frac{dx}{dt}.$$

But $d\upsilon_x/dt=a_x$ and $dx/dt=\upsilon_x$, so

$$\frac{dE}{dt}=\upsilon_x(ma_x+kx).$$

From Eq. (14.41), $ma_x+kx=-b\ dx/dt=-b\upsilon_x$, so

$$\frac{dE}{dt}=\upsilon_x(-b\upsilon_x)=-b\upsilon_x^2 \text{ (damped oscillations).} \qquad (14.45)$$

The right side of Eq. (14.45) is negative whenever the oscillating object is in motion, whether the x-velocity υ_x is positive or negative. This shows that as the object moves, the energy decreases, though not at a uniform rate. The term $-b\upsilon_x^2=(-b\upsilon_x)\upsilon_x$ (force times velocity) is the rate at which the damping force does (negative) work on the system (that is, the damping *power*). This equals the rate of change of the total mechanical energy of the system.

Similar behavior occurs in electric circuits containing inductance, capacitance, and resistance. There is a natural frequency of oscillation, and the resistance plays the role of the damping constant b.

14.7 Forced Oscillations and Resonance

A damped oscillator left to itself will eventually stop moving. But we can maintain a constant-amplitude oscillation by applying a force that varies with time in a periodic way. As an example, consider your cousin Throckmorton on a playground swing. You can keep him swinging with constant amplitude by giving him a push once each cycle. We call this additional force a **driving force**.

Damped Oscillation with a Periodic Driving Force

If we apply a periodic driving force with angular frequency ω_d to a damped harmonic oscillator, the motion that results is called a **forced oscillation** or a *driven oscillation*. It is different from the motion that occurs when the system is simply displaced from equilibrium and then left alone, in which case the system oscillates with a **natural angular frequency** ω' determined by *m*, *k*, and *b*, as in Eq. (14.43). In a forced oscillation, however, the angular frequency with which the mass oscillates is equal to the driving angular frequency ω_d. This does *not* have to be equal to the natural angular frequency ω'. If you grab the ropes of Throckmorton's swing, you can force the swing to oscillate with any frequency you like.

Suppose we force the oscillator to vibrate with an angular frequency ω_d that is nearly *equal* to the angular frequency ω' it would have with no driving force. What happens? The oscillator is naturally disposed to oscillate at $\omega = \omega'$, so we expect the amplitude of the resulting oscillation to be larger than when the two frequencies are very different. Detailed analysis and experiment show that this is just what happens. The easiest case to analyze is a *sinusoidally* varying force—say, $F(t) = F_{\max} \cos \omega_d t$. If we vary the frequency ω_d of the driving force, the amplitude of the resulting forced oscillation varies in an interesting way (Fig. 14.23). When there is very little damping (small *b*), the amplitude goes through a sharp peak as the driving angular frequency ω_d nears the natural oscillation angular frequency ω'. When the damping is increased (larger *b*), the peak becomes broader and smaller in height and shifts toward lower frequencies.

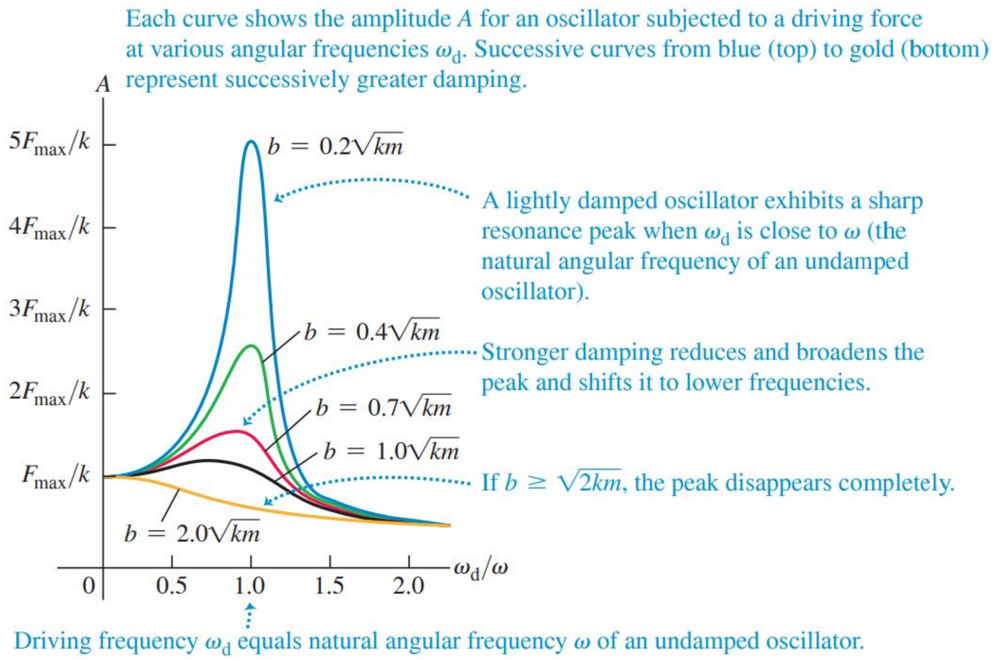


Figure 14.23 - Graph of the amplitude *A* of forced oscillation as a function of the angular frequency ω_d of the driving force. The horizontal axis shows the ratio of ω_d to the angular frequency $\omega = \sqrt{k/m}$ of an undamped oscillator. Each curve has a different value of the damping constant *b*

Using more differential equations than we're ready for, we could find an expression for the amplitude *A* of the forced oscillation as a function of the driving angular frequency. Here is the result:

$$A = \frac{F_{max}}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}} \qquad (14.46)$$

Maximum value of driving force; Amplitude of a forced oscillator; Damping constant; Force constant of restoring force; Mass; Driving angular frequency

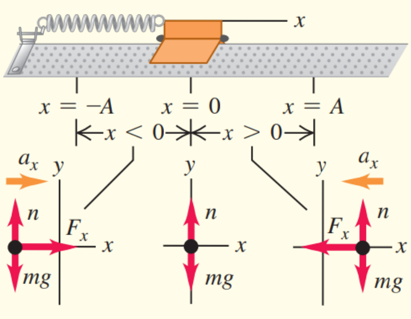
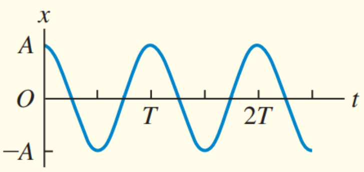
When $k - m\omega_d^2 = 0$, the first term under the radical is zero, so A has a maximum near $\omega_d = \sqrt{k/m}$. The height of the curve at this point is proportional to $1/b$; the less damping, the higher the peak. At the low-frequency extreme, when $\omega_d = 0$, we get $A = F_{max}/k$. This corresponds to a *constant* force F_{max} and a constant displacement $A = F_{max}/k$ from equilibrium, as we might expect.

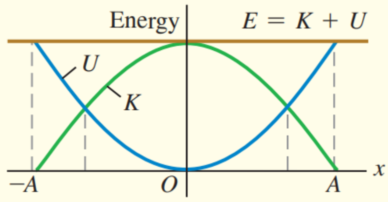
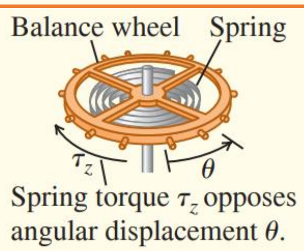
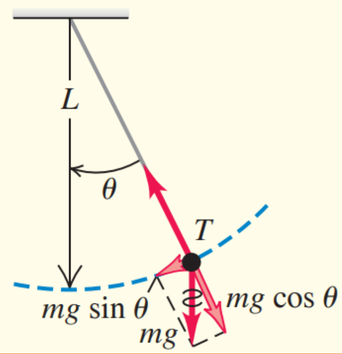
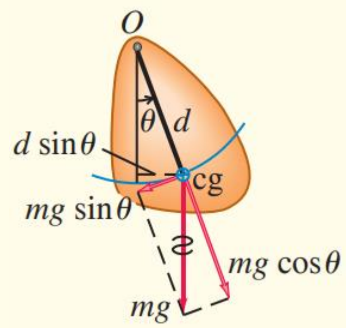
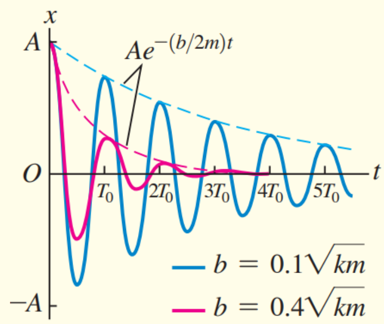
Resonance and Its Consequences

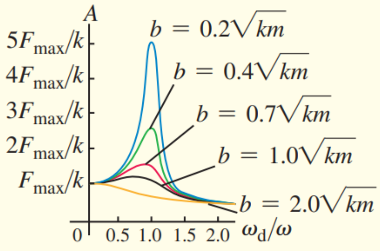
The peaking of the amplitude at driving frequencies close to the natural frequency of the system is called **resonance**. Physics is full of examples of resonance; building up the oscillations of a child on a swing by pushing with a frequency equal to the swing's natural frequency is one. A vibrating rattle in a car that occurs only at a certain engine speed is another example. Inexpensive loudspeakers often have an annoying boom or buzz when a musical note coincides with the natural frequency of the speaker cone or housing. In Chapter 16 we'll study examples of resonance that involve sound. Resonance also occurs in electric circuits, as we'll see in Chapter 31; a tuned circuit in a radio receiver responds strongly to waves with frequencies near its natural frequency. This phenomenon lets us select one radio station and reject other stations.

Resonance in mechanical systems can be destructive. A company of soldiers once destroyed a bridge by marching across it in step; the frequency of their steps was close to a natural frequency of the bridge, and the resulting oscillation had large enough amplitude to tear the bridge apart. Ever since, marching soldiers have been ordered to break step before crossing a bridge. Some years ago, vibrations of the engines of a particular type of airplane had just the right frequency to resonate with the natural frequencies of its wings. Large oscillations built up, and occasionally the wings fell off.

CHAPTER 14: SUMMARY

Periodic motion: Periodic motion is motion that repeats itself in a definite cycle. It occurs whenever an object has a stable equilibrium position and a restoring force that acts when the object is displaced from equilibrium. Period T is the time for one cycle. Frequency f is the number of cycles per unit time. Angular frequency ω is 2π times the frequency	$f = \frac{1}{T}$ $T = \frac{1}{f}$ $\omega = 2\pi f = \frac{2\pi}{T}$	
Simple harmonic motion: If the restoring force F_x in periodic motion is directly proportional to the displacement x, the motion is called simple harmonic motion (SHM). In many cases this condition is satisfied if the displacement from equilibrium is small. The angular frequency, frequency, and period in SHM do	$F_x = -kx$ $a_x = \frac{F_x}{m} = -\frac{k}{m}x$ $\omega = \sqrt{\frac{k}{m}}$ $f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$	

not depend on the amplitude but on only the mass m and force constant *k*. The displacement, velocity, and acceleration in SHM are sinusoidal functions of time; the amplitude *A* and phase angle ϕ of the oscillation are determined by the initial displacement and velocity of the object	$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}$ $x = A\cos(\omega t + \phi)$	
Energy in simple harmonic motion: Energy is conserved in SHM. The total energy can be expressed in terms of the force constant *k* and amplitude *A*	$E = \frac{1}{2}m\upsilon_x^2 + \frac{1}{2}kx^2$ $= \frac{1}{2}kA^2 = \text{constant}$	
Angular simple harmonic motion: In angular SHM, the frequency and angular frequency are related to the moment of inertia *I* and the torsion constant κ	$\omega = \sqrt{\frac{\kappa}{I}}$ and $f = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}}$	
Simple pendulum: A simple pendulum consists of a point mass *m* at the end of a massless string of length *L*. Its motion is approximately simple harmonic for sufficiently small amplitude; the angular frequency, frequency, and period then depend on only *g* and *L*, not on the mass or amplitude	$\omega = \sqrt{\frac{g}{L}}$ $f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$ $T = \frac{2\pi}{\omega} = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}}$	
Physical pendulum:*A* physical pendulum is any object suspended from an axis of rotation. The angular frequency and period for small-amplitude oscillations are independent of amplitude but depend on the mass *m*, distance *d* from the axis of rotation to the center of gravity, and moment of inertia *I* about the axis	$\omega = \sqrt{\frac{mgd}{I}}$ $T = 2\pi\sqrt{\frac{I}{mgd}}$	
Damped oscillations: When a force $F_x = -b\upsilon_x$ is added to a simple harmonic oscillator, the motion is called a damped oscillation. If $b < 2\sqrt{km}$ (called underdamping), the system oscillates with a decaying amplitude and an angular frequency ω' that is lower than it would be without damping. If $b = 2\sqrt{km}$ (called critical damping) or	$x = Ae^{-(b/2m)t}\cos(\omega' t + \phi)$ $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$	

$b > 2\sqrt{km}$ (called overdamping), when the system is displaced it returns to equilibrium without oscillating		
Forced oscillations and resonance: When a sinusoidally varying driving force is added to a damped harmonic oscillator, the resulting motion is called a forced oscillation or driven oscillation. The amplitude is a function of the driving frequency ω_d and reaches a peak at a driving frequency close to the natural frequency of the system. This behavior is called resonance	$A = \dfrac{F_{max}}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}}$	

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МАТЕМАТИЧНІ МОДЕЛІ У ФІЗИЦІ

Конспект лекцій
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денної форми навчання

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